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PROCEEDINGS

OF

THE LONDON MATHEMATICAL SOCIETY

SECOND SERIES

VOLUME 19

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RECORDS OF PROCEEDINGS AT MEETINGS.

SESSION NOVEMBER, 1919–JUNE, 1920.

Thursday, November 13th, 1919.

ANNUAL GENERAL MEETING.

Mr. J. E. CAMPBELL, President, in the Chair.

Present sixteen members and one visitor.

Mr. W. G. Bickley was admitted into the Society.

Messrs. C. Chaffer, C. G. Darwin, H. Freeman, W. V. Lovell, and T. Knox Shaw were elected members.

Messrs. W. N. Bailey, B. B. Baker, M. D. Bhansali, F. Bowman, E. F. Collingwood, C. Fox, P. Fraser, C. W. Gilham, E. A. Milne, E. M. Moors, S. P. Owen, S. Pollard, F. E. Relton, N. M. Shah, W. F. Sheppard, E. H. Smart, G. I. Taylor, G. P. Thomson, F. P. White, B. M. Wilson, and A. E. Williams were nominated for election.

The Treasurer presented his Report. On the motion of Mr. L. J. Mordell, seconded by Mr. T. L. Wren, the Report was received. Lt.-Col. A. Cunningham was appointed Auditor.

The Council and Officers for the ensuing Session were elected. The list is as follows:—President, Mr. J. E. Campbell; Vice-Presidents, Dr. T. J. I'A. Bromwich, Prof. H. M. Macdonald, Major P. A. MacMahon; Treasurer, Dr. A. E. Western; Secretaries, Mr. G. H. Hardy, Dr. G. N. Watson; other members of the Council, Mr. A. L. Dixon, Prof. A. S. Eddington, Prof. L. N. G. Filon, Prof. H. Hilton, Miss H. P. Hudson, Mr. A. E. Jolliffe, Mr. J. E. Littlewood, Prof. A. E. H. Love, Prof. J. W. Nicholson.

Mr. G. H. Hardy read a paper "A Convergence Theorem."

Prof. H. Hilton made an informal communication on "Differential Operators."

Dr. G. N. Watson stated an extension of the results published by Whittaker, *Proc. London Math. Soc.*, Ser. 2, Vol. 14 (1915), pp. 260–268, to the effect that Lamé functions of the third and fourth species satisfy integral equations of which

$$E(a) = \lambda \int_{-2K}^{2K} \operatorname{cn} a \operatorname{dn} a \operatorname{cn} t \operatorname{dn} t \, P_n''(k \operatorname{sn} a \operatorname{sn} t) \, E(t) \, dt$$

is typical.

The following papers were communicated by title from the Chair :—

*The Complex Multiplication of Weierstrassian Elliptic Functions: Mr. W. E. H. Berwick.

Standard Relations of Legendre Functions: Mr. R. Hargreaves.

*The Mathematical Expression of the Principle of Huygens (II): Sir Joseph Larmor.

*Note on a Property of Dirichlet's Series: K. Ananda Rau.

*A New Theory of Measurement: Dr. N. Wiener.

ABSTRACTS.

The Complex Multiplication of Weierstrassian Elliptic Functions

Mr. W. E. H. BERWICK.

Starting with the primitive periods

$$2\omega_1 \equiv 2a\Omega, \qquad 2\omega_2 \equiv 2(b+ci\sqrt{m})\Omega,$$

where a, b, c, m are positive integers, this paper is mainly concerned with a discussion of the function

$$\psi_\mu(u) \equiv \exp[C\mu u^2] \, \sigma(\mu u)/\sigma(u)^{N(\mu)},$$

which is doubly periodic for suitable values of

$$\mu \equiv x+yi\sqrt{m}.$$

The function $\psi_\mu(u)$ is a rational integral function of $\wp(u)$ multiplied by

$$1, \ \wp', \ \sqrt{(\wp-e_1)}, \ \sqrt{(\wp-e_2)}, \text{ or } \sqrt{(\wp-e_3)},$$

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* Printed in this volume.

according to the form of μ; it vanishes when

$$\mu u \equiv 0 \pmod{2\omega_1, 2\omega_2},$$

and its zeros are all simple. All ψ's can be obtained from the recurrence formula

$$\psi_{\lambda+\mu}\psi_{\lambda-\mu}\psi_{\nu+\rho}\psi_{\nu-\rho}+\psi_{\mu+\nu}\psi_{\mu-\nu}\psi_{\lambda+\rho}\psi_{\lambda-\rho}+\psi_{\nu+\lambda}\psi_{\nu-\lambda}\psi_{\mu+\rho}\psi_{\mu-\rho}=0,$$

after the first two or three complex ψ's have been evaluated.

The numerical case when $\omega_2=\omega_1 i\sqrt{5}$ is discussed in some detail. A prime number $p \equiv 20n+1$, 9 is the product of two complex factors μ, μ' in $[i\sqrt{5}]$, and ψ_μ is an irreducible polynomial of degree $\frac{1}{2}(p-1)$. Prime numbers $20n+3$, 7 are decomposable into non-principal prime ideal factors $\mathfrak{p}$, $\mathfrak{p}'$, and $\psi_\mathfrak{p}$ is defined as the highest common factor of ψ_μ, $\psi_{\mu'}$, &c., where μ, μ', &c., are arbitrary integers divisible by $\mathfrak{p}$. By these means it is possible to set up a one-one correspondence between any ideal $\mathfrak{a}$ in a quadratic imaginary field and a unique polynomial $\psi_\mathfrak{a}$. In certain cases, as when $\omega_2/\omega_1 = i\sqrt{5}$, $\psi_\mathfrak{a}$ can be evaluated by a quadratic transformation without the labour of going through the greatest common measure process.

A Convergence Theorem

Mr. G. H. HARDY.

Hilbert proved that *if* Σa_n^2, *where* $a_n > 0$, *is convergent, then*

$$\Sigma\Sigma \frac{a_m a_n}{m+n}$$

is convergent, and later proofs have been given by Wiener and Schur. In a note published in 1915 in the *Messenger of Mathematics*, I showed that Hilbert's theorem is a corollary of the following theorem :—

If Σa_n^2 *is convergent, then* $\Sigma\left(\frac{A_n}{n}\right)^2$, *where* $A_n = a_1+a_2+\ldots+a_n$, *is convergent* :—

and in a later note published in the same journal in 1918, I gave a very simple proof of this latter theorem.

The present communication contains a new proof, equally simple, and preferable because more natural, which is due to Dr. Marcel Riesz; and

a similar proof of the more general theorem in which the index 2 is replaced by an arbitrary index κ greater than 1. The details of the proof will be published in the *Mathematische Zeitschrift*.

An Extension of Warren's and Larmor's Theorems

Prof. H. HILTON.

It was proved by Warren and by Larmor (*Cambridge Phil. Trans.*, Vol 12, p. 455; Vol. 14, p. 128; see also Bateman, *Differential Equations*, p. 197) that, if

$$dx^2+dy^2+dz^2 \equiv a\,du^2+b\,dv^2+c\,dw^2+2f\,dv\,dw+2g\,dw\,du+2h\,du\,dv,$$

so that

$$a=\left(\frac{\partial x}{\partial u}\right)^2+\left(\frac{\partial y}{\partial u}\right)^2+\left(\frac{\partial z}{\partial u}\right)^2,\quad f=\frac{\partial x}{\partial v}\frac{\partial x}{\partial w}+\frac{\partial y}{\partial v}\frac{\partial y}{\partial w}+\frac{\partial z}{\partial v}\frac{\partial z}{\partial w},\ \ldots,$$

then

$$\left(\frac{\partial\phi}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial y}\right)^2+\left(\frac{\partial\phi}{\partial z}\right)^2$$

$$\equiv\frac{1}{\Delta}\left\{A\left(\frac{\partial\phi}{\partial u}\right)^2+B\left(\frac{\partial\phi}{\partial v}\right)^2+C\left(\frac{\partial\phi}{\partial w}\right)^2+2F\frac{\partial\phi}{\partial w}\frac{\partial\phi}{\partial v}+2G\frac{\partial\phi}{\partial w}\frac{\partial\phi}{\partial u}+2H\frac{\partial\phi}{\partial u}\frac{\partial\phi}{\partial v}\right\},$$

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}$$

$$\equiv\frac{1}{\sqrt{\Delta}}\left\{\frac{\partial}{\partial u}\left(\frac{A\frac{\partial\phi}{\partial u}+H\frac{\partial\phi}{\partial v}+G\frac{\partial\phi}{\partial w}}{\sqrt{\Delta}}\right)+\frac{\partial}{\partial v}\left(\frac{H\frac{\partial\phi}{\partial u}+B\frac{\partial\phi}{\partial v}+F\frac{\partial\phi}{\partial w}}{\sqrt{\Delta}}\right)\right.$$

$$\left.+\frac{\partial}{\partial w}\left(\frac{G\frac{\partial\phi}{\partial u}+F\frac{\partial\phi}{\partial v}+C\frac{\partial\phi}{\partial w}}{\sqrt{\Delta}}\right)\right\};$$

where $A\,(\equiv bc-f^2)$, $F\,(\equiv gh-af)$, ..., are the co-factors of $a, f, \ldots$, in

$$\Delta\equiv\begin{vmatrix} a & h & g\\ h & b & f\\ g & f & c\end{vmatrix}.$$

These results can be extended to any number n of variables in the following form :—

$$\sum_{t=1}^{n} \left(\frac{\partial \phi}{\partial x_t}\right)^2 \equiv \frac{1}{\Delta} \Sigma A_{ij} \frac{\partial \phi}{\partial u_i} \frac{\partial \phi}{\partial u_j} \quad (i, j = 1, 2, \ldots, n),$$

$$\sum_{t=1}^{n} \frac{\partial^2 \phi}{\partial x_t^2} \equiv \frac{1}{\sqrt{\Delta}} \sum_{t=1}^{n} \frac{\partial}{\partial u_t} \left\{ \frac{A_{t1} \frac{\partial \phi}{\partial u_1} + A_{t2} \frac{\partial \phi}{\partial u_2} + \ldots + A_{tn} \frac{\partial \phi}{\partial u_n}}{\sqrt{\Delta}} \right\} :$$

where $$\Sigma_t \, dx_t^2 \equiv \Sigma_{i,j} \, a_{ij} \, du_i \, du_j,$$

and A_{ij} is the co-factor of a_{ij} in the determinant Δ whose i-th row and j-th column contain the element a_{ij} ($\equiv a_{ji}$).

On the Mathematical Expression of the Principle of Huygens (II)

Sir JOSEPH LARMOR.

In the previous paper [*Proc. London Math. Soc.*, Vol. 1 (1903), p. 1] the deduction of the most general formula for the diffracted waves was effected by immediate physical considerations, for all types of transmitted effect, such as sound, light, electric waves, or even conduction of heat. The degree of indeterminateness was thus exhibited which can exist, without altering the total effect, in the distribution of the secondary sources over the surface at which the emerging radiant disturbance is resolved. It is now claimed that only one specification is physically permissible, on the ground that the others that are analytically possible would give wrong expressions for the flux of energy from the elements of the surface : thus the problem of Huygens, though mathematically indeterminate, is physically definite. For example, if the surface, at which the resolution into elements is effected, is a wave front for the emerging radiation, the actual rays or paths of energy from it to an external point are those paths of quickest propagation which travel away from the surface towards the exterior side ; inasmuch as along the other such paths, trending at first inwards, which equally could transmit energy without spreading, no energy travels, any of the other possible analytical specifications giving the same total result would involve rays of both these kinds, and therefore having intensities different from the unique actual values. The exact formula for diffraction in oblique directions of the transverse waves of

light, as distinct from the simpler case of pressural waves treated by Kirchhoff and his successors, is now determined and set out: it is found to involve the same factor of obliquity, so that a precisely analogous discussion is applicable.

Thursday, December 11*th*, 1919.

Mr. J. E. CAMPBELL, President, in the Chair.

Present fifteen members.

Messrs. W. N. Bailey, B. B. Baker, M. D. Bhansali, F. Bowman, E. F. Collingwood, C. Fox, P. Fraser, C. W. Gilham, E. A. Milne, E. M. Moors, S. P. Owen, S. Pollard, F. E. Relton, N. M. Shah, W. F. Sheppard, E. H. Smart, G. I. Taylor, G. P. Thomson, F. P. White, B. M. Wilson, and A. E. Williams were elected members of the Society.

Messrs. T. A. Brown, G. F. S. Hills, W. Lindley Hughes, J. E. Jones, G. H. Livens, W. N. Rosevear, and R. O. Street were nominated for election.

Mr. B. M. Wilson was admitted into the Society.

The Auditor (Lt.-Col. A. Cunningham) presented his Report, and the Treasurer's accounts were adopted. A vote of thanks to the Treasurer and Auditor was adopted unanimously.

Mr. L. J. Mordell read a paper "On the Generating Function of the Series $\Sigma F(n)q^n$, where $F(n)$ is the Number of Uneven Classes of Binary Quadratics of Determinant $-n$."

Prof. Love gave an account of a paper by Mr. D. Riabouchinsky (communicated by Prof. Lamb) "On Steady Fluid Motions with Free Surfaces."*

The following papers were communicated, by title, from the Chair:—

*Permutations, Lattice Permutations, and the Hypergeometric Series: Major P. A. MacMahon.

*On Fourier Coefficients of Bounded Functions: Mr. H. Steinhaus (communicated by Mr. G. H. Hardy).

On certain Cyclotomic Series: Prof. L. J. Rogers.

*On a Multiple Integral of Importance in the Theory of Errors: Mr. G. F. S. Hills.

On the Lag of a Thermometer in a Medium whose Temperature is a Linear Function of the Time: Mr. S. P. Owen.

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* Printed in this volume.

ABSTRACTS.

On the Lag of a Thermometer in a Medium whose Temperature is a Linear Function of the Time

Mr. S. P. OWEN.

The problem is of practical interest in connection with thermometers carried by aeroplanes. It is assumed that there is a uniform temperature gradient in the atmosphere, and hence, for a uniform rate of descent or ascent of the aeroplane, the surface conditions may be represented by a temperature At, where t is the time and A is a constant. Some solutions of the problem have been obtained by Mr. A. R. McLeod (*Phil. Mag.*, January 1919) and by Dr. Bromwich (*Phil. Mag.*, April 1919).

In the following, the results of McLeod and Bromwich are obtained by a different method and are extended so as to include the effect of the containing bulb.

Let u be the temperature at distance r from the centre of a spherical bulb of radius c, and θ the temperature lag at this point. Then $u = At - \theta$, and u must satisfy the equation

$$\frac{\partial u}{\partial t} = a^2 \left(\frac{\partial^2 u}{\partial r^2} + \frac{2}{r} \frac{\partial u}{\partial r} \right),$$

where $a^2 = k/\rho\sigma$, and k is the conductivity, ρ the density, and σ the specific heat. Hence θ must satisfy

$$\frac{\partial^2 \theta}{\partial r^2} + \frac{2}{r} \frac{\partial \theta}{\partial r} + \frac{A}{a^2} = \frac{1}{a^2} \frac{\partial \theta}{\partial t},$$

which gives

$$\theta = C + \frac{B}{r} - \frac{Ar^2}{6a^2} + \frac{1}{r} e^{-t(\alpha a/c)^2} \left\{ D \sin \frac{\alpha r}{c} + E \cos \frac{\alpha r}{c} \right\},$$

where C, B, D, E and α are constants.

The conditions to be satisfied are

$$\theta = 0 \quad \text{for} \quad t = 0; \tag{1}$$

for infinite surface conductivity,

$$\theta = 0 \quad \text{for} \quad r = c; \tag{2}$$

for finite surface conductivity,

$$-k \frac{\partial \theta}{\partial r} = h\theta \quad \text{for} \quad r = c, \tag{3}$$

where h is the surface conductivity.

For conditions (1) and (2),

$$\theta = \frac{Ac^2}{15a^2} - \frac{6Ac^2}{a^2\pi^2} \sum_{m=1}^{\infty} \frac{1}{m^4} e^{-t(m\pi/c)^2}. \tag{4}$$

For conditions (1) and (3),

$$\theta = \frac{Ac^2}{a^2}\left\{\frac{1}{15} + \frac{k}{3ch}\right\} - \frac{6Ac^2}{a^2} \sum_{m=1}^{\infty} \frac{c^2h^2}{k^2\alpha_m^2 - ch(k-ch)} \frac{1}{\alpha_m^4} e^{-t(a\alpha_m/c)^2}, \tag{5}$$

where α_m is the m-th root of

$$\tan \alpha = \frac{ka}{k-ch}$$

For a cylindrical bulb

$$\theta = C + B \log r - \frac{Ar^2}{4a^2} + e^{-t(a\alpha)^2}\{DJ_0(ar) + EY_0(ar)\},$$

where J_0 and Y_0 are Bessel's and Neumann's Functions of zero order. Then, if c is the radius of the cylinder, we must have:—for conditions (1) and (2)

$$\theta = \frac{Ac^2}{8a^2} - \frac{4A}{a^2c^2} \sum_{m=1}^{\infty} \frac{1}{\alpha_m^4} e^{-t(a\alpha_m)^2}, \tag{6}$$

where α_m is the m-th root of $J_0(ac) = 0$; and for conditions (1) and (3)

$$\theta = \frac{A}{2a^2}\left\{\frac{c^2}{4} + \frac{kc}{h}\right\} - \frac{4A}{a^2c^2} \sum_{m=1}^{\infty} \frac{h^2}{(k^2\alpha_m^2 + h^2)} \frac{1}{\alpha_m^4} e^{-t(a\alpha_m)^2}, \tag{7}$$

where a is a root of $\quad kaJ_1(ac) - hJ_0(ac) = 0.$

Taking into account the containing vessel, the following *steady* lags are obtained.

For a spherical bulb

$$\theta_1 = A\left\{\frac{b^2-c^2}{6a_2^2} + \frac{c^2}{15a_1^2} + \frac{k_2 b}{3ha_2^2} + \frac{c^3}{3k_2}\left(\frac{1}{c} - \frac{1}{b} + \frac{k_2}{hb^2}\right)\left(\frac{k_1}{a_1^2} - \frac{k_2}{a_2^2}\right)\right\};$$

for a cylindrical bulb

$$\theta_1 = A\left\{\frac{b^2-c^2}{4a_2^2} + \frac{c^2}{8a_1^2} + \frac{k_2 b}{2ha_2^2} + \frac{c^2}{2k_2}\left(\log_e \frac{b}{c} + \frac{k_2}{hb}\right)\left(\frac{k_1}{a_1^2} - \frac{k_2}{a_2^2}\right)\right\};$$

the suffixes 1 and 2 referring to the liquid and vessel respectively, and c being the inner radius and b the outer radius of the vessel.

An approximate solution for the *instantaneous* lag is also obtained,

involving the first term of the exponential series only; giving the results

$$\theta = \theta_1(1-e^{-t\,(a_1a/c)^2}) \tag{8}$$

for the spherical bulb, where a is the first root other than zero of

$$\frac{\tan a}{a} = \frac{k_1(q-p)}{k_2+(q-p)(k_1-k_2)+k_2pq(a_1a/a_2)^2},$$

and where $p = (b-c)/c, \quad q = k_2b/(k_2c-hbc).$

Further, we have $$\theta = \theta_1(1-e^{-t\,(a_1a)^2}) \tag{9}$$

for a cylindrical bulb, where a is the first root of

$$k_1a\frac{J_1(ac)}{J_0(ac)} = \frac{k_2\{1-k_2(b-c)(a_1a/a_2)^2/h\}}{b-c+k_2\{1-(b-c)/c\}/h}.$$

In (8) and (9) $b-c$ is assumed small, and numerical results show that the lags in mercury and alcohol spherical bulbs, giving equal volume expansion per degree, are practically the same. The lag in a mercury cylinder is much less and that of an alcohol cylinder least of all.

Permutations, Lattice Permutations, and the Hypergeometric Series

Major P. A. MACMAHON.

It is proved that

$$\sum_{m_{n,\,n}=0}^{\infty} P(m_{1,\,n},\ m_{2,\,n},\ \ldots,\ m_{n,\,n})\,x_1^{m_{2,\,n}}x_2^{m_{2,\,n}}\ldots x_n^{m_{n,\,n}}$$

$$= P(m_{1,\,n-1},\ m_{2,\,n-1},\ \ldots,\ m_{n-1,\,n-1})\,x_1^{m_{1,\,n-1}}x_2^{m_{2,\,n-1}}\ldots x_{n-1}^{m_{n-1,\,n-1}}$$

$$\times F\left\{\begin{matrix}\frac{1}{n}(M_{n-1}+1), & \frac{1}{n}(M_{n-1}+2), & \ldots, & \frac{1}{n}(M_{n-1}+n), & n^n x_1x_2\ldots x_n\\ 1, & m_{n-1,\,n-1}+1, & \ldots, & m_{1,\,n-1}+1, & \end{matrix}\right\},$$

where

$$m_{1,\,n} = m_1+m_2+\ldots+m_n, \quad m_{2,\,n} = m_2+m_3+\ldots+m_n, \quad \ldots, \quad m_{n,\,n} = m_n,$$

$$M_{n-1} = m_1+2m_2+3m_3+\ldots+(n-1)\,m_{n-1},$$

and $P(m_{1,n}, m_{2,n}, \ldots, m_{n,n})$ denotes the number of permutations of the assemblage of letters

$$x_1^{m_{1,n}} x_2^{m_{2,n}} \ldots x_n^{m_{n,n}}.$$

A similar identity is established for the series

$$\sum_{m_{n,n}=0}^{\infty} LP(m_{1,n}, m_{2,n}, \ldots, m_{n,n})\, x_1^{m_{1,n}} x_2^{m_{2,n}} \ldots x_n^{m_{n,n}},$$

where $LP(m_{1,n}, \ldots, m_{n,n})$ denotes the number of "lattice permutations" of the assemblage (see the author's *Combinatory Analysis*, Vol. 1, Chap. 5).

Thursday, January 15th, 1920.

Prof. A. E. H. LOVE, Ex-President, in the Chair.

Present eleven members.
Captain L. F. Plugge was nominated for election.

The following papers were communicated:—

*The Divisors of Numbers: Major P. A. MacMahon.
Two Theorems on n-ans: Lt.-Col. Allan Cunningham.

ABSTRACT.

Two Theorems on n-ans

Lt.-Col. ALLAN CUNNINGHAM, R.E.

DEFINITIONS.—M.A.P.F. means Maximum Algebraic Prime Factor: $\phi(n)$, $\phi'(n)$ mean MAPF of $(x^n - y^n)$ and $(x^n + y^n)$ respectively: and $\phi(n)$, $\phi'(n)$ are styled "*n-ans.*"

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* Printed in this volume.

THEOREM.—*Every* $\phi(n)$ *and* $\phi'(n)$ *can be expressed in terms of* $C=\phi(3)$ *and* $C'=\phi'(3)$.

For both $\phi(n)$ and $\phi'(n)$ are *symmetric homogeneous* functions of x, y of degree $\tau(n)$ when n is *odd*, and $\tau(2n)$ when n is *even*, where $\tau(r)$ means the *totient* of r.

Hence $\phi(n)$ and $\phi'(n)$ can be expressed as a sum of pairs of terms of the form

$$\Sigma_r A_r(xy)^{\alpha_r}(x^{c_r\beta_r}+y^{c_r\beta_r}),$$

where $c_r=2^{\kappa_r}$, $\beta_r=\omega$ (an *odd* number).

Now $$C=x^2+xy+y^2, \qquad C'=x^2-xy+y^2,$$

where $$x^2+y^2=\tfrac{1}{2}(C+C'), \qquad xy=\tfrac{1}{2}(C-C');$$

and it will be found on trial that the above term $x^{c_r\beta_r}+y^{c_r\beta_r}$ can always be expressed as a function of x^2+y^2 and xy, and therefore as a function of C, C'.

This proves the theorem. Here follow examples :—

n	$\phi(n)$ and $\phi'(n)$, [$\phi(n)$ has +, $\phi'(n)$ has −].
5	$CC'\pm\frac{1}{4}(C^2-C'^2)$
7	$\frac{1}{8}(C+C')(6CC'-C^2-C'^2)\pm\frac{1}{2}CC'(C-C')$
9	$\frac{1}{4}(C+C')(4CC'-C^2-C'^2)\pm\frac{1}{8}(C-C')^3$

n	$\phi'(n)$
2	$\frac{1}{2}(C+C')$
4	$\frac{1}{4}(6CC'-C^2-C'^2)$
6	$\frac{1}{2}(4CC'-C^2-C'^2)$
	$\frac{1}{16}(6CC'-C^2-C'^2)^2-\frac{1}{8}(C-C')^4$
12	$\frac{1}{16}(6CC'-C^2-C'^2)^2-\frac{3}{16}(C-C')^4$

THEOREM.—*If* $\phi(n)$ *and* $\phi'(n)$ *are subject to the condition* $x-y=k$ *(*k *constant), then* $\phi(n)$ *and* $\phi'(n)$ *are both expressible in terms of* xy *and* k^2.

For $x-y=k$ gives $x^2+y^2=k^2+2xy$, and it was shown above that every $\phi(n)$ and $\phi'(n)$ is expressible in terms of (x^2+y^2) and xy. Hence

they are both expressible in terms of k^2 and xy. Here follow examples, and v is written for shortness for xy :

n	$\phi(n)$	$\phi'(n)$
3	k^2+3v	k^2+v
5	$k^4+5v(k^2+v)$	$k^4+v(3k^2+v)$
7	$k^6+7v(k^2+v)^2$	$k^6+v(k^2+v)(5k^2+v)$
9	$k^6+3v(k^2+v)(3k^2+v)$	$k^6+v(6k^4+9k^2v+v^2)$

n	$\phi'(n)$
2	k^2+2v
4	$k^4+4k^2v+2v^2$
6	$k^4+4k^2v+v^2$
8	$k^8+8k^6v+20k^4v^2+16k^2v^3+2v^4$
10	$k^8+8k^6v+19k^4v^2+12k^2v^3+v^4$

Interesting cases occur with $k=1$ and $k=n$.

Ex.—$\phi(n)=k+B^2$, when $nxy=(n\xi\eta)^2$, if $n=3$ or 7. The condition $x-y=k$ involves $\xi^2-n\eta^2=k$.

Thursday, February 12th, 1920.

Mr. J. E. CAMPBELL, President, and later Prof. A. E. H. LOVE, Ex-President, in the Chair.

Present sixteen members and two visitors.

Mr. L. F. Plugge was elected a member.

Mr. E. F. Collingwood was admitted into the Society.

The Secretaries reported that the number of Members at the date of the Annual General Meeting, November 13th, 1919, was 312.

Mr. L. J. Mordell gave an account of a paper by Prof. E. Landau and Mr. A. Ostrowski, "On the Diophantine Equation $ay^2+by+c=dx^n$."*

* Printed in this volume.

Prof. Hardy gave an account of some recent researches, undertaken in collaboration with Mr. J. E. Littlewood, on "The Zeros of Riemann's Zeta Function."

Dr. Watson made an informal communication on "The Zeros of Bessel Functions"; Lt.-Col. Cunningham on "Some Properties of Cubans"; and Mr. T. C. Lewis on "Pentaspherical Coordinates for the Tetrahedron."

The following papers were communicated, by title, from the Chair :—

*A Property of Polynomials whose Roots are Real: Mr. G. S. Le Beau.

†On Canonical Forms: Mr. E. K. Wakeford.

The Zeros of Riemann's Zeta-Function

MR. G. H. HARDY *and* MR. J. E. LITTLEWOOD.

The authors have proved (*Acta Mathematica*, Vol. 41, 1917, pp. 119–196) that the number $N_0(T)$ of zeros of $\zeta(s)$, such that

$$s = \sigma + it, \quad \sigma = \tfrac{1}{2}, \quad -T < t < T,$$

is greater than a constant multiple of $T^{\frac{1}{2}-\epsilon}$, for every positive value of ϵ.

They are now able to show that

$$N_0(T) > KT,$$

where K is a constant. The proof will appear shortly in the *Mathematische Zeitschrift*.

If the Riemann hypothesis is true, the real order of $N_0(T)$ is of course $T \log T$.

Some Properties of Cubans

Lt.-Col. ALLAN CUNNINGHAM, R.E.

DEF.—M.A.P.F. means "Maximum Algebraic Prime Factor."

$$N_n = \text{MAPF of } (x^n - y^n), \qquad N'_n = \text{MAPF of } (x^n + y^n),$$

$$N_{\text{iii}} = (x^3 - y^3)/(x - y), \qquad N'_{\text{iii}} = (x^3 + y^3)/(x + y).$$

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* Printed in this volume. † Printed in Vol. 18.

Now $$N_{\text{iii}} = \frac{x^3-y^3}{x-y} = \frac{z^3+x^3}{z+x} = \frac{z^3+y^3}{z+y} = A^2+3B^2 = 6\varpi+1,$$

or $$C = C' = C'' = Q = L,$$

where $z = x+y$, x is taken prime to y; so one of x, y, z is *even*, two are *odd*. And B is one-half of the even one of x, y, z.

Hence any one of C, C', C'', Q when given determines the rest. The above is an *algebraic identity* peculiar to cubans. Hence L may be prime, or composite.

If $L = L_1L_2L_3 \ldots L_r$ (r unequal factors), it has 2^{r-1} forms Q. Each Q has *its own triplet* C, C', C''. But with n-ans generally, if N_n be *dimorph*, and $n \neq 3$, then N_n is *composite*, and its two n-an forms are not algebraically connected.

Properties latent in one form C, C', C'' are often obvious in one of the other forms.

Ex. i.—Take $$x = \xi^2, \quad y = \eta^2, \quad z = \zeta^2.$$

Then $z = x+y$ involves $\zeta^2 = \xi^2+\eta^2$, and

$$N_{\text{vi}} = \frac{\xi^6-\eta^6}{\xi^2-\eta^2} = \frac{\zeta^6+\xi^6}{\zeta^2+\xi^2} = \frac{\zeta^6+\eta^6}{\zeta^2+\eta^2},$$

a *Dimorph-Sextan*, obviously composite, with two obvious cuban factors. No other n-an has this property.

Ex. ii.—Take $n = 4\nu+3$, and take $x = \xi^2$, $y = n\eta^2$, so that

$$nxy = (n\xi\eta)^2 = \square.$$

Then $$\left.\begin{aligned} N_n &= P'^2+Q'^2 \text{ (an } \textit{Ant-Aurifeuillian)} \\ N_n &= P^2+Q^2 \text{ (an } \textit{Aurifeuillian)} \end{aligned}\right\} \textit{algebraically.}$$

But $N_n \neq P^2-Q^2$, and $N_n \neq P'^2+Q'^2$ *algebraically* (when $n \neq 3$).

But N_{iii}, being equal to C' and C'', may be of the form P^2-Q^2, and N_{iii}, being equal to C, may be of the form P^2+Q^2.

Here follow examples of eight associated series:

$$H = (\eta^3-1)/(\eta-1), \qquad H' = (\eta'^3+1)/(\eta'+1),$$

$$K = (k^3-3^3)/(k-3), \qquad K' = (k'^3+1)/(k'+3),$$

$$Z = (z^3-w^3)/(z-w), \qquad Z' = (z'^3+w'^3)/(z'+w'),$$

$$N = (x^3-y^3)/(x-y), \qquad N' = (x'^3+y'^3)/(x'+y'),$$

with $z-w = z'-w' = 1$ in Z, $Z`$, $x-y = x'-y' = 3$ in $N$, $N`$. Then

$$k = 3\eta \quad \text{gives} \quad K = 9H,$$
$$x = 3z \quad ,, \quad N = 9Z,$$
$$z' = 3z-1 \quad ,, \quad Z` = 3Z,$$
$$x' = 3(x-1) \quad ,, \quad N` = 3N.$$

Every $N \equiv 0 \pmod 3$; and

$$\eta' = \eta+1, \quad z' = \eta', \quad x = \eta'+1, \quad \text{give} \quad H = H` = Z` = \tfrac{1}{3}N, \quad A \sim B = 1,$$

$$k' = k+3, \quad x' = k', \quad \text{give} \quad K = K` = N`, \quad A \sim B = 3.$$

DEF.—A series of *composites* $N_r = L_r M_r$, all of the same type, in which $M_r = L_{r+1}$ for every r ($r = 1, 2, 3, \ldots$), is said to be *in chain*.

Ex. 1.—$\eta' = \eta^2$ gives

$$H`_{\eta^2} = \eta^4+\eta^2+1 = (\eta^2-\eta+1)(\eta^2+\eta+1) = H_\eta H_{\eta+1}.$$

Similarly $H`_{(\eta+1)^2} = H_{\eta+1} H_{\eta+2}$. Thus the series H_{η^2} is in chain, and the chain-factors are the successive H_η, $H_{\eta+1}$.

Ex. 2.—$x' = x^2$ gives

$$N`_{x^2} = x^4-3x^2+9 \quad \text{(a } \textit{Trin-Aurifeuillian}\text{)}$$
$$= (x^2-3x+3)(x^2+3x+3) = \tfrac{1}{3}N_x \tfrac{1}{3}N_{x+3}.$$

Similarly $$N`_{(x+3)^2} = \tfrac{1}{3}N_{x+3} \tfrac{1}{3}N_{x+6}.$$

Now form three series of $\frac{1}{3}N_x$:

(1) $x = 1, 4, 7, 10, \ldots, 3\rho+1,$

(2) $x = 2, 5, 8, 11, \ldots, 3\rho+2,$

(3) $x = 3, 6, 9, 12, \ldots, 3\rho+3,$

and form the corresponding $N`_{x^2}$.

The three series N_{x^2} are seen to be in chain, and the chain-factors are the successive terms of the three $\frac{1}{3}N_x$ series.

Ex. 3.—$\eta' = 3\eta^2$ gives

$$H`_{\eta^2} = 9\eta^4-3\eta^2+1 \quad \text{(a } \textit{Trin-Aurifeuillian}\text{)}$$
$$= (3\eta^2-3\eta+1)(3\eta^2+3\eta+1) = h_\eta h_{\eta+1}, \quad \text{suppose.}$$

Similarly $$H_{(\eta+1)^2} = h_{\eta+1}\, h_{\eta+2}.$$

Thus the series H_{η^2} is in chain, and the chain-factors are the successive h_η, $h_{\eta+1}$.

A Multiple Integral of Importance in the Theory of Statistics

Mr. G. F. S. HILLS.

The multiple integral

$$\iint \dots \int \chi'^k e^{-\frac{1}{2}\chi}\, dx_1 dx_2 \dots dx_n,$$

where χ and χ' are two quadratic forms in the variables $x_1, x_2, \dots, x_n$, χ being essentially positive, and the integration is over the finite space within $\chi = h$, is shown to be equal to

$$(2\pi)^{\frac{1}{2}n}\, k!\, 2^k R_k \int_0^{\frac{1}{2}h} \frac{e^{-v} v^{k+\frac{1}{2}n-1}}{\Gamma(k+\frac{1}{2}n)}\, dv,$$

where R_k is the coefficient of $(-\phi)^k$ in $M^{-\frac{1}{2}}$, M being the discriminant of $\chi+\phi\chi'$.

Taking χ' in the special form $(\xi_1 x_1+\xi_2 x_2+\dots+\xi_n x_n)^2$, the discriminant M takes a simple form, and R_k can be written down; and, by comparing coefficients of $\xi_1^a \xi_2^b \dots \xi_n^j$ on both sides, the value of integrals of type

$$\iint x_1^a x_2^b \dots x_n^j\, e^{-\frac{1}{2}\chi}\, dx_1 dx_2 \dots dx_n$$

can be obtained. These integrals are of importance in the theory of statistics.

The Tetrahedron and Pentaspherical Coordinates

Mr. T. C. LEWIS.

The most convenient system of pentaspherical coordinates to employ for investigating the properties of a tetrahedron is that in which the centres of the spheres of reference are the four vertices of the tetrahedron, A_1, A_2, A_3, A_4, and the centre, A_5, of the associated hyperboloid, of which the four perpendiculars from the vertices upon the opposite faces are

generating lines. The expressions for ρ_1^2, ρ_2^2, ρ_3^2, ρ_4^2, ρ_5^2 in terms of the sides are much simplified, and various results are more easily obtained.

It is known that the straight line joining the circumcentre O to A_5 is bisected at G, the centre of gravity of the tetrahedron. Hence, if a, b, c, d, e, f be the edges, and g, h, i, j the distances of A_5 from the A_1, ..., A_4 respectively, we obtain

$$g^2 = R^2+\tfrac{1}{4}(b^2+c^2+d^2)-\tfrac{1}{4}(a^2+e^2+f^2),$$

and similar expressions for h^2, i^2, j^2; and therefore

$$g^2+h^2+i^2+j^2 = 4R^2.$$

Also $$3\rho_5^2 = p_5 = 3R^2-\tfrac{1}{4}(a^2+b^2+c^2+d^2+e^2+f^2),$$

and $$144V^2\left(1+\frac{2\rho_5^2}{\rho_1^2}\right) = a^2d^2(e^2+f^2-a^2)+b^2e^2(f^2+a^2-e^2) + c^2f^2(a^2+e^2-f^2)-2a^2e^2f^2.$$

And putting, as usual, $a^2 = \rho_2^2+\rho_3^2-2\rho_2\rho_3c_{23}$, ...,

we obtain $2l^2$, viz.,

$$\begin{aligned}
&2\rho_5(\rho_1c_{15}+\rho_2c_{25}+\rho_3c_{35}+\rho_4c_{45})\\
&= 2\rho_1(\rho_2c_{12}+\rho_3c_{13}+\rho_4c_{14}-2\rho_5c_{15}) \quad \text{(and three similar expressions)}\\
&= \rho_2\rho_3c_{23}+\rho_2\rho_4c_{24}+\rho_3\rho_4c_{34}+2\rho_1\rho_5c_{15} \quad \text{(and three similar expressions)}\\
&= \tfrac{2}{3}(\rho_1\rho_2c_{12}+\rho_1\rho_3c_{13}+\rho_1\rho_4c_{14}+\rho_2\rho_3c_{23}+\rho_2\rho_4c_{24}+\rho_3\rho_4c_{34})\\
&= \rho_1^2+\rho_2^2+\rho_3^2+\rho_4^2+4\rho_5^2-4R^2.
\end{aligned}$$

The equation to the circumsphere assumes the simple form

$$\rho_1x_1+\rho_2x_2+\rho_3x_3+\rho_4x_4-2\rho_5x_5+2l^2 = 0,$$

and l vanishes when the system is orthogonal, *i.e.* when the tetrahedron is orthocentric.

The equation to the inscribed sphere is

$$\frac{a_1}{\rho_1}x_1+\frac{a_2}{\rho_2}x_2+\frac{a_3}{\rho_3}x_3+\frac{a_4}{\rho_4}x_4+2Hr = 0,$$

where $a_1 = \rho_1^2/h_1$, h_1 being the altitude of A_1 from the opposite face whose area is a_1; and H is a constant which becomes unity for an orthogonal system, viz.

$$2H = \frac{a_1^2}{\rho_1^2}+\ldots+\frac{a_4^2}{\rho_4^2}+2\Sigma\frac{a_ha_kc_{hk}}{\rho_h\rho_k}-1.$$

The four escribed spheres touching one face of the tetrahedron on the reverse side have their equations as for the inscribed sphere with one of the quantities a_1, a_2, a_3, a_4 taken negatively, and H and r changed thereby to H_1 and r_1.

Proceeding from this it is proved that the sphere through A_2, A_3, A_4, which touches the inscribed sphere, also touches the sphere escribed on $A_2A_3A_4$. If the radius of this sphere be R_1,

$$2R_1 = a_2+a_3+a_4-H(r+r_1)+2r\frac{a_1}{\rho_1}\left(\frac{a_2c_{12}}{\rho_2}+\frac{a_3c_{13}}{\rho_3}+\frac{a_4c_{14}}{\rho_4}\right).$$

The proof of this proposition depends on the identity

$$\begin{aligned}576R^2V^2.16(a_2^2+a_3^2+a_4^2-a_1^2) &= 256a^4a_2^2a_3^2+e^4a_2^2a_4^2+f^4a_3^2a_4^2-b^4a_1^2a_3^2-c^4a_1^2a_2^2\\ &\quad -d^4a_1^2a_4^2+(144V^2)^2\\ &\quad -2\frac{\rho_1^2+2\rho_2^2}{\rho_1^2}144V^2\{144V^2-16(b^2a_3^2+c^2a_2^2+d^2a_1^2)\},\end{aligned}$$

every term in which can be expressed in terms of the squares of the edges.

A Property of Polynomials whose Roots are Real

Mr. G. S. LE BEAU.

Let a_1, a_2, a_3, ..., a_n be the roots, supposed real and in ascending order of magnitude, of a polynomial $f(x)$, and let β_1, β_2, ..., β_{n-1} be the roots, also real and in ascending order of magnitude, of a polynomial $\phi(x)$; further, let the roots of $\phi(x)$ separate those of $f(x)$, so that

$$a_1 < \beta_1 \leqslant a_2 \leqslant \beta_2 \leqslant \ldots \leqslant \beta_{n-1} < a_n.$$

Let $f(x)$ be divided by $\phi(x)$ and let the quotient be $x-\gamma$ and the remainder $-A\psi(x)$, where $\psi(x) = x^{n-2}+\ldots$. Let $f(x)$ be divided by $(x-\gamma)\psi(x)$, and let the quotient be $x-\gamma_1$, and the remainder $-A_1\psi_1(x)$, where

$$\psi_1(x) = x^{n-2}+\ldots.$$

Let $f(x)$ be next divided by $(x-\gamma_1)\psi_1(x)$, and so on. It is shown that the roots of the successive polynomials $\psi(x)$, $\psi_1(x)$, ..., obtained in this way, tend to the limits a_2, a_3, ..., a_{n-1}.

The proof depends upon the facts that, (1) the roots of each remainder separate those of the corresponding divisor, (2) the roots of every divisor separate those of $f(x)$, (3) the value of A cannot decrease at any stage of the operations and hence tends to a definite limit, and (4) the greatest value of the difference between $\frac{1}{2}(a_1+a_n)$ and a root of the divisor cannot increase at any stage of the operations and hence tends to a definite limit.

The result can be applied to obtain rational or semi-rational approximations to the values of an algebraic function of x, valid for ranges of real values of x for which all the values of the function are real.

Thursday, March 11th, 1920.

Mr. J. E. CAMPBELL, President, in the Chair.

Present twelve members.

Messrs. J. L. Burchnall and C. M. Ross were nominated for membership.

Mr. G. S. Le Beau read a paper "A Property of Polynomials whose Roots are Real."*

Mr. E. G. C. Poole read a paper "A Point in the Dynamical Theory of the Tides."*

Dr. Watson and Mr. Mordell made informal communications.

A paper by Mr. B. M. Sen "On Double Surfaces," was communicated by title from the chair.

ABSTRACT.

A Point in the Dynamical Theory of the Tides

Mr. E. G. C. POOLE.

There is a series of critical values of f^2 for which the equation

$$\frac{d}{d\mu}\left(\frac{1-\mu^2}{f^2-\mu^2}\frac{d\zeta}{d\mu}\right)+\beta\zeta=0,$$

* Printed in this volume.

where β is a given positive constant, has solutions regular for all values of μ. Hough has shown how to determine these values by a process of approximation; but it does not appear to have been proved that the values to which his approximation leads exhaust all possible critical values. Such a proof is given in the present paper.

Thursday, April 22nd, 1920.

Mr. J. E. CAMPBELL, President, in the Chair.

Present ten members and a visitor.

Messrs. J. L. Burchnall and C. M. Ross were elected members of the Society.

Messrs. S. G. Soal, H. B. C. Darling, and V. V. Ramana-Šāstrin were nominated for election.

Prof. Hardy communicated a paper, written in collaboration with Mr. J. E. Littlewood, "Some Problems of Diophantine Approximation—The Lattice-Points of a Right-Angled Triangle"; and also made an informal communication on "Collineations".

Prof. Watson made an informal communication on a point in the theory of Bessel functions.

The following papers were communicated by title from the chair:—

*The Influence of Diffusion on the Propagation of Sound Waves in Air: Prof. S. Chapman and Mr. G. H. Livens.

The Three-Bar Sextic Curve: Mr. G. T. Bennett.

The Relation between Apolarity and the Pippian-Quippian Syzygy: Prof. W. P. Milne and Mr. D. G. Taylor.

ABSTRACT.

Some Problems of Diophantine Approximation—The Lattice-Points of a Right-Angled Triangle

G. H. HARDY *and* J. E. LITTLEWOOD.

Suppose that ω and ω' are two positive numbers whose ratio $\theta = \omega/\omega'$ is irrational, and that $N(\eta)$ is the number of lattice points (points with

* Printed in this volume.

integral coordinates, *Gitterpunkte*) inside the triangle

$$x > 0, \quad y > 0, \quad x\omega + y\omega' < \eta;$$

and let $$N(\eta) = \frac{\eta^2}{2\omega\omega'} - \frac{\eta}{2\omega} - \frac{\eta}{2\omega'} + R(\eta).$$

Then $R(\eta) = o(\eta)$; and this is all that is true for *every* irrational θ. If

$$\theta = \frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \ldots,$$

and a_n is less than a constant (in particular if θ is *quadratic*), then $R(\eta) = O(\log \eta)$, and this result also is the best possible of its kind. Further, if θ is *algebraic*, $R(\eta) = O(\eta^\alpha)$, where $\alpha < 1$.

Thursday, May 13th, 1920.

Mr. J. E. CAMPBELL, President, in the Chair.

Present seventeen members.

Messrs. S. G. Soal, H. B. C. Darling, and V. V. Ramana-Šastrin were elected members of the Society.

Messrs. E. S. Littlejohn and C. V. Hanumanta Rao were nominated for membership.

Mr. W. E. H. Berwick was admitted into the Society.

Mr. H. W. Richmond read two papers (1) "Historical Note on some Canonical Forms quoted by Mr. Wakeford," (2) "Historical Note on Cayley's Theorems on the Intersections of Algebraic Curves." The President, Mr. Dixon, and Mr. Jolliffe, took part in a discussion of these papers.

Mr. A. E. Jolliffe read a paper "The Pascal Lines of a Hexagon."

Mr. T. Stuart read a paper "The Lowest Parametric Solutions of a Dimorph Sextan Equation in the Rational, Irrational, and Complex Fields."

ABSTRACTS.

An Historical Note upon certain Canonical Forms

H. W. RICHMOND.

In the course of Mr. E. K. Wakeford's paper "On Canonical Forms,"* six examples are quoted to illustrate the possibility or impossibility of expressing quantics as the sum of powers of linear forms. These with the exception of number (2) are also given in my paper with the same title, mentioned in the footnote, p. 403, where references for numbers (1), (3), (5) will be found.

When my paper was published in April 1902, I believed numbers (4 and (6) to be new; but Prof. F. Morley pointed out to me that number (4) —the unique expression of a ternary quintic as the sum of seven fifth powers—had previously been studied by Hilbert in a "Lettre adressée à M. Hermite," published in *Liouville*, Ser. 4, Vol. 4 (1888), pp. 249–256. My paper was written after I had spent a good deal of time and labour in deciding (6), whether or no the sum of seven cubes is a possible form for the locus in space of four dimensions given by the general cubic equation; and then saw that the method I had discovered for this special problem was capable of other applications, chiefly though not exclusively to the expression of quantics as a sum of powers of linear forms. In much the same way Hilbert at the end of his letter to Hermite explains how a principle he has used can be applied to establish (5), Sylvester's Pentahedral Form for a cubic surface; and as a second example adds the statement that it will also establish (4), thus anticipating me by fourteen years.

A little later theorems (4) and (6) were again proved independently in two papers by Palatini. In his first paper (*Atti Accad. Torino*, November 30, 1902) Palatini considers the quantic of degree 3 in five variables, and shows, as I had done in (6), that it cannot in general be expressed as the sum of seven cubes. He briefly discusses properties of the special cubics which can be so represented, and the ways of expressing the general cubic as the sum of eight or more cubes. In his second paper (*Atti d. R. Accad. dei Lincei*, May 17, 1903) he notes that the quartic in five variables cannot be expressed as the sum of fourteen cubes, a result obvious by Wakeford's beautiful and powerful method, since a quadric in space of

* *Proc. London Math. Soc.*, Ser. 2, Vol. 18 (1920), pp. 403–410 (p. 407).

four dimensions can be made to pass through fourteen points, and this taken twice is a quartic having the fourteen points as double points.

In this important paper Palatini sets out to discuss the representation of a ternary quantic as the sum of powers of linear forms, and succeeds in establishing general results for such forms of any order, similar to those of Sylvester for binary forms. When n is not a multiple of 3, the number of constants in a ternary n-ic $(n+1)(n+2)/2$ is a multiple of 3. Palatini proves that for values of n greater than 4 the ternary n-ic can be expressed as the sum of $(n+1)(n+2)/6$ linear forms, uniquely if $n = 5$, and in a finite number of ways if $n > 5$; he also gives results for values of n which are divisible by 3.

The Pascal Lines of a Hexagon

A. E. JOLLIFFE.

In this communication the Pascal lines of six points on a conic were arranged in groups corresponding to triangles formed by taking alternate sides of any hexagon determined by the six points in any order. Among other properties it was shown that there is a (1, 1)-correspondence between the fifteen triangles that can be so formed, and the fifteen I lines and fifteen i points associated with the six points. Some relations between the I point and i line corresponding to any triangle and the trinodal quartic, whose nodal tangents are the lines joining the vertices of the triangle to the points where the opposite sides meet the conic, were also indicated.

Thursday, June 10th, 1920.

Mr. J. E. CAMPBELL, President, in the Chair.

Present nineteen members and two visitors.

Messrs. E. S. Littlejohn and C. V. Hanumanta Rao were elected members of the Society.

Mr. N. Sen was nominated for election.

Messrs. C. Fox, E. S. Littlejohn, C. M. Ross, and G. I. Taylor were admitted into the Society.

The President announced that the De Morgan Medal had been awarded to Prof. E. W. Hobson.

The President and Major MacMahon referred to the loss experienced by the Society in the death of Mr. S. Ramanujan.

Mr. G. I. Taylor read two papers (1) "Tidal Oscillations in Gulfs and Rectangular Basins," (2) "Diffusion by Continuous Movements."

Prof. M. J. M. Hill read a paper "The Irreducibility of the Solution of an Algebraic Differential Equation."

The following papers were communicated by title from the chair :—

*Proofs of certain Identities and Congruences enunciated by Mr. S. Ramanujan: Mr. H. B. C. Darling.

*Functions of Limiting Matrices: Mr. F. B. Pidduck.

The Relation between Apolarity and a certain Porism of the Cubic Curve: Prof. W. P. Milne.

(1) A Note on the Maximum Number of Cusps of a Plane Algebraical Curve, (2) A Note on Plane Unicursal Curves: Mr. C. Fox.

The Solutions of certain Systems of Indeterminate Equations: Dr. T. Stuart.

ABSTRACTS.

On the Differential Equation of the First Order derived from an Irreducible Algebraic Primitive

Prof. M. J. M. HILL.

The primitive is supposed to contain one arbitrary constant c to the degree n, the coefficients of the different powers of c being rational functions of x and y.

It is supposed that it is not possible to break it up into others, whose degree in c is less than n, and which have the coefficients of the different powers of c also rational functions of x and y.

It may happen however that if some function of c, which may be called C, is chosen, it is possible to express the primitive as a function of

* Printed in this volume.

C of a degree less than n, the coefficients of the various powers of C being rational functions of x and y.

So far as the relation between x and y is concerned the two primitives are identical, but *a strict adherence to the rules of elimination* makes the left-hand side of the differential equation derived from the primitive in c an exact power of the left-hand side of the primitive in C.

E.g. the differential equation derived from the primitive

$$y - c^2x - c^4 = 0,$$

is

$$(y - px - p^2)^2 = 0,$$

whereas if we take $C = c^2$, the primitive becomes

$$y - Cx - C^2 = 0,$$

and the differential equation

$$y - px - p^2 = 0.$$

The theorem proved in this communication is that if $f(x, y, c) = 0$, an equation of degree n in c and having coefficients which are rational functions of x and y, cannot be transformed into another

$$\phi(x, y, C) = 0,$$

an equation of degree lower than n in C, and having coefficients which are rational functions of x and y, then the differential equation derived from $f(x, y, c) = 0$ is irreducible.

If however the above-mentioned transformation is possible, then the left-hand side of the differential equation derived from $f(x, y, c) = 0$ is an exact power of the left-hand side of the differential equation derived from $\phi(x, y, C) = 0$.

Functions of Limiting Matrices

Mr. F. B. PIDDUCK.

The Frobenius-Sylvester law of congruity suffices theoretically to determine any function of a matrix. Hence it is of interest to treat the case of equal roots as a limit, for which purpose the notation of Grassmann appears the most flexible and powerful. Limiting forms and their func-

tions are found by a uniform process, $f(z)$ being supposed holomorphic in a region containing the roots. A further limiting problem arises when roots move up to singularities of $f(z)$, and it is shown that coalescence of two or more roots in a branch-point may give rise to arbitrary constants in the explicit expression of a multiform function.

An Historical Note on the Intersections of Curves

Mr. H. W. RICHMOND.

A curious historical point arises in connection with this subject, in the problem of making a plane curve of given order pass through a given set of points. First studied by Cayley in 1843, the subject has been developed and extended at various times; but I have not seen it pointed out that the final result arrived at is this—that the whole theory is contained in and may be derived from an algebraical result discovered by Jacobi in 1835, eight years before Cayley's paper was published. At the same time I feel that it is improbable that the fact has not been noticed, since all the papers to which I have to refer are well known.

We restrict ourselves to the simple case when the points are all distinct.

To consider the possibility of defining a curve by imposing on it the condition that it must pass through certain points is a natural problem in algebraic geometry. In his *Higher Plane Curves*, Salmon places it quite early in his second chapter, giving results which lead up to Cayley's Theorem in § 34. [Cayley took his degree in 1842, and his paper, published a year later, must have been one of the very earliest he wrote; it bears the number "5" in his collected works, in which the order is roughly chronological.] What Cayley proves amounts to this, that a curve (C_r), of degree r, can in certain cases be made to pass through the mn (distinct) intersections of C_m and C_n by imposing upon it a smaller number of conditions than mn. In fact if $r = m+n-\gamma$, then the number of conditions is $mn-\delta$, where $\delta = \frac{1}{2}(\gamma-1)(\gamma-2)$. Cayley saw that he could make the curve pass through the mn intersections by taking an equation of the form

$$C_r \equiv A_{r-m}C_m + B_{r-n}C_n = 0, \qquad (1)$$

the symbols A, B, C representing functions of the coordinates of orders

shown by their suffixes. He counted up the number of free constants in this equation (*i.e.* in A and B) and found that he had fulfilled the conditions at a sacrifice of $mn-\delta$ degrees of freedom in the C_r. His statement that every C_r through $nm-\delta$ of the points must go through the remaining δ is too sweeping. It is true in general, but there are exceptional cases. Cayley's method gives no clue to them.

In 1886, Bacharach (*Math. Annalen*, Bd. 26) discovered what were the exceptional cases. Bacharach was able to base his work on the theorem of Nöther, that a C_r through the mn points *must* have an equation of the form (1). The number δ in Cayley's theorem is just one more than enough to define a curve of order $\gamma-3$. The δ points therefore do not, as a rule, lie on a $C_{\gamma-3}$. If they do not, Cayley's theorem is true; but if they do lie on a $C_{\gamma-3}$, it is not true. Later in the paper Bacharach states that a theorem having no exceptions and covering all the cases when the mn points are distinct may be enunciated in the form

Every C_{m+n-3} through all but one of the mn points must go through the last point.

Thus, if a C_r is made to pass through $mn-\delta$ of the points, and a $C_{\gamma-3}$ is made to pass through $\delta-1$ of the remaining points, the two curves together form a curve of order $m+n-3$ through all but one of the points. This last point, says Bacharach, is bound to lie either on the C_r or on the $C_{\gamma-3}$. If it does not lie on the latter curve, it must lie on the former, and Cayley's theorem will hold. If it does lie on the latter, there is no reason that the C_r should go through it. A variety of other special cases arise, but all are covered by the theorem concerning curves of order $m+n-3$.

Between the days of Cayley and Bacharach a different type of result had been proved by Paul Serret in the *Nouvelles Annales* and in his book *Géométrie de direction*, 1869. Serret's theorems are best known in this country by two papers of W. K. Clifford, one read before this Society in 1869 and the other first published in his collected works, Nos. 13 and 14. Serret proves (it is really almost obvious, but he derives a number of very interesting results from his formula) that, in order that a group of N points having coordinates (a_s, b_s, c_s) should possess the property that every C_r through all but one of them must go through the last, it is necessary and sufficient that a relation

$$\Sigma_1^N k_s (ua_s+vb_s+wc_s)^r \equiv 0 \qquad (2)$$

should hold, *i.e.* that a linear syzygy should connect the r-th powers of

the equations of these points. Putting together the results of Cayley, Bacharach, and Serret, we see that the whole theory will follow if we can show that a relation

$$\Sigma_1^{mn} k_s (ua_s + vb_s + wc_s)^{m+n-3} = 0 \qquad (3)$$

connects the equations of the mn points (a_s, b_s, c_s) of intersection of a C_m and a C_n.

This result was proved by Jacobi in 1835 (*Ges. Werke*, Vol. 3, p. 292, or Forsyth, *Theory of Functions*, 2nd edition, p. 574). The value of k_s is found by Jacobi. If $U = 0$, $V = 0$ are the equations of two curves and xyz a common point, then

$$xU_x + yU_y + zU_z = 0,$$

$$xV_x + yV_y + zV_z = 0,$$

$$x : y : z :: U_y V_z - V_y U_z : U_z V_x - V_z U_x : U_x V_y - V_x U_y;$$

or

$$\left.\begin{aligned} k(U_y V_z - V_y U_z) &= x \\ k(U_z V_x - V_z U_x) &= y \\ k(U_x V_y - V_x U_y) &= z \end{aligned}\right\}. \qquad (4)$$

If in (4) we give to xyz the values $a_s b_s c_s$, the coordinates of the mn different points of intersection, the values of k obtained are those of k_s in (3). Also (3) depends, as it should, only on the ratios $a_s : b_s : c_s$.

From this identity of Jacobi, as interpreted by Paul Serret, the whole Cayley-Bacharach theory follows. Jacobi's identity may be differentiated with respect to u or v or w repeatedly, and so gives similar identities of lower order than $m+n-3$. To obtain results of order $m+n-\gamma$ we use a differential operator

$$F\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial v}, \frac{\partial}{\partial w}\right),$$

where F is any polynomial of order $\gamma-3$, and so obtain the result

$$\Sigma_1^{mn} k_s F(a_s, b_s, c_s)(ua_s + vb_s + wc_s)^{m+n-\gamma} = 0.$$

Thus if $F(x, y, z) = 0$ is the equation of a $C_{\gamma-3}$ through certain of the mn intersections, F vanishes for these points and we have a relation or syzygy of Serret's type of order $m+n-\gamma$ among the remainder of the mn points. Hence a $C_{m+n-\gamma}$ through all but one of the remainder of the mn points

(*i.e.* the points which do not lie on F) must go through the last, as Bacharach showed.

The Bacharach theorem and the Jacobi-Serret identity are, in fact, all but equivalent. Where they differ it would seem that the identity has the advantage. Special or difficult cases are easier to settle when we have definite algebraic result to start from; for example, if some of the mn points approach indefinitely near to one another, we can consider this as a limiting case. The Jacobi result also shows more clearly what happens when certain of the points occupy exceptional positions; *e.g.* if 9 of them lie at the intersections of two cubics, we have two syzygies of order $m+n-6$ in the remaining $mn-9$ points.

LIBRARY

Presents.

BETWEEN December 31st, 1919, and December 31st, 1920, the following presents were made to the Library by their respective authors and publishers:—

Buhl, A.—Géométrie et Analyse des Intégrales doubles.
Desvallées, H.—Tables logarithmiques et trigonométriques à quatre décimales.
Hilton, Harold.—Plane Algebraic Curves.
Inghirami, G.—Tables des Nombres premiers et de la Décomposition des Nombres de 1 à 100,000.
Lebon, Ernest.—Table de Caractéristiques de Base 30030, donnant en un seul coup d'œil les facteurs premiers des nombres, tome 1, fasc. 1.
Leveugle, H.—Précis de Calcul géométrique.
d'Ocagne, Lt.-Col.—Principes usuels de Nomographie.
Poussin, C. de la Vallée.—Leçons sur l'Approximation d'une Variable réelle.
See, T. J. J.—New Theory of the Aether, first and second papers.

Åbo Academy: Acta Humaniora, no. 1.
Amsterdam: Société Mathématique, Œuvres Complètes de Thomas Jan Stieltjes; Revue semestrielle des Publications Mathématiques, tome 28, 1ère partie.
Amsterdam: Royal Academy of Sciences, Proceedings, vol. 18, parts 1, 2; vol. 19, parts 1, 2; Verhandelungen, deel 12, nos. 1, 2, 3.
Athenæum: nos. 4679–4731.
Brussels: Académie Royale de Belgique, Bulletin de la Classe des Sciences, 1914, nos. 5–12; 1919, nos. 1–12; 1920, no. 1. Tables Générales des Bulletins, 1899–1910, 1911–1914; Annexe aux Bulletins, 1915. Catalogue onomastique des Accroissements de la Bibliothèque, 1883–1914. Tables des Notices Biographiques, 1835–1919; Programme des Concours Annuels, 1915, et Fondations Académiques.
Brussels: Académie Royale des Sciences, Annuaire, 81me–85me années, 1915–1919.
Journal für die Reine und Angewandte Mathematik, band 150, hefte 1–4.
Koutchino: Bulletin de l'Institut Aérodynamique, fasc. 6.
London: Conjoint Board of Scientific Societies, Confirmed Minutes of Special Meetings, November 12th, 1919, and January 8th, 1920; Third Annual Report, 1919.
Madrid: Junta para Ampliación de Estudios e Investigaciones Científicas, Publicaciones del Laboratorio y Seminario Matemático, tome 3, memoria 4.
Mathematical Gazette, vol. 9, no. 143; vol. 10, nos. 144–149.
Nautical Almanac, 1922.
Paris: L'Enseignement Mathématique, 20me année, no. 6; 21me année, nos. 1, 2.
Rassegna di Matematica e Fisica, anno 1, no. 1.
Royal Aeronautical Society: A Glossary of Aeronautical Terms.
Sendai: Tôhoku Imperial University, Science Reports, vol. 8, no. 3; vol. 9, nos. 1–4.
Sendai: Tôhoku Mathematical Journal, vol. 16, no. 3; vol. 17, nos. 1–4; vol. 18, nos. 1, 2.
Tokyo: Imperial University, College of Science, vol. 41, article 9; vol. 37, article 5.

Tokyo: Physico-Mathematical Society of Japan, Proceedings, 3rd series, vol. 1, nos. 10, 11; vol. 2, nos. 1–7, 9, 10.

U.S. Coast and Geodetic Survey: Important Publications of the U.S. Coast and Geodetic Survey appearing since January 1st, 1914.

Exchanges.

Acta Mathematica: vol. 42, nos. 3, 4.

American Journal of Mathematics, vol. 42, nos. 3, 4.

Amsterdam: Nieuw Archief voor Wiskunde, deel 13, stuk 2.

Amsterdam: Wiskundige Opgaven, deel 13, stuk 2.

Annals of Mathematics: vol. 19, no. 1.

Benares: Mathematical Society, Proceedings, vol. 1; vol. 2, pt. 1.

Berlin: Mathematische Zeitschrift, band 5, hefte 3–5; band 6, hefte 3, 4; band 7, hefte 1–4.

Boston (Mass.): American Academy of Arts and Sciences, Proceedings, vol. 50, no. 1; vol. 55, no. 7.

Bulletin des Sciences Mathématiques, vol. 43, nos. 10–12; vol. 44, nos. 1–10.

Calcutta: Mathematical Society, Bulletin, vol. 10, nos. 3, 4; vol. 11, nos. 1, 2.

Cambridge Philosophical Society, Proceedings, vol. 19, pt. 6; vol. 20, pt. 1; Transactions, vol. 22, nos. 15–22.

Cape Town: South African Journal of Science, vol. 16, nos. 1–5; vol. 17, no. 1.

Catania: Accademia Gioenia, Bolletino, fasc. 47, 48.

Copenhagen: K. Danske Vidensk. Selskabs Mathematisk-fysiske Meddelelser, vol. 1, nos. 13–15; vol. 2, nos. 4–11.

Edinburgh Mathematical Society, Proceedings, vols. 37, 38; vol. 40, pt. 1.

Edinburgh: Royal Society, Proceedings, vol. 39, pt. 3.

Florence: Biblioteca Nazionale Centrale, Bollettino, nos. 223–233.

Journal des Mathématiques pures et appliquées, 8me série, tome 2, année 1919.

Lancaster, Pa.: American Mathematical Society, Bulletin, vol. 26, nos. 4–10; vol. 27, nos. 1, 2; Transactions, vol. 20, no. 4; vol. 21, nos. 1–4.

La Plata: Universidad Nacional, Contribución al Estudio de las Ciencias físicas y matemáticos, nos. 37, 43, 45, 46.

London: Institute of Actuaries, vol. 52, pt. 1.

London: Institute of Naval Architects, 13 papers read at the Spring and Summer Meetings, 1920.

London: Physical Society, Proceedings, vol. 32, pts. 1–5.

London: Royal Astronomical Society, Monthly Notices, vol. 80.

London: Royal Society, Philosophical Transactions, vol. 220, nos. 575–581; vol. 221, nos. 582–591. Proceedings, vol. 96, no. 679–vol. 98, no. 690.

Madras: Indian Mathematical Society, Journal, vol. 11, no. 6; vol. 12, nos. 1–3.

Naples: Accademia delle Scienze fisiche e matematico, Rendiconto, vol. 22, fasc. 7; vol. 26, fasc. 3, 4, 6.

National Physical Laboratory: Report for 1919; Supplementary Report for 1918; Collected Researches, vol. 14, 1920.

Nature: vol. 104, nos. 2618–2625; vol. 105; vol. 106, nos. 2653–2668.

Nouvelles Annales de Mathématiques: 4me série, Jan.–Nov., 1920.

Oporto: Academia Polytechnica, Annaes Scientificos, vol. 13, nos. 1, 2, 3.

Palermo: Rendiconto del Circolo Matematico, tomo 43, fasc. 1, 2; tomo 44, fasc. 1; Supplement, tomo 11, fasc. 1 (anni 1919–20).

Paris: Société Mathématique de France, Bulletin, tome 47, fasc. 1–4; tome 48, fasc. 1, 2; Comptes Rendus des Séances, année 1919.

Philadelphia: American Philosophical Society, Proceedings, vol. 58, nos. 1–7; vol. 59, nos. 1–4.

Pisa: Annali della R. Scuola Normale, Scienze fisiche e matematiche, vol. 13.
Prague: Casopis pro pestování mathematiky a fisiky, Rocnik 49, Cislo 1, 2, 3.
Revista de matemáticas y Físicas elementales, año 1, nos. 1–8, 12; año 2, nos. 1–6.
Rome: Reale Accademia dei Lincei, Atti, vol. 28, fasc. 10–12; vol. 29; vol. 30, fasc. 1–6.
Toronto: Royal Canadian Institute, Transactions, vol. 12, pt. 2.
Turin: Reale Accademia delle Scienze, Atti, vol. 55, nos. 1–16.
Washington: National Academy of Sciences, Proceedings, vol. 5, no. 12; vol. 6, nos. 1–9; National Research Council, Bulletin, vol. 1, pt. 1, no. 1.
Washington: U.S. Naval Observatory, Annual Report, 1919; American Ephemeris and Nautical Almanac, 1921, 1922.
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Mathematische Annalen, band 77, hefte 1–4; band 78, hefte 1–4; band 79, hefte 1–4; band 80, heft 1; band 81, hefte 1–4.
Messenger of Mathematics, vol. 48, nos. 11, 12; vol. 49, nos. 1–12.
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Quarterly Journal of Mathematics, vol. 48, no. 4.

OBITUARY NOTICES

HIERONYMUS GEORG ZEUTHEN.

By the death of Prof. Hieronymus Georg Zeuthen, on January 6th, 1920, in the eighty-first year of his age, the Society has lost one of the oldest of its honorary members—a member of forty-five years standing—for it was in January 1875 that Drs. Klein, Kronecker, and Zeuthen were elected foreign members of the Society. The writer of this notice had not the privilege of personal acquaintance with Prof. Zeuthen, and wishes gratefully to acknowledge his obligation to the kindness of Prof. C. Juel, of Copenhagen, who has allowed him to quote from the memoir of Prof. Zeuthen which was read before the Royal Danish Academy of Science, and has also supplied a list of Prof. Zeuthen's publications.

Zeuthen was born in Jutland in 1839, and entered the University of Copenhagen as a student in 1857. His earliest productions were papers contributed to the Danish *Tidsskrift for Matematik*, which was founded in 1859; these were written during his student days or the years immediately following. His first work of importance was his dissertation for his Doctorate. He had gone, in 1863, to Paris to study under Chasles, the mathematician who undoubtedly exerted a greater influence upon him than any other. Chasles is the founder of Enumerative Geometry and of the Theory of Characteristics, and it was in these subjects that Zeuthen's powers first revealed themselves. His earliest work in this field was his Doctor's Thesis of 1865, translated and published in the *Nouvelles Annales de Mathématiques* in 1866, with the title "A New Method of Determining the Characteristics of Systems of Conics"—a work whose merit was immediately recognised. Zeuthen next studied surfaces of the second order and determined the characteristics in the elementary systems of such surfaces. It may be mentioned that, on learning that Chasles was writing on the same subject, Zeuthen withheld his results from publication, sending them in a closed envelope to the Danish Academy of Science, with the instructions that it should not be opened until after the publication of Chasles' treatise. Continuing investigations of a similar kind Zeuthen produced in 1873 his comprehensive "General Properties of Systems of Plane Curves with Application to

determine Characteristics in the Elementary Systems of the Fourth Order." The subject is one which has never attracted so much attention in this country as it has abroad. As a type of the results which Zeuthen obtained we may extract an example from Chap. XV of Pascal's *Repertorio di Mathematiche Superiori*, Vol. II, 1900. Since nine conditions determine a plane cubic curve, it was to be expected that a finite number of such curves will pass through r given points and touch $9-r$ given lines. Zeuthen determined the number of such curves, corresponding to values 9, 8, 7, ..., 0 of r, viz. 1, 4, 16, 64, 256, 976, 3424, 9766, 21004, 33616.

A variety of such results will be found quoted in this chapter; reference should also be made to Zeuthen's article "Abzählende Methoden," in the *Encyklopädie der Mathematischen Wissenschaften*, Bd. III, Heft 2 (1906), pp. 257–312. For quadric surfaces, determined by nine conditions of passing through certain points, touching certain planes, and touching certain lines, thirty separate cases have to be considered and the numbers of solutions in various cases range from 1 up to 128.

Other important works by Zeuthen of this period bear testimony to the brilliance of his powers. They include several which deal with the genus (or deficiency) of algebraic curves and allied matters. There is his beautiful geometrical proof, *Comptes Rendus*, Vol. 70 (1870), p. 743, that the genera of two curves whose points are in (1, 1)-correspondence must be equal (a theorem already proved by Riemann from consideration of Riemann surfaces, and algebraically by Clebsch and Gordan). Zeuthen's proof was obtained independently of a very similar proof published a few months earlier by Bertini [*Giorn. di Mat., Battaglini*, Vol. 7 (1869), p. 105]. The method of proof is as follows. If a moving point M of a given curve C and a moving point M_1 of a second given curve C_1 are in (1, 1)-correspondence, the intersection of the lines AM and A_1M_1 joining M and M_1 to two fixed points A and A_1 traces an algebraic curve; and by considering the class of this curve as calculated from the number of tangents to it from A and A_1, respectively, the theorem that the genus of C is equal to that of C_1 follows at once. It would be difficult to devise any proof more simple and fundamental than this. Zeuthen proceeded to use his method to extend the theorem to cases where the points of two curves are in multiple correspondence, and so established what is known as "Zeuthen's extended theorem upon genus". Here Zeuthen's geometrical method led to a result which had not previously been recognised, although it was remarked later that the theorem could be obtained from the classical theory. It is therefore fitting that the theorem should bear Zeuthen's name. Continuing to work in the same field Zeuthen applied the principles of correspondence to solid geometry, and in 1871 discovered a number

which is invariant in any point for point transformation of one algebraic surface into another. The value of this discovery was not, and in fact could not be, recognised at the time; but more than twenty years later, when such properties of surfaces were investigated by more modern methods by the Italian school of mathematicians, the invariant was rediscovered in 1895 by Segre, and now is known as the Zeuthen-Segre invariant of the surface. See Prof. H. F. Baker's Presidential Address to this Society (1912), *Proc. London Math. Soc.*, Vol. 12, p. 33. or *Encyklopädie*, Vol. III, Cap. 6, *b.*, p. 701.

Space will not permit more than a mention of Zeuthen's work upon cubic surfaces, or of his contribution to Vol. 10, Ser. 1, of the *Proceedings* of this Society in 1879. An arresting paper is that entitled "Sur les différentes formes de courbes planes du quatrième ordre" (*Math. Annalen*, Vol. 7, pp. 410–432), in which Zeuthen first examines the distinction (pointed out by v. Staudt) between the odd and even branches of a curve, and proves in a very simple manner the theorems concerning the intersections of two branches. He then shows that of the twenty-eight double tangents of a curve without nodes there are always four which are real, and either touch the *same* branch twice or are isolated, *i.e.* are real lines having two imaginary contacts with the curve; the eight points of contact of these four double tangents lie on a conic. All other real double tangents touch two different branches, each two branches external to one another necessarily giving rise to four double tangents. It is hardly to be doubted that this paper largely inspired the striking discoveries of Klein and Harnack, published in two famous papers in Vol. 10 of the *Math. Annalen*; and Klein's results as to the form of cubic surfaces are closely connected with it.

From about 1880 onwards Zeuthen's interests turned more and more towards the history of mathematics, chiefly, but by no means wholly, in classical times. He had published a short paper in 1876 on "Brahmegupta's trapeziums", but from 1880 he found a richer field for study in tracing the development of Greek mathematics. Thus it is probable that the name of Zeuthen is better known at the present day as a historian of mathematics than as an original discoverer in the subject. We will not here attempt to give a detailed account of his many writings upon Archimedes, Euclid, Apollonius, Diophantus, &c., or of those dealing with the later times of Descartes, Cardan, Fermat, Newton, Barrow. His most important historical work, *Die Lehre von den Kegelschnitten in Altertum*, was published in 1886.

To the end of his life Zeuthen continued to publish papers on mathematical, and for the most part geometrical or historical, subjects. In the

year 1919 (a year before his death) he published two papers—one on the origin of Algebra, and the other on the explanation of a paradox in Enumerative Geometry.

Almost the whole of Zeuthen's life was passed in Copenhagen, where he was for many years Professor at the University. The number of Zeuthen's publications amounts to nearly two hundred, and include besides the numerous articles in various periodicals, elementary textbooks, textbooks for students at the University or Polytechnic, papers read at various International Congresses in mathematics or philosophy, and (in addition to the history of Conic Sections already referred to) a *History of Mathematics* (1883), and a *History of Mathematics in the Sixteenth and Seventeenth Centuries* (1903). Until the last year of his life he was Secretary to the Danish Academy of Science.

H. W. R.

SRINIVASA RAMANUJAN.

I.

SRINIVASA RAMANUJAN, who died at Kumbakonam on April 26th, 1920, had been a member of the Society since 1917. He was not a man who talked much about himself, and until recently I knew very little of his early life. Two notices, by P. V. Seshu Aiyar and R. Ramachandra Rao, two of the most devoted of Ramanujan's Indian friends, have been published recently in the *Journal of the Indian Mathematical Society*; and Sir Francis Spring has very kindly placed at my disposal an article which appeared in the *Madras Times* of April 5th, 1919. From these sources of information I can now supply a good many details with which I was previously unacquainted. Ramanujan (Srinivasa Iyengar Ramanuja Iyengar, to give him for once his proper name) was born on December 22nd, 1887, at Erode in southern India. His father was an accountant (*gumasta*) to a cloth merchant at Kumbakonam, while his maternal grandfather had served as *amin* in the Munsiff's (or local judge's) Court at Erode. He first went to school at five, and was transferred before he was seven to the Town High School at Kumbakonam, where he held a "free scholarship", and where his extraordinary powers appear to have been recognised immediately. "He used", so writes an old schoolfellow to Mr. Seshu Aiyar, "to borrow Carr's *Synopsis of Pure Mathematics* from the College library, and delight in verifying some of the formulæ given there. . . . He used to entertain his friends with his theorems and formulæ, even in those early days. . . . He had an extraordinary memory and could easily repeat the complete lists of Sanscrit roots (*atmanepada* and *parasmepada*); he could give the values of $\sqrt{2}$, π, e, ... to any number of decimal places. . . . In manners, he was simplicity itself. . . ."

He passed his matriculation examination to the Government College at Kumbakonam in 1904; and secured the "Junior Subraniam Scholarship". Owing to weakness in English, he failed in his next examination and lost his scholarship; and left Kumbakonam, first for Vizagapatam and then for Madras. Here he presented himself for the "First Examination in Arts" in December 1906, but failed and never tried again. For the next few years he continued his independent work in mathematics, "jotting down his results in two good-sized notebooks": I have one of these note

books in my possession still. In 1909 he married, and it became necessary for him to find some permanent employment. I quote Mr. Seshu Aiyar:

To this end, he went to Tirukoilur, a small sub-division town in South Arcot District, to see Mr. V. Ramaswami Aiyar, the founder of the Indian Mathematical Society, but Mr. Aiyar, seeing his wonderful gifts, persuaded him to go to Madras. It was then after some four years' interval that Mr. Ramanujan met me at Madras, with his two well-sized note-books referred to above. I sent Ramanujan with a note of recommendation to that true lover of Mathematics, Dewan Bahadur R. Ramachandra Rao, who was then District Collector at Nellore, a small town some eighty miles north of Madras. Mr. Rao sent him back to me saying it was cruel to make an intellectual giant like Ramanujan rot at a mofussil station like Nellore, and recommended his stay at Madras, generously undertaking to pay Mr. Ramanujan's expenses for a time. This was in December 1910. After a while, other attempts to obtain for him a scholarship having failed, and Ramanujan himself being unwilling to be a burden on anybody for any length of time, he decided to take up a small appointment under the Madras Port Trust in 1911.

But he never slackened his work at Mathematics. His earliest contribution to the *Journal of the Indian Mathematical Society* was in the form of questions communicated by me in Vol. III (1911). His first long article on 'Some Properties of Bernoulli's Numbers' was published in the December number of the same volume. Mr. Ramanujan's methods were so terse and novel and his presentation was so lacking in clearness and precision, that the ordinary reader, unaccustomed to such intellectual gymnastics, could hardly follow him. This particular article was returned more than once by the Editor before it took a form suitable for publication. It was during this period that he came to me one day with some theorems on Prime Numbers, and when I referred him to Hardy's Tract on *Orders of Infinity*, he observed that Hardy had said on p. 36 of his Tract 'the exact order of $\rho(x)$ [defined by the equation

$$\rho(x) = \pi(x) - \int_2^x \frac{dt}{\log t},$$

where $\pi(x)$ denotes the number of primes less than x], has not yet been determined', and that he himself had discovered a result which gave the order of $\rho(x)$. On this I suggested that he might communicate his result to Mr. Hardy, together with some more of his results.

This passage brings me to the beginning of my own acquaintance with Ramanujan. But before I say anything about the letters which I received from him, and which resulted ultimately in his journey to England, I must add a little more about his Indian career. Dr. G. T. Walker, F.R.S., Head of the Meteorological Department, and formerly Fellow and Mathematical Lecturer of Trinity College, Cambridge, visited Madras for some official purpose some time in 1912; and Sir Francis Spring, K.C.I.E., the Chairman of the Madras Port Authority, called his attention to Ramanujan's work. Dr. Walker was far too good a mathematician not to recognise its quality, little as it had in common with his own. He brought Ramanujan's case to the notice of the Government and the University of Madras. A research studentship, "Rs. 75 *per mensem* for a period of two years", was awarded him; and he became, and remained for the rest of his life, a professional mathematician.

II.

Ramanujan wrote to me first on January 16th, 1913, and at fairly regular intervals until he sailed for England in 1914. I do not believe that his letters were entirely his own. His knowledge of English, at that stage of his life, could scarcely have been sufficient, and there is an occasional phrase which is hardly characteristic. Indeed I seem to remember his telling me that his friends had given him some assistance. However, it was the mathematics that mattered, and that was very emphatically his.

Madras, 16th January 1913

"Dear Sir

I beg to introduce myself to you as a clerk in the Accounts Department of the Port Trust Office at Madras on a salary of only £20 per annum. I am now about 23 years of age. I have had no university education but I have undergone the ordinary school course. After leaving school I have been employing the spare time at my disposal to work at Mathematics. I have not trodden through the conventional regular course which is followed in a university course, but I am striking out a new path for myself. I have made a special investigation of divergent series in general and the results I get are termed by the local mathematicians as 'startling'.

Just as in elementary mathematics you give a meaning to a^n when n is negative and fractional to conform to the law which holds when n is a positive integer, similarly the whole of my investigations proceed on giving a meaning to Eulerian Second Integral for all values of n. My friends who have gone through the regular course of university education tell me that $\int_0^\infty x^{n-1}e^{-x}dx = \Gamma(n)$ is true only when n is positive. They say that this integral relation is not true when n is negative. Supposing this is true only for positive values of n and also supposing the definition $n\Gamma(n) = \Gamma(n+1)$ to be universally true, I have given meanings to these integrals and under the conditions I state the integral is true for all values of n negative and fractional. My whole investigations are based upon this and I have been developing this to a remarkable extent so much so that the local mathematicians are not able to understand me in my higher flights.

Very recently I came across a tract published by you styled *Orders of Infinity* in page 36 of which I find a statement that no definite expression has been as yet found for the no of prime nos less than any

given number. I have found an expression which very nearly approximates to the real result, the error being negligible. I would request you to go through the enclosed papers. Being poor, if you are convinced that there is anything of value I would like to have my theorems published. I have not given the actual investigations nor the expressions that I get but I have indicated to the lines on which I proceed. Being inexperienced I would very highly value any advice you give me. Requesting to be excused for the trouble I give you.

I remain Dear sir Yours truly

S. Ramanujan

P.S. My address is S. Ramanujan, Clerk Accounts Department, Port Trust, Madras, India."

I quote now from the "papers enclosed," and from later letters :—

"In page 36 it is stated that 'the no of prime nos less than $x = \int_2^x \frac{dt}{\log t} + \rho(x)$ where the precise order of $\rho(x)$ has not been determined. . . .'

I have observed that $\rho(e^{2\pi x})$ is of such a nature that its value is very small when x lies between 0 and 3 (its value is less than a few hundreds when $x = 3$) and rapidly increases when x is greater than 3. . . .

The difference between the no of prime nos of the form $4n-1$ and which are less than x and those of the form $4n+1$ less than x is infinite when x becomes infinite. . . .

The following are a few examples from my theorems :—

(1) The nos of the form $2^p 3^q$ less than $n = \frac{1}{2} \frac{\log(2n) \log(3n)}{\log 2 \log 3}$ where p and q may have any positive integral value including 0.

(2) Let us take all nos containing an odd no of dissimilar prime divisors viz.

2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 30, 31, 37, 41, 42, 43, 47 &c

(*a*) The no of such nos less than $n = \frac{3n}{\pi^2}$.

(*b*) $\frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots + \frac{1}{30^2} + \frac{1}{31^2} + \ldots = \frac{9}{2\pi^2}$.

(*c*) $\frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \&c. = \frac{15}{2\pi^4}$.

(3) Let us take the no of divisors of natural nos viz.

1, 2, 2, 3, 2, 4, 2, 4, 3, 4, 2 &c (1 having 1 divisor, 2 having 2, 3 having 2, 4 having 3, 5 having 2, &c).

The sum of such nos to n terms

$$= n(2\gamma - 1 + \log n) + \tfrac{1}{2} \text{ of the no of divisors of } n$$

where $\gamma = \cdot 5772156649\ldots$, the Eulerian Constant.

(4) 1, 2, 4, 5, 8, 9, 10, 13, 16, 17, 18 &c are nos which are either themselves sqq. or which can be expressed as the sum of two sqq.

The no of such nos greater than A and less than B

$$= K \int_A^B \frac{dx}{\sqrt{\log x}} + \theta(x)^* \quad \text{where} \quad K = \cdot 764\ldots$$

and $\theta(x)$ is very small when compared with the previous integral. K and $\theta(x)$ have been exactly found though complicated. . . ."

Ramanujan's theory of primes was vitiated by his ignorance of the theory of functions of a complex variable. It was (so to say) what the theory might be if the Zeta-function had no complex zeros. His methods of proof depended upon a wholesale use of divergent series. He disregarded entirely all the difficulties which are involved in the interchange of double limit operations: he did not distinguish, for example, between the sum of a series Σa_n and the value of the Abelian limit

$$\lim_{x \to 1} \Sigma a_n x^n,$$

or that of any other limit which might be used for similar purposes by a modern analyst. There are regions of mathematics in which the precepts of modern rigour may be disregarded with comparative safety, but the Analytic Theory of Numbers is not one of them, and Ramanujan's Indian work on primes, and on all the allied problems of the theory, was definitely wrong. That his proofs should have been invalid was only to be expected. But the mistakes went deeper than that, and many of the actual results were false. He had obtained the dominant terms of the classical formulæ, although by invalid methods; but none of them are such close approximations as he supposed.

This may be said to have been Ramanujan's one great failure. And yet I am not sure that, in some ways, his failure was not more wonderful than any of his triumphs. Consider, for example, problem (4). The dominant term, which Ramanujan gives correctly, was first obtained by

* This should presumably be $\theta(B)$.

Landau in 1908. The correct order of the error term is still unknown. Ramanujan had none of Landau's weapons at his command; he had never seen a French or German book; his knowledge even of English was insufficient to enable him to qualify for a degree. It is sufficiently marvellous that he should have even dreamt of problems such as these, problems which it has taken the finest mathematicians in Europe a hundred years to solve, and of which the solution is incomplete to the present day.

"... IV. Theorems on integrals. The following are a few examples

$$(1)\quad \int_0^\infty \frac{1+\left(\frac{x}{b+1}\right)^2}{1+\left(\frac{x}{a}\right)^2}\cdot\frac{1+\left(\frac{x}{b+2}\right)^2}{1+\left(\frac{x}{a+1}\right)^2}\cdots \&c\ dx$$

$$= \frac{\sqrt{\pi}}{2}\cdot\frac{\Gamma(a+\frac{1}{2})}{\Gamma(a)}\cdot\frac{\Gamma(b+1)}{\Gamma(b+\frac{1}{2})}\cdot\frac{\Gamma(b-a+\frac{1}{2})}{\Gamma(b-a+1)}.$$

...

(3) If
$$\int_0^\infty \frac{\cos nx}{e^{2\pi\sqrt{x}}-1}\,dx = \phi(n),$$

then
$$\int_0^\infty \frac{\sin nx}{e^{2\pi\sqrt{x}}-1}\,dx = \phi(n)-\frac{1}{2n}+\phi\left(\frac{\pi^2}{n}\right)\sqrt{\frac{2\pi^3}{n^3}}.$$

$\phi(n)$ is a complicated function. The following are certain special values

$$\phi(0)=\frac{1}{12};\quad \phi\left(\frac{\pi}{2}\right)=\frac{1}{4\pi};\quad \phi(\pi)=\frac{2-\sqrt{2}}{8};\quad \phi(2\pi)=\frac{1}{16};$$

$$\phi\left(\frac{2\pi}{5}\right)=\frac{8-8\sqrt{5}}{16};\quad \phi\left(\frac{\pi}{5}\right)=\frac{6+\sqrt{5}}{4}-\frac{5\sqrt{10}}{8};\quad \phi(\infty)=0;$$

$$\phi\left(\frac{2\pi}{3}\right)=\frac{1}{3}-\sqrt{3}\left(\frac{3}{16}-\frac{1}{8\pi}\right).$$

$$(4)\quad \int_0^\infty \frac{dx}{(1+x^2)(1+r^2x^2)(1+r^4x^2)\ldots \&c} = \frac{\pi}{2(1+r+r^3+r^6+r^{10}+\&c)}$$

where 1, 3, 6, 10 &c are sums of natural nos.

$$(5)\quad \int_0^\infty \frac{\sin 2nx}{x(\cosh \pi x+\cos \pi x)}\,dx = \frac{\pi}{4}-2\left(\frac{e^{-n}\cos n}{\cosh\frac{\pi}{2}}-\frac{e^{-3n}\cos 3n}{3\cosh\frac{3\pi}{2}}\ldots \&c\right).$$

...

V. Theorems on summation of series;* *e.g.*

(1) $\frac{1}{1^3}\cdot\frac{1}{2}+\frac{1}{2^3}\cdot\frac{1}{2^2}+\frac{1}{3^3}\cdot\frac{1}{2^3}+\frac{1}{4^3}\cdot\frac{1}{2^4}+\&c$

$$=\tfrac{1}{6}(\log 2)^3-\frac{\pi^2}{12}\log 2+\left(\frac{1}{1^3}+\frac{1}{3^3}+\frac{1}{5^3}+\&c\right).$$

(2) $1+9.(\tfrac{1}{4})^4+17.\left(\frac{1.5}{4.8}\right)^4+25.\left(\frac{1.5.9}{4.8.12}\right)^4+\&c=\frac{2\sqrt{2}}{\sqrt{\pi}.\{\Gamma(\frac{3}{4})\}^2}.$

(3) $1-5.(\tfrac{1}{2})^3+9.\left(\frac{1.3}{2.4}\right)^3-\&c=\frac{2}{\pi}.$

(4) $\frac{1^{13}}{e^{2\pi}-1}+\frac{2^{13}}{e^{4\pi}-1}+\frac{3^{13}}{e^{6\pi}-1}+\&c=\frac{1}{24}.$

(5) $\frac{\coth\pi}{1^7}+\frac{\coth 2\pi}{2^7}+\frac{\coth 3\pi}{3^7}+\&c=\frac{19\pi^7}{56700}$

(6) $\frac{1}{1^5\cosh\frac{\pi}{2}}-\frac{1}{3^5\cosh\frac{3\pi}{2}}+\frac{1}{5^5\cosh\frac{5\pi}{2}}-\&c=\frac{\pi^5}{768}.$

. . .

VI. Theorems on transformation of series and Integrals, *e.g.*

(1) $\pi\left(\frac{1}{2}-\frac{1}{\sqrt{1}+\sqrt{3}}+\frac{1}{\sqrt{3}+\sqrt{5}}-\frac{1}{\sqrt{5}+\sqrt{7}}+\&c\right)$

$$=\frac{1}{1\sqrt{1}}-\frac{1}{3\sqrt{3}}+\frac{1}{5\sqrt{5}}-\&c.$$

. . .

(3) $1-\frac{x^2\lfloor\underline{3}}{(\lfloor\underline{1}\,\lfloor\underline{2})^3}+\frac{x^4\lfloor\underline{6}}{(\lfloor\underline{2}\,\lfloor\underline{4})^3}-\frac{x^6\lfloor\underline{9}}{(\lfloor\underline{3}\,\lfloor\underline{6})^3}+\&c$

$$=\left\{1+\frac{x}{(\lfloor\underline{1})^3}+\frac{x^2}{(\lfloor\underline{2})^3}+\&c.\right\}\left\{1-\frac{x}{(\lfloor\underline{1})^3}+\frac{x^2}{(\lfloor\underline{2})^3}-\&c\right\}.$$

. . .

(6) If $\alpha\beta=\pi^2$, then $\frac{1}{\sqrt[4]{\alpha}}\left\{1+4\alpha\int_0^\infty\frac{xe^{-\alpha x^2}}{e^{2\pi x}-1}\,dx\right\}$

$$=\frac{1}{\sqrt[4]{\beta}}\left\{1+4\beta\int_0^\infty\frac{xe^{-\beta x^2}}{e^{2\pi x}-1}\,dx\right\}.$$

* There is always more in one of Ramanujan's formulæ than meets the eye, as anyone who sets to work to verify those which look the easiest will soon discover. In some the interest lies very deep, in others comparatively near the surface; but there is not one which is not curious and entertaining.

(7) $n\left(e^{-n^2}-\frac{e^{-\frac{n^2}{3}}}{3\sqrt{3}}+\frac{e^{-\frac{n^2}{5}}}{5\sqrt{5}}-\&c\right)$

$$=\sqrt{\pi}\,(e^{-n\sqrt{\pi}}\sin n\sqrt{\pi}-e^{-n\sqrt{3\pi}}\sin n\sqrt{3\pi}+\&c).$$

(8) If n is any positive integer excluding 0

$$\frac{1^{4n}}{(e^{\pi}-e^{-\pi})^2}+\frac{2^{4n}}{(e^{2\pi}-e^{-2\pi})^2}\ldots\ \&c=\frac{n}{\pi}\left\{\frac{B_{4n}}{8n}+\frac{1^{4n-1}}{e^{2\pi}-1}+\frac{2^{4n-1}}{e^{4\pi}-1}\ldots\ \&c\right\}$$

where $B_2=\frac{1}{6}$, $B_4=\frac{1}{30}$, &c.

VII. Theorems on approximate integration and summation of series.

. . .

(2) $1+\frac{x}{\lfloor 1}+\frac{x^2}{\lfloor 2}+\frac{x^3}{\lfloor 3}+\ldots+\frac{x^x}{\lfloor x}\theta=\frac{e^x}{2}$

where $\theta=\frac{1}{3}+\frac{4}{135(x+k)}$ where k lies between $\frac{8}{45}$ and $\frac{2}{21}$.

(3) $1+\left(\frac{x}{\lfloor 1}\right)^5+\left(\frac{x^2}{\lfloor 2}\right)^5+\left(\frac{x^3}{\lfloor 3}\right)^5+\&c=\frac{\sqrt{5}}{4\pi^2}\cdot\frac{e^{5x}}{5x^2-x+\theta}$

where θ vanishes when $x=\infty$.

(4) $\frac{1^2}{e^x-1}+\frac{2^2}{e^{2x}-1}+\frac{3^2}{e^{3x}-1}+\frac{4^2}{e^{4x}-1}+\&c$

$$=\frac{2}{x^3}\left(\frac{1}{1^3}+\frac{1}{2^3}+\frac{1}{3^3}+\&c\right)-\frac{1}{12x}-\frac{x}{1440}+\frac{x^3}{181440}$$

$$+\frac{x^5}{7257600}+\frac{x^7}{159667200}+\&c\ \text{ when } x \text{ is small.}$$

(*Note.*—x may be given values from 0 to 2).

(5) $\frac{1}{1001}+\frac{1}{1002^2}+\frac{3}{1003^3}+\frac{4^2}{1004^4}+\frac{5^3}{1005^5}+\&c$

$$=\frac{1}{1000}-10^{-440}\times 1{\cdot}0125 \text{ nearly.}$$

(6) $\int_0^a e^{-x^2}dx=\frac{\sqrt{\pi}}{2}-\frac{e^{-a^2}}{2a}+\frac{1}{a}+\frac{2}{2a}+\frac{3}{a}+\frac{4}{2a}+\&c.$

(7) The coefficient of x^n in $\frac{1}{1-2x+2x^4-2x^9+2x^{16}-\&c}$

$$=\text{the nearest integer to } \frac{1}{4n}\left\{\cosh(\pi\sqrt{n})-\frac{\sinh(\pi\sqrt{n})}{\pi\sqrt{n}}\right\}.*$$

* This is quite untrue. But the formula is extremely interesting for a variety of reasons.

IX. Theorems on continued fractions, a few examples are :—

(1) $$\frac{4}{x}+\frac{1^2}{2x}+\frac{3^2}{2x}+\frac{5^2}{2x}+\frac{7^2}{2x}+\&c = \left\{\frac{\Gamma\left(\frac{x+1}{4}\right)}{\Gamma\left(\frac{x+3}{4}\right)}\right\}^2.$$

. . .

(4) If $$u=\frac{x}{1}+\frac{x^5}{1}+\frac{x^{10}}{1}+\frac{x^{15}}{1}+\frac{x^{20}}{1}+\&c$$

and $$v=\frac{\sqrt[5]{x}}{1}+\frac{x}{1}+\frac{x^2}{1}+\frac{x^3}{1}+\&c$$

then $$v^5 = u\cdot\frac{1-2u+4u^2-3u^3+u^4}{1+3u+4u^2+2u^3+u^4}.$$

(5) $$\frac{1}{1}+\frac{e^{-2\pi}}{1}+\frac{e^{-4\pi}}{1}+\frac{e^{-6\pi}}{1}+\&c = \left(\sqrt{\frac{5+\sqrt{5}}{2}}-\frac{\sqrt{5}+1}{2}\right)\sqrt[5]{e^{2\pi}}.$$

(6) $$\frac{1}{1}-\frac{e^{-\pi}}{1}+\frac{e^{-2\pi}}{1}-\frac{e^{-3\pi}}{1}+\&c = \left(\sqrt{\frac{5-\sqrt{5}}{2}}-\frac{\sqrt{5}-1}{2}\right)\sqrt[5]{e^{\pi}}.$$

(7) $$\frac{1}{1}+\frac{e^{-\pi\sqrt{n}}}{1}+\frac{e^{-2\pi\sqrt{n}}}{1}+\frac{e^{-3\pi\sqrt{n}}}{1}+\&c$$ can be exactly found if n be any positive rational quantity. . . ."

27 February 1913

". . . I have found a friend in you who views my labours sympathetically. This is already some encouragement to me to proceed. . . . I find in many a place in your letter rigorous proofs are required and you ask me to communicate the methods of proof. . . . I told him* that the sum of an infinite no of terms of the series $1+2+3+4+\ldots = -\frac{1}{12}$ under my theory. If I tell you this you will at once point out to me the lunatic asylum as my goal. . . . What I tell you is this. Verify the results I give and if they agree with your results . . . you should at least grant that there may be some truths in my fundamental basis. . . .

To preserve my brains I want food and this is now my first consideration. Any sympathetic letter from you will be helpful to me here to get a scholarship either from the University or from Government. . . .

1. The no of prime nos. less than $e^a = \int_0^\infty \frac{a^x\,dx}{xS_{x+1}\,\Gamma(x+1)}$

where $$S_{x+1}=\frac{1}{1^{x+1}}+\frac{1}{2^{x+1}}+\ldots.$$

* Referring to a previous correspondence.

2. The no of prime nos. less than $n =$

$$\frac{2}{\pi}\left\{\frac{2}{B_2}\left(\frac{\log n}{2\pi}\right)+\frac{4}{3B_4}\left(\frac{\log n}{2\pi}\right)^3+\frac{6}{5B_6}\left(\frac{\log n}{2\pi}\right)^5+\text{etc}\right.$$

where $B_2=\frac{1}{6}$; $B_4=\frac{1}{30}$ &c, the Bernoullian nos. . . .

For practical calculations

$$\int_\mu^n \frac{dx}{\log x} = n\left(\frac{1}{\log n}+\frac{\lfloor 1}{(\log n)^2}+\ldots+\frac{\lfloor k-1}{(\log n)^k}\theta\right)$$

where $\theta=\frac{2}{3}-\delta+\frac{1}{\log n}\left\{\frac{4}{135}-\frac{\delta^2(1-\delta)}{3}\right\}$

$$+\frac{1}{(\log n)^2}\left\{\frac{8}{2835}+\frac{2\delta(1-\delta)}{135}-\frac{\delta(1-\delta^2)(2-3\delta^2)}{15}\right\}+\text{etc}$$

where $\delta = k-\log n$. . . .

The order of $\theta(x)$ which you asked in your letter is $\sqrt{\left(\frac{x}{\log x}\right)}$.

. . .

(1) If $F(x)=\cfrac{1}{1+}\cfrac{x}{1+}\cfrac{x^2}{1+}\cfrac{x^3}{1+}\cfrac{x^4}{1+}\cfrac{x^5}{1+}\&c$

then $\left\{\frac{\sqrt{5}+1}{2}+e^{-\frac{2\alpha}{5}}F(e^{-2\alpha})\right\}\left\{\frac{\sqrt{5}+1}{2}+e^{-\frac{2\beta}{5}}F(e^{-2\beta})\right\}=\frac{5+\sqrt{5}}{2}$,

with the conditions $\alpha\beta=\pi^2$. . . .

$$e.g.\ \cfrac{1}{1+}\cfrac{e^{-2\pi\sqrt{5}}}{1+}\cfrac{e^{-4\pi\sqrt{5}}}{1+}\text{etc}\ \ldots = e^{\frac{\pi}{\sqrt{5}}}\left(\frac{\sqrt{5}}{1+\sqrt[5]{5^3\left(\frac{\sqrt{5}-1}{2}\right)^3-1}}-\frac{\sqrt{5}+1}{2}\right)$$

The above theorem is a particular case of a theorem on the c.f.

$$\cfrac{1}{1+}\cfrac{ax}{1+}\cfrac{ax^2}{1+}\cfrac{ax^3}{1+}\cfrac{ax^4}{1+}\cfrac{ax^5}{1+}\&c.$$

which is a particular case of the c.f.

$$\cfrac{1}{1+}\cfrac{ax}{1+bx+}\cfrac{ax^2}{1+bx^2+}\cfrac{ax^3}{1+bx^3+}\&c$$

which is a particular case of a general theorem on c.f.

(2) i. $4\int_0^\infty \frac{xe^{-x\sqrt{5}}}{\cosh x}dx = \cfrac{1}{1+}\cfrac{1^2}{1+}\cfrac{1^2}{1+}\cfrac{2^2}{1+}\cfrac{2^2}{1+}\cfrac{3^2}{1+}\cfrac{3^2}{1+}\&c$

ii. $4\int_0^\infty \frac{x^2e^{-x\sqrt{3}}}{\cosh x}dx = \cfrac{1}{1+}\cfrac{1^3}{1+}\cfrac{1^3}{3+}\cfrac{2^3}{1+}\cfrac{2^3}{5+}\cfrac{3^3}{1+}\cfrac{3^3}{7+}\&c$

(3) $1-5.\left(\frac{1}{2}\right)^5+9.\left(\frac{1.3}{2.4}\right)^5-13.\left(\frac{1.3.5}{2.4.6}\right)^5+\&c=\frac{2}{\{\Gamma(\frac{3}{4})\}^4}$

. . .

(6) If $v = \dfrac{x}{1} + \dfrac{x^3+x^6}{1} + \dfrac{x^6+x^{12}}{1} + \dfrac{x^9+x^{18}}{1} + \&c.$

then i. $x\left(1+\dfrac{1}{v}\right) = \dfrac{1+x+x^3+x^6+x^{10}+\&c}{1+x^9+x^{27}+x^{54}+x^{90}+\&c}$

ii. $x^3\left(1+\dfrac{1}{v^3}\right) = \left(\dfrac{1+x+x^3+x^6+x^{10}+\&c}{1+x^3+x^9+x^{18}+x^{30}+\&c}\right)^4$

(7) If n is any odd integer,

$$\frac{1}{\cosh\frac{\pi}{2n}+\cos\frac{\pi}{2n}} - \frac{1}{3\left(\cosh\frac{3\pi}{2n}+\cos\frac{3\pi}{2n}\right)} + \frac{1}{5\left(\cosh\frac{5\pi}{2n}+\cos\frac{5\pi}{2n}\right)} \ldots \&c = \frac{\pi}{8}.$$

. . .

(10) If $F(\alpha, \beta, \gamma, \delta, \epsilon) = 1 + \dfrac{\alpha}{\lfloor 1} \cdot \dfrac{\beta}{\delta} \cdot \dfrac{\gamma}{\epsilon} + \dfrac{\alpha(\alpha+1)}{\lfloor 2} \cdot \dfrac{\beta(\beta+1)}{\delta(\delta+1)} \times \dfrac{\gamma(\gamma+1)}{\epsilon(\epsilon+1)} + \&c.$

then $F(\alpha, \beta, \gamma, \delta, \epsilon) = \dfrac{\Gamma(\delta)\,\Gamma(\delta-\alpha-\beta)}{\Gamma(\delta-\alpha)\,\Gamma(\delta-\beta)} \cdot F(\alpha, \beta, \epsilon-\gamma, \alpha+\beta-\delta+1, \epsilon)$

$$+ \frac{\Gamma(\delta)\,\Gamma(\epsilon)\,\Gamma(\alpha+\beta-\delta)\,\Gamma(\delta+\epsilon-\alpha-\beta-\gamma)}{\Gamma(\alpha)\,\Gamma(\beta)\,\Gamma(\epsilon-\gamma)\,\Gamma(\delta+\epsilon-\alpha-\beta)}$$

$$\times F(\delta-\alpha,\ \delta-\beta,\ \delta+\epsilon-\alpha-\beta-\gamma,\ \delta-\alpha-\beta+1,\ \delta+\epsilon-\alpha-\beta).$$

. . .

(13) $$\frac{a}{1+n} + \frac{a^2}{3+n} + \frac{(2a)^2}{5+n} + \frac{(3a)^2}{7+n} + \ldots$$

$$= 2a \int_0^1 z^{\frac{n}{\sqrt{(1+a^2)}}} \frac{dz}{\{\sqrt{(1+a^2)}+1\} + z^2\{\sqrt{(1+a^2)}-1\}}.$$

(14) If $F(\alpha, \beta) = \alpha + \dfrac{(1+\beta)^2+k}{2\alpha} + \dfrac{(3+\beta)^2+k}{2\alpha} + \dfrac{(5+\beta)^2+k}{2\alpha} + \ldots,$

then $$F(\alpha, \beta) = F(\beta, \alpha).$$

(15) If $F(\alpha, \beta) = \dfrac{a}{n} + \dfrac{\beta^2}{n} + \dfrac{(2a)^2}{n} + \dfrac{(3\beta)^2}{n} + \ldots$

then $$F(\alpha, \beta) + F(\beta, \alpha) = 2F\{\tfrac{1}{2}(\alpha+\beta), \sqrt{(\alpha\beta)}\}$$

. . .

(17) If $F(k) = 1 + \left(\dfrac{1}{2}\right)^2 k + \left(\dfrac{1.3}{2.4}\right)^2 k^2 + \ldots$ and $F(1-k) = \sqrt{(210)}\,F(k)$,

then $k = (\sqrt{2}-1)^4(2-\sqrt{3})^2(\sqrt{7}-\sqrt{6})^4(8-3\sqrt{7})^2(\sqrt{10}-3)^4(4-\sqrt{15})^4 \times (\sqrt{15}-\sqrt{14})^2(6-\sqrt{35})^2$.

. . .

(20) If $F(\alpha) = \int_0^{\frac{1}{2}\pi} \frac{d\phi}{\sqrt{\{1-(1-\alpha)\sin^2\phi\}}} \Big/ \int_0^{\frac{1}{2}\pi} \frac{d\phi}{\sqrt{\{1-\alpha\sin^2\phi\}}}$

and $F(\alpha) = 3F(\beta) = 5F(\gamma) = 15F(\delta)$,

then i. $[(\alpha\delta)^{\frac{1}{2}} + \{(1-\alpha)(1-\delta)\}^{\frac{1}{2}}][(\beta\gamma)^{\frac{1}{2}} + \{(1-\beta)(1-\gamma)\}^{\frac{1}{2}}] = 1$

. . .

v. $(\alpha\beta\gamma\delta)^{\frac{1}{2}} + \{(1-\alpha)(1-\beta)(1-\gamma)(1-\delta)\}^{\frac{1}{2}} + \{16\alpha\beta\gamma\delta(1-\alpha)(1-\beta)(1-\gamma)(1-\delta)\}^{\frac{1}{4}} = 1$

. . .

(21) If $F(\alpha) = 3F(\beta) = 13F(\gamma) = 39F(\delta)$

or $F(\alpha) = 5F(\beta) = 11F(\gamma) = 55F(\delta)$

or $F(\alpha) = 7F(\beta) = 9F(\gamma) = 63F(\delta)$

then $\frac{\{(1-\alpha)(1-\delta)\}^{\frac{1}{2}} - (\alpha\delta)^{\frac{1}{2}}}{\{(1-\beta)(1-\gamma)\}^{\frac{1}{2}} - (\beta\gamma)^{\frac{1}{2}}} = \frac{1 + \{(1-\alpha)(1-\delta)\}^{\frac{1}{2}} + (\alpha\delta)^{\frac{1}{2}}}{1 + \{(1-\beta)(1-\gamma)\}^{\frac{1}{2}} + (\beta\gamma)^{\frac{1}{2}}}$.

. . .

(23) $(1+e^{-\pi\sqrt{1353}})(1+e^{-3\pi\sqrt{1353}})(1+e^{-5\pi\sqrt{1353}})\ldots$

$= \sqrt[4]{2}\, e^{-\frac{1}{24}\pi\sqrt{1353}} \times \sqrt{\frac{1}{2}\sqrt{\left(\frac{569+99\sqrt{33}}{8}\right)} + \sqrt{\left(\frac{561+99\sqrt{33}}{8}\right)}\Big\}}$

$\times \sqrt{\frac{1}{2}\sqrt{\left(\frac{25+3\sqrt{33}}{8}\right)} + \sqrt{\left(\frac{17+3\sqrt{33}}{8}\right)}\Big\}} \times \sqrt[4]{\left(\frac{\sqrt{123}+11}{\sqrt{2}}\right)}$

$\times \sqrt[8]{(10+3\sqrt{11})} \times \sqrt[8]{(26+15\sqrt{3})} \times \sqrt[12]{\left(\frac{6817+321\sqrt{451}}{\sqrt{2}}\right)}$

. . ."

17 April 1913

". . . I am a little pained to see what you have written. . . .* I am not in the least apprehensive of my method being utilized by others. On the contrary my method has been in my possession for the last eight years and I have not found anyone to appreciate the method. As I wrote in my last letter I have found a sympathetic friend in you and I am willing to place unreservedly in your hands what little I have. It was on

* Ramanujan might very reasonably have been reluctant to give away his secrets to an English mathematician, and I had tried to reassure him on this point as well as I could.

account of the novelty of the method I have used that I am a little diffident even now to communicate my own way of arriving at the expressions I have already given. . . .

. . . I am glad to inform you that the local University has been pleased to grant me a scholarship of £60 per annum for two years and this was at the instance of Dr. Walker, F.R.S., Head of the Meteorological Department in India, to whom my thanks are due. . . . I request you to convey my thanks also to Mr. Littlewood, Dr Barnes, Mr. Berry and others who take an interest in me. . . ."

III.

It is unnecessary to repeat the story of how Ramanujan was brought to England. There were serious difficulties; and the credit for overcoming them is due primarily to Prof. E. H. Neville, in whose company Ramanujan arrived in April 1914. He had a scholarship from Madras of £250, of which £50 was allotted to the support of his family in India, and an exhibition of £60 from Trinity. For a man of his almost ludicrously simple tastes, this was an ample income; and he was able to save a good deal of money which was badly wanted later. He had no duties and could do as he pleased; he wished indeed to qualify for a Cambridge degree as a research student, but this was a formality. He was now, for the first time in his life, in a really comfortable position, and could devote himself to his researches without anxiety.

There was one great puzzle. What was to be done in the way of teaching him modern mathematics? The limitations of his knowledge were as startling as its profundity. Here was a man who could work out modular equations, and theorems of complex multiplication, to orders unheard of, whose mastery of continued fractions was, on the formal side at any rate, beyond that of any mathematician in the world, who had found for himself the functional equation of the Zeta-function, and the dominant terms of many of the most famous problems in the analytic theory of numbers; and he had never heard of a doubly periodic function or of Cauchy's theorem, and had indeed but the vaguest idea of what a function of a complex variable was. His ideas as to what constituted a mathematical proof were of the most shadowy description. All his results, new or old, right or wrong, had been arrived at by a process of mingled argument, intuition, and induction, of which he was entirely unable to give any coherent account.

It was impossible to ask such a man to submit to systematic instruction, to try to learn mathematics from the beginning once more. I was

afraid too that, if I insisted unduly on matters which Ramanujan found irksome, I might destroy his confidence or break the spell of his inspiration. On the other hand there were things of which it was impossible that he should remain in ignorance. Some of his results were wrong, and in particular those which concerned the distribution of primes, to which he attached the greatest importance. It was impossible to allow him to go through life supposing that all the zeros of the Zeta-function were real. So I had to try to teach him, and in a measure I succeeded, though obviously I learnt from him much more than he learnt from me. In a few years' time he had a very tolerable knowledge of the theory of functions and the analytic theory of numbers. He was never a mathematician of the modern school, and it was hardly desirable that he should become one; but he knew when he had proved a theorem and when he had not. And his flow of original ideas showed no symptom of abatement.

I should add a word here about Ramanujan's interests outside mathematics. Like his mathematics, they showed the strangest contrasts. He had very little interest, I should say, in literature as such, or in art, though he could tell good literature from bad. On the other hand, he was a keen philosopher, of what appeared, to followers of the modern Cambridge school, a rather nebulous kind, and an ardent politician, of a pacifist and ultra-radical type. He adhered, with a severity most unusual in Indians resident in England, to the religious observances of his caste; but his religion was a matter of observance and not of intellectual conviction, and I remember well his telling me (much to my surprise) that all religions seemed to him more or less equally true. Alike in literature, philosophy, and mathematics, he had a passion for what was unexpected, strange, and odd; he had quite a small library of books by circle-squarers and other cranks.

It was in the spring of 1917 that Ramanujan first appeared to be unwell. He went into the Nursing Home at Cambridge in the early summer, and was never out of bed for any length of time again. He was in sanatoria at Wells, at Matlock, and in London, and it was not until the autumn of 1918 that he showed any decided symptom of improvement. He had then resumed active work, stimulated perhaps by his election to the Royal Society, and some of his most beautiful theorems were discovered about this time. His election to a Trinity Fellowship was a further encouragement; and each of those famous societies may well congratulate themselves that they recognised his claims before it was too late. Early in 1919 he had recovered, it seemed, sufficiently for the voyage home to India, and the best medical opinion held out hopes of a permanent restoration. I was rather alarmed by not hearing from him for a con-

siderable time; but a letter reached me in February 1920, from which it appeared that he was still active in research.

University of Madras
12th January 1920

"I am extremely sorry for not writing you a single letter up to now. . . . I discovered very interesting functions recently which I call 'Mock' ϑ-functions. Unlike the 'False' ϑ-functions (studied partially by Prof. Rogers in his interesting paper) they enter into mathematics as beautifully as the ordinary ϑ-functions. I am sending you with this letter some examples. . . .

Mock ϑ-functions

$$\phi(q) = 1 + \frac{q}{1+q^2} + \frac{q^4}{(1+q^2)(1+q^4)} + \dots$$

$$\psi(q) = \frac{q}{1-q} + \frac{q^4}{(1-q)(1-q^3)} + \frac{q^9}{(1-q)(1-q^3)(1-q^5)} + \dots$$

. . .

Mock ϑ-functions (of 5th order)

$$f(q) = 1 + \frac{q}{1+q} + \frac{q^4}{(1+q)(1+q^2)} + \frac{q^9}{(1+q)(1+q)(1+q^3)} + \dots$$

. . .

Mock ϑ-functions (of 7th order)

(i) $$1 + \frac{q}{1-q^2} + \frac{q^4}{(1-q^3)(1-q^4)} + \frac{q^9}{(1-q^4)(1-q^5)(1-q^6)} + \dots$$

. . ."

He said little about his health, and what he said was not particularly discouraging; and I was quite unprepared for the news of his death.

IV.

Ramanujan published the following papers in Europe:—

(1) "Some definite integrals", *Messenger of Mathematics*, Vol. 44 (1914), pp. 10–18.
(2) "Some definite integrals connected with Gauss's sums", *ibid.*, pp. 75–85.
(3) "Modular equations and approximations to π", *Quarterly Journal of Mathematics*, Vol. 45 (1914), pp. 350–372.
(4) "New expressions for Riemann's functions $\zeta(s)$ and $\Xi(t)$", *ibid.*, Vol. 46 (1915) pp. 253–261.
(5) "On certain infinite series", *Messenger of Mathematics*, Vol. 45 (1915), pp. 11–15.
(6) "Summation of a certain series", *ibid.*, pp. 157–160.
(7) "Highly composite numbers", *Proc. London Math. Soc.*, Ser. 2, Vol. 14 (1915) pp. 347–409.

(8) "Some formulæ in the analytic theory of numbers", *Messenger of Mathematics*, Vol. 45 (1916), pp. 81–84.

(9) "On certain arithmetical functions", *Trans. Cambridge Phil. Soc.*, Vol. 22 (1916), No. 9, pp. 159–184.

(10) "Some series for Euler's constant", *Messenger of Mathematics*, Vol. 46 (1916), pp. 73–80.

(11) "On the expression of numbers in the form $ax^2+by^2+cz^2+dt^2$", *Proc. Cambridge Phil. Soc.*, Vol. 19 (1917), pp. 11–21.

*(12) "Une formule asymptotique pour le nombre des partitions de n", *Comptes Rendus*, 2 Jan. 1917.

*(13) "Asymptotic formulæ concerning the distribution of integers of various types", *Proc. London Math. Soc.*, Ser. 2, Vol. 16 (1917), pp. 112–132.

*(14) "The normal number of prime factors of a number n", *Quarterly Journal of Mathematics*, Vol. 48 (1917), pp. 76–92.

*(15) "Asymptotic formulæ in Combinatory Analysis", *Proc. London Math. Soc.*, Ser. 2, Vol. 17 (1918), pp. 75–115.

*(16) "On the coefficients in the expansions of certain modular functions", *Proc. Roy. Soc.*, (A), Vol. 95 (1918), pp. 144–155.

(17) "On certain trigonometrical sums and their applications in the theory of numbers", *Trans. Camb. Phil. Soc.*, Vol. 22 (1918), pp. 259–276.

(18) "Some properties of $p(n)$, the number of partitions of n", *Proc. Camb. Phil. Soc.*, Vol. 19 (1919), pp. 207–210.

(19) "Proof of certain identities in Combinatory Analysis", *ibid.*, pp. 214–216.

(20) "A class of definite integrals", *Quarterly Journal of Mathematics*, Vol. 48 (1920), pp. 294–309.

(21) "Congruence properties of partitions", *Math. Zeitschrift*, Vol. 9 (1921), pp. 147–153.

Of these those marked with an asterisk were written in collaboration with me, and (21) is a posthumous extract from a much larger unpublished manuscript in my possession.† He also published a number of short notes in the *Records of Proceedings* at our meetings, and in the *Journal of the Indian Mathematical Society*. The complete list of these is as follows :

Records of Proceedings at Meetings.

*(22) "Proof that almost all numbers n are composed of about $\log\log n$ prime factors", 14 Dec. 1916.

*(23) "Asymptotic formulæ in Combinatory Analysis", 1 March, 1917.

(24) "Some definite integrals", 17 Jan., 1918.

(25) "Congruence properties of partitions", 13 March, 1919.

(26) "Algebraic relations between certain infinite products", 13 March, 1919.

Journal of the Indian Mathematical Society.

(A) Articles and Notes.

(27) "Some properties of Bernoulli's numbers", Vol. 3 (1911), pp. 219–235.

(28) "On Q. 330 of Prof. Sanjaná", Vol. 4 (1912), pp. 59–61.

(29) "A set of equations", Vol. 4 (1912), pp. 94–96.

† All of Ramanujan's manuscripts passed through my hands, and I edited them very carefully for publication. The earlier ones I rewrote completely. I had no share of any kind in the results, except of course when I was actually a collaborator, or when explicit acknowledgment is made. Ramanujan was almost absurdly scrupulous in his desire to acknowledge the slightest help.

(30) "Irregular numbers", Vol. 5 (1913), pp. 105–107.
(31) "Squaring the circle", Vol. 5 (1913), pp. 132–133.
(32) "On the integral $\int_0^x \text{arc tan } t.\frac{dt}{t}$", Vol. 7 (1915), pp. 93–96.
(33) "On the divisors of a number", Vol. 7 (1915), pp. 131–134.
(34) "The sum of the square roots of the first n natural numbers", Vol. 7 (1915), pp. 173–175.
(35) "On the product $\pi\left[1+\frac{x^2}{(a+nd)^2}\right]$", Vol. 7 (1915), pp. 209–212.
(36) "Some definite integrals", Vol. 11 (1919), pp. 81–88.
(37) "A proof of Bertrand's postulate", Vol. 11 (1919), pp. 181–183.
(38) (Communicated by S. Narayana Aiyar), Vol. 3 (1911), p. 60.

(B) Questions proposed and solved.

Nos. 260, 261, 283, 289, 294, 295, 298, 308, 353, 358, 386, 427, 441, 464, 489, 507, 541, 546, 571, 605, 606, 629, 642, 666, 682, 700, 723, 724, 739, 740, 753, 768, 769, 783, 785.

(C) Questions proposed but not solved as yet.

Nos. 284, 327, 359, 387, 441, 463, 469, 524, 525, 526, 584, 661, 662, 681, 699, 722, 738, 754, 770, 784, 1049, 1070, and 1076.

Finally, I may mention the following writings by other authors, concerned with Ramanujan's work.

"Proof of a formula of Mr. Ramanujan", by G. H. Hardy (*Messenger of Mathematics*, Vol. 44, 1915, pp. 18–21).
"Mr. S. Ramanujan's mathematical work in England", by G. H. Hardy (Report to the University of Madras, 1916, privately printed).
"On Mr. Ramanujan's empirical expansions of modular functions", by L. J. Mordell (*Proc. Camb. Phil. Soc.*, Vol. 19, 1917, pp. 117–124).
"Life sketch of Ramanujan" (editorial in the *Journal of the Indian Math. Soc.*, Vol. 11, 1919, p. 122).
"Note on the parity of the number which enumerates the partitions of a number", by P. A. MacMahon (*Proc. Camb. Phil. Soc.*, Vol. 20, 1921, pp. 281–283).
"Proof of certain identities and congruences enunciated by S. Ramanujan", by H. B. C. Darling (*Proc. London Math. Soc.*, Ser. 2, Vol. 19, 1921, pp. 350–372).
"On a type of modular relation", by L. J. Rogers (*ibid.*, pp. 387–397).

It is plainly impossible for me, within the limits of a notice such as this, to attempt a reasoned estimate of Ramanujan's work. Some of it is very intimately connected with my own, and my verdict could not be impartial; there is much too that I am hardly competent to judge; and there is a mass of unpublished material, in part new and in part anticipated, in part proved and in part only conjectured, that still awaits analysis. But it may be useful if I state, shortly and dogmatically, what seems to me Ramanujan's finest, most independent, and most characteristic work.

His most remarkable papers appear to me to be (3), (7), (9), (17), (18), (19), and (21). The first of these is mainly Indian work, done before he came to England; and much of it had been anticipated. But there is

much that is new, and in particular a very remarkable series of algebraic approximations to π. I may mention only the formulæ

$$\pi = \frac{63}{25}\frac{17+15\sqrt{5}}{7+15\sqrt{5}}, \quad \frac{1}{2\pi\sqrt{2}} = \frac{1103}{99^2},$$

correct to 9 and 8 places of decimals respectively.

The long memoir (7) represents work, perhaps, in a backwater of mathematics, and is somewhat overloaded with detail; but the elementary analysis of "highly composite" numbers—numbers which have more divisors than any preceding number—is exceedingly remarkable, and shows very clearly Ramanujan's extraordinary mastery over the algebra of inequalities. Papers (9) and (17) should be read together, and in connection with Mr. Mordell's paper mentioned above; for Mr. Mordell afterwards proved a great deal that Ramanujan conjectured. They contain, in particular, exceedingly remarkable contributions to the theory of the representation of numbers by sums of squares. But I am inclined to think that it was in the theory of partitions, and the allied parts of the theories of elliptic functions and continued fractions, that Ramanujan shows at his very best. It is in papers (18), (19), and (21), and in the papers of Prof. Rogers and Mr. Darling that I have quoted, that this side of his work (so far as it has been published) is to be found. It would be difficult to find more beautiful formulæ than the "Rogers-Ramanujan" identities, proved in (19); but here Ramanujan must take second place to Prof. Rogers; and, if I had to select one formula from all Ramanujan's work, I would agree with Major MacMahon in selecting a formula from (18), viz.

$$p(4)+p(9)\,x+p(14)\,x^2+\ldots = 5\,\frac{\{(1-x^5)(1-x^{10})(1-x^{15})\ldots\}^5}{\{(1-x)(1-x^2)(1-x^3)\ldots\}^6},$$

where $p(n)$ is the number of partitions of n.

I have often been asked whether Ramanujan had any special secret; whether his methods differed in kind from those of other mathematicians; whether there was anything really abnormal in his mode of thought. I cannot answer these questions with any confidence or conviction; but I do not believe it. My belief is that all mathematicians think, at bottom, in the same kind of way, and that Ramanujan was no exception. He had, of course, an extraordinary memory. He could remember the idiosyncrasies of numbers in an almost uncanny way. It was Mr. Littlewood (I believe) who remarked that "every positive integer was one of his personal friends." I remember once going to see him when he was lying ill at Putney. I had ridden in taxi-cab No. 1729, and remarked that the number (7.13.19) seemed to me rather a dull one, and that I hoped it was not an unfavourable omen. "No," he replied, "it is a very interesting

number; it is the smallest number expressible as a sum of two cubes in two different ways." I asked him, naturally, whether he knew the answer to the corresponding problem for fourth powers; and he replied, after a moment's thought, that he could see no obvious example, and thought that the first such number must be very large.* His memory, and his powers of calculation, were very unusual, but they could not reasonably be called "abnormal". If he had to multiply two large numbers, he multiplied them in the ordinary way; he would do it with unusual rapidity and accuracy, but not more rapidly or more accurately than any mathematician who is naturally quick and has the habit of computation. There is a table of partitions at the end of our paper (15). This was, for the most part, calculated independently by Ramanujan and Major MacMahon; and Major MacMahon was, in general, slightly the quicker and more accurate of the two.

It was his insight into algebraical formulæ, transformations of infinite series, and so forth, that was most amazing. On this side most certainly I have never met his equal, and I can compare him only with Euler or Jacobi. He worked, far more than the majority of modern mathematicians, by induction from numerical examples; all of his congruence properties of partitions, for example, were discovered in this way. But with his memory, his patience, and his power of calculation, he combined a power of generalisation, a feeling for form, and a capacity for rapid modification of his hypotheses, that was often really startling, and made him, in his own peculiar field, without a rival in his day.

It is often said that it is much more difficult now for a mathematician to be original than it was in the great days when the foundations of modern analysis were laid; and no doubt in a measure it is true. Opinions may differ as to the importance of Ramanujan's work, the kind of standard by which it should be judged, and the influence which it is likely to have on the mathematics of the future. It has not the simplicity and the inevitableness of the very greatest work; it would be greater if it were less strange. One gift it has which no one can deny, profound and invincible originality. He would probably have been a greater mathematician if he had been caught and tamed a little in his youth; he would have discovered more that was new, and that, no doubt, of greater importance. On the other hand he would have been less of a Ramanujan, and more of a European professor, and the loss might have been greater than the gain.

G. H. H.

* Euler gave $542^4 + 103^4 = 359^4 + 514^4$ as an example. See Sir T. L. Heath's *Diophantus of Alexandria*, p. 380.

PHILIP EDWARD BERTRAND JOURDAIN.

(Born October 16th, 1879; Died October 1st, 1919.)

The death in 1919 of Philip Edward Bertrand Jourdain is a loss that will be widely felt by those who knew his work, and a cause of sincere grief to his many friends. Jourdain, in spite of severe disabilities, accomplished many things in his short life. At a very early age he showed mechanical and mathematical ability; and he went up to Trinity College, Cambridge in 1898, although he was already a cripple. His academic career shows (as is not unnatural) the strangest contrasts. He was ploughed in the Mathematical Tripos, and compelled to take a Pass Degree. He was honourably mentioned in the ensuing Smith's Prize competition, and in 1904 he was awarded the Allen studentship for research.

Apart from his own personal contributions to mathematics, Jourdain was an important figure in mathematical circles. His disinterested and efficient work in abstracting mathematical papers for the *Revue Semestrielle*, and in writing the "Recent Advances" in *Science Progress*, are examples of his labours for the advancement of mathematics. His extensive correspondence on mathematical subjects with eminent mathematicians of all nationalities shows that he was in touch with mathematical thought all over the world. The plans which he was lately elaborating for the advancement of science were to ensure the translation of all scientific papers and articles into English and French. As another example of his activities, we may refer to his attempts to republish the works of Newton. Jourdain was recognised as the leading authority on Newton, and had done a large amount of research, with a view to the publication of a new edition. This is hardly the place to describe in detail his other activities, such as his editorship of *The Monist* and the *International Journal of Ethics*, his many researches into the history of science, and his important work on Induction and Probability, which was in course of publication in *Mind*. We must, however, mention that Jourdain had in preparation a large work on *The History of Mathematical Thought*. It is quite evident that, with his intimate knowledge of the lives of the older mathematicians, his wide knowledge of foreign languages, and his keen interest in the evolution of abstract ideas, he was the ideal author of such a book.

Among the papers produced by Jourdain in his short career one of the most important is his article "On the General Theory of Functions" (*Journal für Mathematik*, Bd. 128, Heft 3). Related to this are his paper "On the Question of the Existence of Transfinite Numbers," published in the *Proceedings* of this Society (1907), and a series of articles (1909–1913) in the *Archiv der Mathematik und Physik* on "The Development of the Theory of Transfinite Numbers." These and a number of papers in the *Mathematische Annalen*, *Messenger of Mathematics*, *Quarterly Journal*, and various other periodicals, dealt with the general theory of aggregates and relations. Jourdain was also one of the large number of people who have attempted to prove "the axiom of Zermelo" or multiplicative axiom, so notorious in mathematical logic and the general theory of aggregates.

An example of the other side of his work is to be found in an early article in *The Monist*, entitled "On some Points in the Foundations of Mathematical Physics" (1908). Jourdain used his results stated in his article "On the General Theory of Functions" to attack the problems of causality in physics. This was the first of a series of papers in which Jourdain applied the conceptions of modern logic to mathematical physics. Other papers by Jourdain include a paper "On those Principles of Mechanics which depend upon Processes of Variation" (*Math. Ann.*, 1908), and two articles on "The Influence of Fourier's Theory on the Conduction of Heat on the Development of Pure Mathematics" (*Scientia*, 1917). He was also the author of two separate publications, "The Nature of Mathematics" (1912) and "The Principle of Least Action" (1913).

Jourdain's work lay in regions still unfamiliar to many mathematicians, and still distracted by controversy, and opinions will differ as to the permanent value of his accomplishment. There can be no difference of opinion as to the value of a life lived with such invincible courage and inspired by so disinterested a devotion to mathematical science.

D. M. W.

ERRATA.

P. 177, l. 24, "The Principle of Huygens":—Sir George Stokes did verify that there would be no backward propagation; see § 34 of his memoir. J. L.

P. 384, "The Theory of a Thin Elastic Plate":—In the expression (8), when $u < u_0$, the sign of the whole expression must be changed, in addition to the change in λ. At the end of § 9 similar changes must be made in those elements of the integral for which $t < y$. A. C. D.

PAPERS

PUBLISHED IN THE

PROCEEDINGS OF THE LONDON MATHEMATICAL SOCIETY

ON LAMBERT'S SERIES

By K. Ananda-Rau.

[Received April 1st, 1919.—Read April 24th, 1919.]

1. *Introduction.*

Let the radius of convergence of the power series

$$F(x) = \sum_1^\infty \frac{a_n x^n}{n}$$

be unity, so that the Lambert's series

$$\psi(x) = \sum_1^\infty \frac{a_n x^n}{1-x^n}$$

is also convergent for $|x| < 1$. The investigation contained in this paper is concerned with the asymptotic behaviour of $\psi(x)$ as $x \to 1$ through real values less than 1, and its relation with the behaviour of $F(x)$ as $x \to 1$ on the one hand, and on the other with the asymptotic behaviour of the partial sum

$$A_n = \frac{a_1}{1} + \frac{a_2}{2} + \ldots + \frac{a_n}{n}$$

as $n \to \infty$. I consider the first of these questions in § 2, and the second in §§ 3 and 4. Confining ourselves for the moment to the former, the problem presents itself from two standpoints. We may either postulate the behaviour of $F(x)$ and deduce that of $\psi(x)$: or, we may do the

converse. The former aspect, which is the simpler of the two, is adopted in this paper. If, for instance, we suppose that

$$\lim_{x\to 1} F(x) = l,$$

then it is proved that under certain restrictions, which $F(x)$ and its derivative $f(x)$ must satisfy,

$$\psi(x) \sim \frac{l}{1-x}; \tag{1}$$

in other words, if we agree to say that, when (1) is true, the series $\Sigma \frac{a_n}{n}$ is summable (L), then summability (A) implies under certain restrictions summability (L). This result is given in Theorem 2.2. In Theorem 2.1, instead of taking as hypothesis summability (A) of the series $\Sigma \frac{a_n}{n}$, I suppose that

$$f(x) \sim \phi(x),$$

where $\phi(x)$ is a positive increasing function, which satisfies certain conditions, and deduce that

$$(1-x)\psi(x) \sim \int^x \phi(t)\,dt.$$

The proofs of these two theorems are so similar that I have considered it sufficient to give a full proof of Theorem 2.1 alone.

I may perhaps say a word here about the nature of the question of § 2 considered from the converse standpoint: that is, postulating the behaviour of $\psi(x)$, and deducing that of $F(x)$. The problem is then very much more difficult; Mr. Hardy has kindly pointed out to me its close connection with the Prime Number Theory. He also tells me that he and Mr. Littlewood have proved (on assuming the Prime Number Theorem) that, *if a series is summable* (L), *then it is also summable* (A). I understand that their paper on the subject is to be published in this volume of the *Proceedings*.

§ 3 contains an Abelian result connecting the relation between the behaviour of A_n and that of $\psi(x)$. Theorem 3 proved in § 3 is the analogue of the following theorem on power series due to Lasker and Pringsheim*:

Theorem 1.—*Let the power series*

$$f(x) = \sum_0^\infty c_n x^n$$

* Lasker, *Phil. Trans. Roy. Soc.*, (A), Vol. 196 (1901), p. 444; Pringsheim, *Acta Mathematica*, Vol. 28 (1904), p. 29.

be convergent for $|x|<1$, *and let*

$$s_n = c_0+c_1+\ldots+c_n,$$

$$L(u) = (\log u)^{\alpha_1}(\log\log u)^{\alpha_2}\ldots(l_q u)^{\alpha_q},$$

where the α*'s are real. Suppose further that, as* $n\to\infty$,

$$s_n \sim An^{\alpha}L(n),$$

where $A\neq 0$, *the indices* α, α_1, α_2, ..., α_q, *being such that* $n^{\alpha}L(n)$ *tends to a positive limit or to infinity. Then*

$$f(x) \sim \frac{A\Gamma(\alpha+1)}{(1-x)^{\alpha}}L\left(\frac{1}{1-x}\right),$$

as $x\to 1$.

The proof of Theorem 3 depends partly on an application of the above theorem, and partly on the results proved in § 2.

§ 4 contains a simple Tauberian theorem of the *special* or *o*-type; namely, that, *if* $a_n\to 0$, *and*

$$\psi(x) \sim \frac{s}{1-x},$$

then $\Sigma\frac{a_n}{n}$ *converges to the sum* s.* The proof follows from a straightforward modification of Tauber's proof of the converse of Abel's theorem.

2. *Relation between the behaviours of* $\psi(x)$ *and* $f(x)$.

I shall begin this section by proving a few preliminary lemmas.

LEMMA 2.1.—*Suppose that the power series*

$$f(x) = \sum_1^{\infty} a_n x^n$$

is convergent for $|x|<1$, *so that*

$$\psi(x) = \sum_1^{\infty}\frac{a_n x^n}{1-x^n}$$

* This theorem was previously known to Messrs. Hardy and Littlewood, and I am indebted to them for their permission to have it published here.

is also convergent for $|x|<1$. *Then, whatever positive integer* r *may be,*

$$\psi(x)=\sum_{n=1}^{\infty}\frac{a_n x^{(r+1)n}}{1-x^n}+\sum_{n=1}^{r}f(x^n),$$

and

$$\psi(x)=\sum_{n=1}^{\infty}f(x^n).$$

We have, for $|x|<1$,

$$\Sigma\frac{a_n x^n}{1-x^n}-\Sigma\frac{a_n x^{2n}}{1-x^n}=\Sigma a_n x^n,$$

$$\Sigma\frac{a_n x^{2n}}{1-x^n}-\Sigma\frac{a_n x^{3n}}{1-x^n}=\Sigma a_n x^{2n},$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots$$

$$\Sigma\frac{a_n x^{rn}}{1-x^n}-\Sigma\frac{a_n x^{(r+1)n}}{1-x^n}=\Sigma a_n x^{rn}.$$

Adding these equations, we get

$$\psi(x)-\Sigma\frac{a_n x^{(r+1)n}}{1-x^n}=\sum_{n=1}^{r}f(x^n),$$

which is equivalent to the first result to be proved.

Next, since $|x|<1$,

$$|1-x^n|\geqslant 1-|x|^n\geqslant 1-|x|,$$

and so

$$\left|\Sigma\frac{a_n x^{(r+1)n}}{1-x^n}\right|\leqslant\frac{1}{1-|x|}\Sigma|a_n x^{(r+1)n}|\to 0,$$

as $r\to\infty$.

Therefore,*

$$\psi(x)=\sum_{n=1}^{\infty}f(x^n).$$

LEMMA 2.2. — *Suppose that* $\phi(t)$ *is a positive increasing function defined for* $0<x_0\leqslant t\leqslant x<1$, *and that* m *is the positive integer such that*

$$x^{m+1}<x_0\leqslant x^m.$$

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* This result has been known for a long time. The usual proof given is by transforming the series for $\psi(x)$ into a double series and then rearranging the terms. See, for example, Knopp, "Über Lambertsche Reihen," *Journal für Math.*, Vol. 142 (1913), pp. 283–315.

Then $$\int_{x_0}^{x} \phi(t)\,dt-(1-x)\sum_{1}^{m} x^{n+1}\phi(x^n) \leqslant (1-x)^2 \sum_{1}^{m} x^n\phi(x^n),$$

and $$(1-x)\sum_{1}^{m} x^n\phi(x^n) < (1-x)\,\phi(x)+\int_{x_0}^{x} \phi(t)\,dt.$$

We have $$\int_{x^2}^{x} \phi(t)\,dt \leqslant (x-x^2)\,\phi(x) = x\,(1-x)\,\phi(x),$$

$$\int_{x^3}^{x^2} \phi(t)\,dt \leqslant (x^2-x^3)\phi(x^2) = x^2(1-x)\,\phi(x^2),$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$

$$\int_{x_0}^{x^m} \phi(t)\,dt \leqslant (x^m-x_0)\,\phi(x^m) < x^m(1-x)\,\phi(x^m).$$

Adding, we get $$\int_{x_0}^{x} \phi(t)\,dt \leqslant (1-x)\sum_{1}^{m} x^n\phi(x^n);$$

and so $$\int_{x_0}^{x} \phi(t)\,dt-(1-x)\sum_{1}^{m} x^{n+1}\phi(x^n)$$

$$\leqslant (1-x)\sum_{1}^{m} x^n\phi(x^n)-(1-x)\sum_{1}^{m} x^{n+1}\phi(x^n)$$

$$= (1-x)^2\sum_{1}^{m} x^n\phi(x^n),$$

which is the first result.

Next, $$\int_{x^2}^{x} \phi(t)\,dt \geqslant (x-x^2)\,\phi(x^2) = x(1-x)\,\phi(x^2),$$

$$\int_{x^3}^{x^2} \phi(t)\,dt \geqslant (x^2-x^3)\phi(x^3) = x^2(1-x)\phi(x^3),$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$

$$\int_{x^m}^{x^{m-1}} \phi(t)\,dt \geqslant (x^{m-1}-x^m)\,\phi(x^m) = x^{m-1}(1-x)\phi(x^m).$$

Adding, we obtain

$$(1-x)\sum_{2}^{m} x^{n-1}\phi(x^n) \leqslant \int_{x^m}^{x} \phi(t)\,dt \leqslant \int_{x_0}^{x} \phi(t)\,dt$$

and so $$(1-x)\sum_1^m x^n\phi(x^n) < (1-x)\sum^m x^{n-1}\phi(x^n)$$

$$= (1-x)\phi(x)+(1-x)\sum_2^m x^{n-1}\phi(x^n)$$

$$\leqslant (1-x)\phi(x)+\int_{x_0}^x \phi(t)\,dt,$$

which is the second result of the lemma.

We are now in a position to prove the following theorem :—

THEOREM 2.1.—*Suppose that $f(x)$ and $\psi(x)$ are as defined in Lemma* 2.1, *and that*

$$f(x) \sim \phi(x),$$

as $x \to 1$, $\phi(x)$ being a positive increasing function satisfying the following conditions :

$$\text{(i)} \lim_{x\to 1} \int^x \phi(t)\,dt = \infty,$$

$$\text{(ii)} \quad (1-x)\phi(x) = o\int^x \phi(t)\,dt.$$

Then $$(1-x)\psi(x) \sim \int^x \phi(t)\,dt.$$

Let $\phi(t)$ be defined for $1 > t \geqslant a > 0$. Let

$$\int_a^x \phi(t)\,dt = \Phi(x).$$

Since $$f(x) \sim \phi(x),$$

it follows that $$\frac{f(x)}{x} \sim \phi(x),$$

and $$\int_0^x \frac{f(t)}{t}\,dt \sim \Phi(x).$$

Let $\epsilon > 0$. There is an $x_0 = x_0(\epsilon) > a$, such that, for $1 > x \geqslant x_0$,

(2.11) $$(1-x)\phi(x) < \epsilon\Phi(x),$$

(2.12) $$\left|\frac{f(x)}{x} - \phi(x)\right| < \epsilon\phi(x),$$

and

(2.13) $$\left|\int_0^x \frac{f(t)}{t}\,dt - \Phi(x)\right| < \epsilon\Phi(x).$$

Having fixed x_0 we can find $h > 0$, such that

$$h < \text{Min}\,(1-x_0,\ \epsilon), \tag{2.21}$$

$$\left| \frac{f(x_0)}{x_0}\, h \right| < \epsilon, \tag{2.22}$$

and that for every sequence

$$x_0 > x_1 > x_2 \ldots > x_n > \ldots \quad (x_n \to 0),$$

which satisfies the conditions

$$x_r - x_{r+1} < h \quad (r = 0,\ 1,\ 2,\ \ldots),$$

the inequality

$$\left| \int_0^{x_0} \frac{f(t)}{t}\, dt - \sum_0^\infty \frac{f(x_r)}{x_r}\,(x_r - x_{r+1}) \right| < \epsilon \tag{2.23}$$

is also satisfied, provided, of course, the series on the left is convergent.

Now, take x so that $0 < 1-x < h$, and *a fortiori* $x > x_0$ from (2.21). Let m be the positive integer such that

$$x^{m+1} < x_0 \leqslant x^m.$$

Take $\quad x_1 = x^{m+1},\ \ x_2 = x^{m+2},\ \ \ldots,\ \ x_r = x^{m+r},\ \ \ldots.$

Then $\quad x_0 - x_1 \leqslant x^m - x^{m+1} = x^m(1-x) < h;$

and for $r = 1,\ 2,\ \ldots,$

$$x_r - x_{r+1} = x^{m+r}(1-x) < h.$$

We have

$$\sum_0^\infty \frac{f(x_r)}{x_r}\,(x_r - x_{r+1}) = \frac{f(x_0)}{x_0}\,(x_0 - x_1) + (1-x) \sum_{r=1}^\infty \frac{f(x^{m+r})}{x^{m+r}}\, x^{m+r}$$

$$= \frac{f(x_0)}{x_0}\,(x_0 - x_1) + (1-x) \sum_{n=1}^\infty f(x^n) - (1-x) \sum_{n=1}^m f(x^n);$$

so that, by Lemma 2.1,

$$(1-x)\,\psi(x) = \sum_0^\infty \frac{f(x_r)}{x_r}\,(x_r - x_{r+1}) + (1-x) \sum_{n=1}^m f(x^n) - \frac{f(x_0)}{x_0}\,(x_0 - x_1).$$

Hence

$$(2.3)\quad |(1-x)\psi(x)-\Phi(x)|$$

$$= \left| \int_0^x \frac{f(t)}{t}\,dt - \Phi(x) - \int_{x_0}^x \frac{f(t)}{t}\,dt - \int_0^{x_0} \frac{f(t)}{t}\,dt \right.$$

$$\left. + \sum_0^\infty \frac{f(x_r)}{x_r}(x_r - x_{r+1}) + (1-x)\sum_1^m f(x^n) - \frac{f(x_0)}{x_0}(x_0 - x_1) \right|$$

$$\leqslant \left| \int_0^x \frac{f(t)}{t}\,dt - \Phi(x) \right| + \left| \int_0^{x_0} \frac{f(t)}{t}\,dt - \sum_0^\infty \frac{f(x_r)}{x_r}(x_r - x_{r+1}) \right|$$

$$+ \left| \frac{f(x_0)}{x_0}\,h \right| + \left| \int_{x_0}^x \frac{f(t)}{t}\,dt - (1-x)\sum_1^m f(x^n) \right|$$

$$< \epsilon\Phi(x) + \epsilon + \epsilon + \left| \int_{x_0}^x \frac{f(t)}{t}\,dt - (1-x)\sum_1^m f(x^n) \right|,$$

by using (2.13), (2.23), and (2.22). It now remains to estimate the last term in the above inequality. It is not greater than

$$(2.4)\quad \left| \int_{x_0}^x \phi(t)\,dt - (1-x)\sum_1^m x^{n+1}\phi(x^n) \right| + \int_{x_0}^x \left| \frac{f(t)}{t} - \phi(t) \right| dt$$

$$+ (1-x)\sum_1^m |x^{n+1}\phi(x^n) - f(x^n)|.$$

In the first place, it follows from Lemma 2.2 that

$$\int_{x_0}^x \phi(t)\,dt - (1-x)\sum_1^m x^{n+1}\phi(x^n) < (1-x)^2 \sum_1^m x^n \phi(x^n)$$

$$< \epsilon[(1-x)\,\phi(x) + \Phi(x)]$$

$$< \epsilon(1+\epsilon)\,\Phi(x).$$

Also, by the same lemma,

$$(1-x)\sum_1^m x^{n+1}\phi(x^n) - \int_{x_0}^x \phi(t)\,dt < (1-x)\sum_1^m x^n \phi(x^n) - \int_{x_0}^x \phi(t)\,dt$$

$$< (1-x)\,\phi(x)$$

$$< \epsilon\Phi(x).$$

Hence

$$(2.51)\qquad \left| \int_{x_0}^x \phi(t)\,dt - (1-x)\sum_1^m x^{n+1}\phi(x^n) \right| = o[\Phi(x)].$$

In the second place,

$$\int_{x_0}^{x} \left| \frac{f(t)}{t} - \phi(t) \right| dt < \epsilon\Phi(x). \tag{2.52}$$

Lastly,

$$\begin{aligned}(1-x)\sum_1^m |\, x^{n+1}\phi(x^n) - f(x^n)\,| \tag{2.53}\\ \leqslant (1-x)\sum_1^m |\, x^n\phi(x^n) - f(x^n)\,| + (1-x)^2 \sum_1^m x^n\phi(x^n)\\ < \epsilon(1-x)\sum_1^m x^n\phi(x^n) + (1-x)^2\sum_1^m x^n\phi(x^n)\\ = o[\Phi(x)],\end{aligned}$$

as above. Combining (2.3), (2.4), (2.51), (2.52), and (2.53), we see that the theorem is established.

By an easy modification of the proof of the above theorem we can establish the following:

THEOREM 2.2.—*Suppose that $f(x)$ and $\psi(x)$ have the same meaning as above, and that*

$$\lim_{x\to 1} \Sigma \frac{a_n x^n}{n}$$

exists. Further, as $x \to 1$, let

$$f(x) = O[\phi(x)],$$

where $\phi(x)$ is a positive increasing function satisfying the following conditions:

(i) $\int^1 \phi(t)\,dt$ *is convergent,**

(ii) $(1-x)\,\phi(x) \to 0$, *as* $x \to 1$.

Then

$$\lim_{x\to} (1-x)\,\Sigma \frac{a_n x^n}{1-x^n} = \lim_{x\to 1} \Sigma \frac{a_n x^n}{n}.$$

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* This condition combined with $f(x) = O[\phi(x)]$ implies the existence of

$$\lim_{x\to 1} \Sigma \frac{a_n x^n}{n}$$

3. *An Abelian Theorem.*

This section contains the analogue for Lambert's series of the theorem of Lasker and Pringsheim on Power Series (Theorem 1 of the Introduction). We shall require a few lemmas to start with.

LEMMA 3 . 1.—*Suppose that* $\sum_1^\infty \frac{a_n x^n}{1-x^n}$ *is convergent for* $|x|<1$, *and that, as* $n\to\infty$,

$$\frac{a_1}{1}+\frac{a_2}{2}+\ldots+\frac{a_n}{n}\sim\frac{b_1}{1}+\frac{b_2}{2}+\ldots+\frac{b_n}{n},$$

where the b's are positive and $\Sigma\frac{b_n}{n}$ *is divergent. Then, as* $x\to 1$,

$$\Sigma\frac{a_n x^n}{1-x^n}\sim\Sigma\frac{b_n x^n}{1-x^n}.$$

The proof of this lemma is so similar to that of the corresponding result for power series, that it is hardly necessary to write it out at length. We have only to observe that we may put

$$(1-x)\psi(x)=(1-x)\Sigma\frac{a_n x^n}{1-x^n}=\Sigma\frac{a_n}{n}v_n(x),$$

and

$$(1-x)\chi(x)=(1-x)\Sigma\frac{b_n x^n}{1-x^n}=\Sigma\frac{b_n}{n}v_n(x),$$

where

$$v_n(x)=(1-x)\frac{nx^n}{1-x^n}$$

has the following properties:

(i) for a fixed n, $\lim_{x\to 1} v_n(x)=1$;

(ii) for a fixed x in the interval (0, 1) the sequence $\{v_n(x)\}$ is a decreasing one.

This last follows from the fact that, for $0\leqslant x\leqslant 1$,

$$v_n(x)-v_{n+1}(x)=\frac{(1-x)^2x^n}{(1-x^n)(1-x^{n+1})}(n-1-x-x^2-\ldots-x^{n-1})\geqslant 0.$$

A straightforward application of Abel's Partial Summation Lemma on

lines employed in the proof of the corresponding result for power series gives the result required.*

LEMMA 3.2.—*Let* $\eta > 0$, $\lambda > 1-\eta > 0$, *and let*

$$\omega(x) = \frac{1}{x} \exp\left[-\tfrac{1}{2}(1-x)^{1-\eta}\right],$$

where $0 < x < 1$. *Then*

$$\lim_{x \to 1} \frac{(1-x)^{\lambda}}{1-\omega} = 0.$$

We have

$$1-\omega = 1-\frac{1}{x}\left[1-\tfrac{1}{2}(1-x)^{1-\eta}+\tfrac{1}{4}(1-x)^{2-2\eta}-\ldots\right]$$

$$= (1-x)^{1-\eta}\left[-\frac{1}{x}(1-x)^{\eta}+\frac{1}{2x}-\frac{1}{4x}(1-x)^{1-\eta}\ldots\right] \sim \tfrac{1}{2}(1-x)^{1-\eta}.$$

Hence $$\frac{(1-x)^{\lambda}}{1-\omega} \sim 2(1-x)^{\lambda-1+\eta} \to 0,$$

since $$\lambda > 1-\eta.$$

LEMMA 3.3.—*Let*

$$L(u) = (\log u)^{a_1}(\log\log u)^{a_2} \ldots (l_q u)^{a_q},$$

where the a's are any real numbers. Let

$$0 < x < 1, \quad 0 < \eta < 1, \quad 1 \leqslant p < \frac{1}{(1-x)^{\eta}}, \quad l_q\left(\frac{1}{1-x^p}\right) > 1.$$

Further, let $\epsilon_1 = \dfrac{\eta}{1-\eta}$, *and, for* $m = 2, 3, \ldots, q$, *let*

$$\epsilon_m = \log(1+\epsilon_{m-1}).$$

Then $$L\left(\frac{1}{1-x}\right) < L\left(\frac{1}{1-x^p}\right) \prod_{m=1}^{q} (1+\epsilon_m)^{|a_m|},$$

and $$L\left(\frac{1}{1-x^p}\right) < L\left(\frac{1}{1-x}\right) \prod_{m=1}^{q} (1+\epsilon_m)^{|a_m|}.$$

Since $$l_q\left(\frac{1}{1-x^p}\right) > 1,$$

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* For the details of the proof in the case of power series, see Bromwich, *Infinite Series*, pp. 131, 132. See, also, Example 15, p. 171, of the same book.

it follows that, for $m \leqslant q$,

$$l_m\left(\frac{1}{1-x^p}\right) > 1.$$

Now
$$\frac{1-x^p}{1-x} \leqslant p < \frac{1}{(1-x)^\eta},$$

$$\frac{1}{1-x^p} > \frac{1}{(1-x)^{1-\eta}},$$

$$\left(\frac{1}{1-x^p}\right)^{1+\epsilon_1} > \left(\frac{1}{1-x}\right)^{(1-\eta)(1+\epsilon_1)} = \frac{1}{1-x};$$

so that
$$l_1\left(\frac{1}{1-x}\right) < (1+\epsilon_1)\, l_1\left(\frac{1}{1-x^p}\right).$$

Next,
$$l_2\left(\frac{1}{1-x}\right) - l_2\left(\frac{1}{1-x^p}\right) = \log \frac{l_1\left(\frac{1}{1-x}\right)}{l_1\left(\frac{1}{1-x^p}\right)} < \log(1+\epsilon_1)$$

$$= \epsilon_2 < \epsilon_2\, l_2\left(\frac{1}{1-x^p}\right),$$

from which it follows that

$$l_2\left(\frac{1}{1-x}\right) < (1+\epsilon_2)\, l_2\left(\frac{1}{1-x^p}\right).$$

By repeating the argument, it is clear that we can prove that, for $m = 1, 2, \ldots, q$,

(3.01)
$$l_m\left(\frac{1}{1-x}\right) < (1+\epsilon_m)\, l_m\left(\frac{1}{1-x^p}\right).$$

Now, if α_m is positive, we obtain from the above inequality

$$\left[l_m\left(\frac{1}{1-x}\right)\right]^{\alpha_m} < (1+\epsilon_m)^{\alpha_m}\left[l_m\left(\frac{1}{1-x^p}\right)\right]^{\alpha_m},$$

On the other hand, if α_m is negative, we have, since

$$\frac{1}{1-x} > \frac{1}{1-x^p},$$

(3.02)
$$l_m\left(\frac{1}{1-x}\right) > l_m\left(\frac{1}{1-x^p}\right),$$

$$\left[l_m\left(\frac{1}{1-x}\right)\right]^{\alpha_m} < \left[l_m\left(\frac{1}{1-x^p}\right)\right]^{\alpha_m} < (1+\epsilon_m)^{|\alpha_m|}\left[l_m\left(\frac{1}{1-x^p}\right)\right]^{\alpha_m};$$

so that, in any case, we have for $m = 1, 2, \ldots, q$,

$$\left[l_m\left(\frac{1}{1-x}\right)\right]^{\alpha_m} < (1+\epsilon_m)^{|\alpha_m|}\left[l_m\left(\frac{1}{1-x^p}\right)\right]^{\alpha_m}.$$

Combining these q inequalities we get the first result of the lemma. To get the second result we proceed similarly, using in this case (3 . 01) when $\alpha_m < 0$, and (3 . 02) when $\alpha_m \geqslant 0$, and then combining the q inequalities so obtained.

We can now prove the following theorem :—

THEOREM 3.—*Let*

$$A_n = \frac{a_1}{1} + \frac{a_2}{2} + \ldots + \frac{a_n}{n},$$

and

$$L(u) = (\log u)^{\alpha_1} (\log\log u)^{\alpha_2} \ldots (l_q u)^{\alpha_q},$$

where the α's are real. Suppose that, as $n \to \infty$,

$$A_n \sim A n^\alpha L(n),$$

*where $A \neq 0$, and the indices $\alpha, \alpha_1, \alpha_2, \ldots, \alpha_q$, are such that $n^\alpha L(n)$ tends to a positive limit or to infinity. Then**

$$\psi(x) = \sum_1^\infty \frac{a_n x^n}{1-x^n} \sim \frac{A\alpha\Gamma(\alpha+1)\,\zeta(\alpha+1)}{(1-x)^{\alpha+1}} L\left(\frac{1}{1-x}\right),$$

as $x \to 1$.

The proof of the theorem is divided into two parts, according as α is positive or zero. We shall first consider the case when $\alpha > 0$.

In consequence of Lemma 3 . 1, we may suppose, without loss of generality, that

$$\frac{a_n}{n} = A\left[n^\alpha L(n) - (n-1)^\alpha L(n-1)\right],$$

supposing that $a_n = 0$ for those values of n at the beginning for which $L(n-1)$ is not positive. We shall also take, for simplicity, $A = 1$.

Now $a_n = \left[n^{\alpha+1} L(n) - (n-1)^{\alpha+1} L(n-1)\right] - (n-1)^\alpha L(n-1)$,

and so $a_1 + a_2 + \ldots + a_n \sim n^{\alpha+1} L(n) - \dfrac{n^{\alpha+1}}{\alpha+1} L(n) = \dfrac{\alpha}{\alpha+1} n^{\alpha+1} L(n)$.

---

* When $\alpha = 0$, we must naturally replace $\alpha\zeta(\alpha+1)$ by unity.

Hence, by the theorem of Lasker-Pringsheim,*

$$f(x) = \Sigma a_n x^n \sim \frac{a\Gamma(a+1)}{(1-x)^{a+1}} L\left(\frac{1}{1-x}\right). \tag{3.11}$$

Let $\Delta > 0$ be such that, for $0 < 1-y < \Delta$,

$$l_q\left(\frac{1}{1-y}\right) > 1. \tag{3.12}$$

Let $\epsilon > 0$. There exists a $\delta_1 < \Delta$, such that, for $0 < 1-y < \delta_1$,

$$\left| f(y) - \frac{a\Gamma(a+1)}{(1-y)^{a+1}} L\left(\frac{1}{1-y}\right) \right| < \frac{\epsilon}{(1-y)^{a+1}} L\left(\frac{1}{1-y}\right).$$

Therefore, whatever positive integer p may be,

$$\left| f(x^p) - \frac{a\Gamma(a+1)}{(1-x^p)^{a+1}} L\left(\frac{1}{1-x^p}\right) \right| < \frac{\epsilon}{(1-x^p)^{a+1}} L\left(\frac{1}{1-x^p}\right), \tag{3.13}$$

provided only $1-x^p < \delta_1$.

Let $\delta_2 > 0$ be defined by

$$\frac{1}{(1-\delta_2)^{a+1}} = 1+\epsilon. \tag{3.14}$$

Let $\delta = \text{Min}\,(\delta_1, \delta_2, \tfrac{1}{2})$.

Now, if $\epsilon_1 > 0$, and if, for $m = 2, 3, \ldots, q$,

$$\log(1+\epsilon_{m-1}) = \epsilon_m,$$

then clearly $$\prod_{m=1}^{q} (1+\epsilon_m)^{|a_m|} \to 1,$$

as $\epsilon_1 \to 0$; hence, we can take ϵ_1 so small that

$$\prod_{1}^{q} (1+\epsilon_m)^{|a_m|} < 1+\epsilon. \tag{3.21}$$

Having fixed ϵ_1 let us take $\eta = \dfrac{\epsilon_1}{1+\epsilon_1}$.

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* Quoted in the Introduction (Theorem 1).

Let $r = r(x)$ be the positive integer defined by

$$(3.22) \qquad \begin{cases} r < \dfrac{1}{(1-x)^\eta}, \\ r+1 \geqslant \dfrac{1}{(1-x)^\eta}. \end{cases}$$

It is obvious that $r(x) \to \infty$ as $x \to 1$. Hence we can find a positive $X_1 < 1$, so that, for $1 > x > X_1$,

$$(3.23) \qquad \zeta(a+1) - \left(\frac{1}{1^{a+1}} + \frac{1}{2^{a+1}} + \ldots + \frac{1}{r^{a+1}}\right) < \epsilon.$$

Let $\tau > 0$ be so small that

$$\lambda = \frac{a}{a+\tau} > 1-\eta.$$

Then, by Lemma 3.2, we can find a positive $X_2 < 1$, such that, for $1 > x > X_2$,

$$(3.24) \qquad \frac{(1-x)^a}{(1-\omega)^{a+\tau}} = \left[\frac{(1-x)^\lambda}{1-\omega}\right]^{a+\tau} < \epsilon,$$

$\omega = \omega(x)$ having the same meaning as in the lemma.

Next, since $\eta < 1$,

$$\frac{1}{(1-x)^\eta} \log \frac{1}{x}\Big) \sim (1-x)^{1-\eta} \to 0,$$

and so,

$$x^{1/(1-x)^\eta} \to 1.$$

We can therefore find X_3 so that for $x > X_3$,

$$x^{1/(1-x)^\eta} > 1-\delta;$$

and remembering (3.22) we shall then have

$$(3.25) \qquad x^r > x^{1/(1-x)^\eta} > 1-\delta.$$

Lastly, since

$$\log\left(\frac{1}{x}\right) \sim (1-x),$$

we can find X_4 so that, for $1 > x > X_4$,

$$(3.26) \qquad \log\left(\frac{1}{x}\right) > \tfrac{1}{2}(1-x).$$

Let

$$X = \text{Max}\,(X_1, X_2, X_3, X_4).$$

We shall now show that, for $1 > x > X$, and $p = 1, 2, \ldots, r$,

$$(3.3)\quad \left| f(x^p) - \frac{a\Gamma(a+1)}{p^{a+1}(1-x)^{a+1}} L\left(\frac{1}{1-x}\right) \right| < \frac{K\epsilon}{p^{a+1}(1-x)^{a+1}} L\left(\frac{1}{1-x}\right),$$

K being here, as elsewhere, a positive constant depending only on a.

In the first place, since

$$x > X \geqslant X_3,$$

we have $$x^p \geqslant x^r > 1-\delta \geqslant 1-\delta_1;$$

and it follows from (3.13) that*

$$(3.31)\quad \left| f(x^p) - \frac{a\Gamma(a+1)}{(1-x^p)^{a+1}} L\left(\frac{1}{1-x^p}\right) \right| < \frac{\epsilon}{(1-x^p)^{a+1}} L\left(\frac{1}{1-x^p}\right)$$

$$< \frac{\epsilon 2^{a+1}(1+\epsilon)}{p^{a+1}(1-x)^{a+1}} L\left(\frac{1}{1-x}\right),$$

since $(1-x^p) \geqslant px^p(1-x) \geqslant px^r(1-x) > p(1-\delta)(1-x) \geqslant \frac{p}{2}(1-x)$,

and

$$(3.32)\quad L\left(\frac{1}{1-x^p}\right) \leqslant L\left(\frac{1}{1-x}\right)(1+\epsilon).$$

In the second place, since

$$1-x^p \geqslant px^p(1-x),$$

and $$x^p \geqslant x^r > 1-\delta \geqslant 1-\delta_2,$$

$$(3.33)\quad \frac{1}{(1-x^p)^{a+1}} - \frac{1}{p^{a+1}(1-x)^{a+1}} < \frac{1}{p^{a+1}(1-x)^{a+1}}\left[\frac{1}{x^{p(a+1)}} - 1\right]$$

$$< \frac{1}{p^{a+1}(1-x)^{a+1}}\left[\frac{1}{(1-\delta_2)^{a+1}} - 1\right]$$

$$= \frac{\epsilon}{p^{a+1}(1-x)^{a+1}}.$$

Therefore $$\frac{1}{(1-x^p)^{a+1}} < \frac{1}{p^{a+1}(1-x)^{a+1}} + \frac{\epsilon}{p^{a+1}(1-x)^{a+1}},$$

so that, from (3.32) and (3.33),

$$(3.34)\quad \frac{1}{(1-x^p)^{a+1}} L\left(\frac{1}{1-x^p}\right) - \frac{1}{p^{a+1}(1-x)^{a+1}} L\left(\frac{1}{1-x}\right)$$

$$< \frac{K\epsilon}{p^{a+1}(1-x)^{a+1}} L\left(\frac{1}{1-x}\right).$$

* By Lemma 3.3 (whose conditions are clearly satisfied) and from the manner in which ϵ_1 has been chosen.

Further, by Lemma 3 . 3, we have

$$L\left(\frac{1}{1-x}\right) < L\left(\frac{1}{1-x^p}\right)\prod_1^q (1+\epsilon_m)^{|a_m|} < L\left(\frac{1}{1-x^p}\right)(1+\epsilon).$$

Also
$$\frac{1}{p^{\alpha+1}(1-x)^{\alpha+1}} \leqslant \frac{1}{(1-x^p)^{\alpha+1}}.$$

Hence

$$(3\,.\,35)\qquad \frac{1}{p^{\alpha+1}(1-x)^{\alpha+1}} L\left(\frac{1}{1-x}\right) - \frac{1}{(1-x^p)^{\alpha+1}} L\left(\frac{1}{1-x^p}\right)$$
$$< L\left(\frac{1}{1-x^p}\right)\frac{\epsilon}{(1-x^p)^{\alpha+1}} < L\left(\frac{1}{1-x}\right)\frac{\epsilon\,.\,2^{\alpha+1}(1+\epsilon)}{p^{\alpha+1}(1-x)^{\alpha+1}}.$$

From (3 . 34) and (3 . 35), we get

$$(3\,.\,36)\qquad \left|\frac{1}{(1-x^p)^{\alpha+1}} L\left(\frac{1}{1-x^p}\right) - \frac{1}{p^{\alpha+1}(1-x)^{\alpha+1}} L\left(\frac{1}{1-x}\right)\right|$$
$$< \frac{K\epsilon}{p^{\alpha+1}(1-x)^{\alpha+1}} L\left(\frac{1}{1-x}\right).$$

From (3 . 31) and (3 . 36) we get (3 . 3).

Now, by Lemma 2 . 1,

$$\psi(x) = \sum_{n=1}^{\infty} \frac{a_n x^{(r+1)n}}{1-x^n} + \sum_{p=1}^{r} f(x^p).$$

$$(3\,.\,41)\qquad \left|\psi(x) - \frac{a\Gamma(a+1)\zeta(a+1)}{(1-x)^{a+1}} L\left(\frac{1}{1-x}\right)\right|$$
$$\leqslant \left|\Sigma \frac{a_n x^{(r+1)n}}{1-x^n}\right| + \sum_1^r \left|f(x^p) - \frac{1}{p^{a+1}} \frac{a\Gamma(a+1)}{(1-x)^{a+1}} L\left(\frac{1}{1-x}\right)\right|$$
$$+ \frac{a\Gamma(a+1)}{(1-x)^{a+1}} L\left(\frac{1}{1-x}\right) \sum_{r+1}^{\infty} \frac{1}{n^{a+1}}.$$

In the first place, from (3 . 22) and from $x > X_4$, we have

$$x^{r+1} \leqslant x^{1/(1-x)^\eta} = \exp\left[-\frac{1}{(1-x)^\eta}\log\left(\frac{1}{x}\right)\right] < \exp\left[-\tfrac{1}{2}\frac{1}{(1-x)^\eta}(1-x)\right]$$
$$= x\omega.$$

Remembering that the a's are positive and that

$$1-x^n \geqslant nx^n(1-x),$$

$$A_n \sim n^\alpha L(n),$$

we easily see that

$$\left|\Sigma \frac{a_n x^{(r+1)n}}{1-x^n}\right| \leqslant \frac{1}{1-x}\Sigma\frac{a_n\omega^n}{n} \sim \frac{\Gamma(\alpha+1)}{(1-x)(1-\omega)^\alpha} L\left(\frac{1}{1-\omega}\right)$$

$$< \frac{K}{(1-x)(1-\omega)^{\alpha+\tau}};$$

and, by (3 . 24),

$$\frac{1}{(1-\omega)^{\alpha+\tau}} < \frac{\epsilon}{(1-x)^\alpha}.$$

Hence

$$\left|\Sigma \frac{a_n x^{(r+1)n}}{1-x^n}\right| < \frac{K\epsilon}{(1-x)^{\alpha+1}}. \tag{3 . 42}$$

In the second place, by (3 . 3),

$$\sum_1^r \left|f(x^p) - \frac{1}{p^{\alpha+1}} \frac{a\Gamma(a+1)}{(1-x)^{\alpha+1}} L\left(\frac{1}{1-x}\right)\right| \tag{3 . 43}$$

$$< \frac{K\epsilon}{(1-x)^{\alpha+1}} L\left(\frac{1}{1-x}\right) \sum_1^r \frac{1}{p^{\alpha+1}} < \frac{K\epsilon\zeta(\alpha+1)}{(1-x)^{\alpha+1}} L\left(\frac{1}{1-x}\right).$$

Lastly, by (3 . 23),

$$\frac{a\Gamma(\alpha+1)}{(1-x)^{\alpha+1}} L\left(\frac{1}{1-x}\right) \sum_{r+1}^\infty \frac{1}{n^{\alpha+1}} < \frac{K\epsilon}{(1-x)^{\alpha+1}} L\left(\frac{1}{1-x}\right). \tag{3 . 44}$$

Using (3 . 42), (3 . 43), and (3 . 44) in (3 . 41), we easily see that the theorem is established in the case $\alpha > 0$.

It now remains to consider the case when $\alpha = 0$. When this is so, $L(u)$ tends to a positive limit or to infinity. In the former case the theorem is well known.* We may therefore suppose that $L(u) \to \infty$.

In consequence of Lemma 3 . 1 there is no loss of generality in taking

$$\frac{a_n}{n} = L(n) - L(n-1),$$

supposing that $a_n = 0$ for those values of n at the beginning for which $L(n-1)$ is not positive.

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* See, for example, Hardy, *Proc. London Math. Soc.*, Ser. 2, Vol. 4 (1906), p. 253.

Now $$a_n = n[L(n)-L(n-1)]$$
$$= [nL(n)-(n-1)L(n-1)]-L(n-1);$$

so, $$a_1+a_2+\ldots+a_n = nL(n)-\Sigma L(n-1)+o[nL(n)] = o[nL(n)].$$

Hence $$\frac{f(x)}{1-x} = \Sigma(a_1+a_2+\ldots+a_n)x^n = o[\Sigma nL(n)x^n]$$
$$= o\left[\frac{1}{(1-x)^2}L\left(\frac{1}{1-x}\right)\right],$$

and so, $$f(x) = o\left[\frac{1}{1-x}L\left(\frac{1}{1-x}\right)\right].$$

Further, since $$A_n \sim L(n),$$

$$\int_0^x f(t)dt \sim \Sigma\frac{a_n x^n}{n} \sim L\left(\frac{1}{1-x}\right).$$

Therefore $$f(x) = o\left[\frac{1}{1-x}\int_0^x f(t)\,dt\right].$$

Also, $f(x)$ is a positive increasing function. We can therefore take $f(x)$ for $\phi(x)$ in Theorem 2.1. We then get

$$(1-x)\psi(x) \sim \int_0^x f(t)\,dt \sim L\left(\frac{1}{1-x}\right),$$

which is the result to be established.

4. *A Tauberian Theorem.*

I shall close the paper with a simple Tauberian theorem.

Theorem 4.—*Suppose that* $a_n \to 0$, *and that*

$$\lim_{x\to 1}(1-x)\Sigma\frac{a_n x^n}{1-x^n} = l.$$

Then $\Sigma\dfrac{a_n}{n}$ *converges to the sum* l.

Since $$1-x^n \leqslant n(1-x),$$

$$\frac{nx^n}{1-x^n} \geqslant \frac{x^n}{1-x},$$

$$(4.1)\quad \frac{1}{1-x}-\frac{nx^n}{1-x^n} \leqslant \frac{1}{1-x}-\frac{x^n}{1-x} = \frac{1-x^n}{1-x} \leqslant \frac{n(1-x)}{1-x} = n.$$

We have

$$\sum_{n=1}^{\nu} \frac{a_n}{n}\left[1-\frac{(1-x)\,nx^n}{1-x^n}\right] \leqslant (1-x)\sum_{n=1}^{\nu}\frac{|a_n|}{n}\left[\frac{1}{1-x}-\frac{nx^n}{1-x^n}\right]$$

$$\leqslant (1-x)\sum_{n=1}^{\nu}|a_n|,$$

by (4 . 1). Further,

$$\left|\sum_{n=\nu+1}^{\infty}\frac{a_n(1-x)\,x^n}{1-x^n}\right| \leqslant \frac{H_\nu}{1-x^\nu}(1-x)\sum_{0}^{\infty}x^n = \frac{H_\nu}{1-x^\nu},$$

where H_ν is the upper limit of the moduli of $a_{\nu+1}, a_{\nu+2}, \ldots$. Therefore

$$\sum_{n=1}\frac{a_n}{n}-(1-x)\sum_{n=1}^{\infty}\frac{a_n x^n}{1-x^n} \leqslant (1-x)\sum_{n=1}^{\nu}|a_n|+\frac{H_\nu}{1-x^\nu}.$$

Let us take $x=1-\dfrac{1}{\nu}$, so that when $x\to 1$, $\nu\to\infty$. Then, remembering that

$$1-x^\nu = 1-\left(1-\frac{1}{\nu}\right)^\nu > 1-\frac{1}{e},$$

we get

$$\left|\overline{\lim_{\nu\to\infty}}\sum_{1}^{\nu}\frac{a_n}{n}-\lim_{x\to 1}(1-x)\psi(x)\right| \leqslant \frac{1}{\nu}\sum_{1}^{\nu}|a_n|+\frac{eH_\nu}{e-1} = o(1),$$

since $a_n\to 0$. The theorem is therefore proved.

ON A TAUBERIAN THEOREM FOR LAMBERT'S SERIES, AND SOME FUNDAMENTAL THEOREMS IN THE ANALYTIC THEORY OF NUMBERS

By G. H. Hardy *and* J. E. Littlewood.

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1. It is known that, if the series

$$\Sigma \frac{a_n}{n} \quad (n = 1, 2, \ldots) \tag{1.1}$$

converges to the sum A, then the series

$$f(x) = \Sigma \frac{a_n x^n}{1-x^n} \tag{1.2}$$

is convergent for all values of x whose modulus is less than unity, and

$$f(x) \sim \frac{A}{1-x} \tag{1.3}$$

when $x \to 1$ by real values, or, more generally, along any regular path which does not touch the unit circle.* Our object is to prove a theorem which is related to this theorem as Theorem 11 of our paper "Tauberian Theorems concerning power series and Dirichlet's series whose coefficients are positive"† is related to Abel's standard theorem as to the continuity of a power series.

We quote this theorem, and Theorem 9 of the same paper,‡ as we shall have to appeal to them frequently in this note.

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* For reference to the literature connected with this theorem and its extensions see Hardy, "Note on Lambert's series", *Proc. London Math. Soc.*, Ser. 2, Vol. 13, 1913, pp. 192–198.

† *Proc. London Math. Soc.*, Ser. 2, Vol. 13, 1913, pp. 174–191.

‡ We have combined Theorems 9 and 12, and specialised their conditions in an unimportant manner.

THEOREM 9. — *If* (i) $a_n > -Kn^{\alpha-1}(\log n)^\beta$, *where* $\alpha > 0$ *or* $\alpha = 0$, $\beta > 0$; (ii) *the series* $\Sigma a_n x^n$ *is convergent for* $|x| < 1$;

(iii) $$f(x) = \Sigma a_n x^n \sim \frac{A}{(1-x)^\alpha}\left\{\log\left(\frac{1}{1-x}\right)\right\}^\beta$$

when $x \to 1$; *then*

$$s_n = a_0 + a_1 + \ldots + a_n \sim \frac{A}{\Gamma(1+\alpha)} n^\alpha (\log n)^\beta$$

when $n \to \infty$.

THEOREM 11. — *If* $a_n > -K/n$ *and* $f(x) \to A$, *then* Σa_n *converges to the sum* A.

The second theorem corresponds to the case $\alpha = 0$, $\beta = 0$.

The theorem which we propose to prove now is as follows :—

THEOREM A. — *If* (i) $a_n > -K$, *where* K *is a constant*; (ii) *the series* (1.2) *is convergent when* $|x| < 1$; (iii) *the relation* (1.3) *is true when* $x \to 1$ *by real values*; *then the series* (1.1) *converges to the sum* A.

This theorem is distinguished from all the "Tauberian" theorems which we have proved before, in that we cannot prove it by elementary methods. Our proof, in fact, involves the assumption of one of the deeper results in the theory of the distribution of primes, which have never been proved except by the use of the methods of complex function theory. And it would seem that this dependence upon the theory of functions is not due merely to the imperfections of our analysis. For it is easy (as we shall show in § 5) to present the *Primzahlsatz* or "Prime-number Theorem" itself as a *corollary* of the theorem; so that an elementary proof of the theorem would be tantamount to an elementary proof of the Prime-number Theorem, and would involve something like a revolution in the logical arrangement of the ideas of the Analytic Theory of Numbers.

2. It is plain that we may suppose $A = 0$; if this condition is not satisfied we replace a_1 by $a_1 - A$. We may also replace x by e^{-y}, and suppose that $y \to 0$ by positive values. The theorem may then be restated in the following form :—

If $a_n > -K$ *and*

$$\phi(y) = \Sigma \frac{a_n e^{-ny}}{1-e^{-ny}} = o\left(\frac{1}{y}\right), \tag{2.1}$$

then

$$\Sigma \frac{a_n}{n} = 0. \tag{2.2}$$

3. The result which we take from the Analytic Theory of Numbers is as follows. Suppose that $\mu(n)$ denotes, as usual, the arithmetical function which is equal to zero if n contains any squared factor, and to $(-1)^\rho$ if n is the product of ρ different primes.* Let

$$g(n) = \sum_1^n \frac{\mu(m)}{m}.$$

Then

$$g(n) = O\left\{\frac{1}{(\log n)^2}\right\}. \tag{3.1}$$

For a proof of this result we may refer to Landau's *Handbuch.*†

4. The assumption that (1.2) is convergent for $|x| < 1$ is equivalent to the assumption that

$$a_n = O(e^{\epsilon n})$$

for every positive value of ϵ. This assumption involves the convergence of

$$\psi(y) = \Sigma a_n e^{-ny}$$

for $y > 0$, the absolute convergence of the double series used in the ensuing transformation, and the legitimacy of the inversions on which it depends.

We have $$\phi(y) = \sum_n a_n \sum_m e^{-mny} = \Sigma \psi(my),$$

and so $$\Sigma \mu(q)\, \phi(qy) = \sum_q \mu(q) \sum_m \psi(mqy) = \Sigma c_r \psi(ry),$$

where $$c_r = \sum_{q|r} \mu(q),$$

* As unity is not counted as a prime, $\mu(1) = 1$.

† P. 570 and pp. 593–597.

the notation implying as usual that the summation extends over all divisors of r. But c_r is unity if $r = 1$ and zero otherwise.* Hence

$$\psi(y) = \Sigma \mu(q)\, \phi(qy)\,; \tag{4.1}$$

and so

$$\begin{aligned} \chi(y) = \Sigma \frac{a_n}{n} e^{-ny} &= \int_y^\infty \psi(t)\, dt \\ &= \Sigma \mu(q) \int_y^\infty \phi(qt)\, dt = \Sigma \frac{\mu(q)}{q} \int_{qy}^\infty \phi(u)\, du \\ &= \Sigma g(q) \int_{qy}^{(q+1)y} \phi(u)\, du \\ &= \Sigma g(q)\, h(q), \end{aligned} \tag{4.2}$$

say.

We write

$$S = \Sigma g(q)\, h(q) = \sum_1^{\delta/y} + \sum_{\delta/y}^{1/y} + \sum_{1/y}^{\infty} = S_1 + S_2 + S_3, \tag{4.3}$$

say, δ being a (small) positive constant.†

In S_1 we have $(q+1)\, y < 2\delta$, and so

$$\begin{aligned} |h(q)| &= y\, |\phi\{(q+\theta)\, y\}| \quad (0 < \theta < 1) \\ &< \epsilon_\delta / q, \end{aligned}$$

since $\phi(y) = o(1/y)$, ϵ_δ being a number which tends to zero with δ. Hence

$$|S_1| < K\epsilon_\delta \sum_1^{\delta/y} \frac{1}{q(\log q)^2} < K\epsilon_\delta. \tag{4.4}$$

In S_2 we have $$\delta \leqslant qy < (q+1)\, y \leqslant 2,$$

and so $$|\phi(u)| < G_\delta, \quad |h(q)| < H_\delta y,$$

G_δ and H_δ being functions of δ alone. Hence

$$|S_2| < H_\delta y \sum_{\delta/y}^{1/y} \frac{1}{(\log q)^2} < \frac{H_\delta}{\{\log(\delta/y)\}^2}. \tag{4.4}$$

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* Landau, *Handbuch*, p. 575.

† We denote by $$\sum_\alpha^\beta u_q,$$

where α and β are not necessarily integers, a sum extended over the range $\alpha \leqslant q <$.

Finally, in S_3 we have $qy > 1$, and so

$$|\phi(u)| < Ke^{-u}, \quad |h(q)| < Kye^{-qy},$$

$$|S_3| < Ky \sum_{1/y}^{\infty} \frac{e^{-qy}}{(\log q)^2} < \frac{K}{\{\log(1/y)\}^2} = o(1), \tag{4.6}$$

as $y \to 0$. From (4.3), (4.4), (4.5), and (4.6) it follows, by choice first of δ and then of y, that

$$\chi(y) \to 0 \tag{4.7}$$

when $y \to 0$. But the coefficient $b_n = a_n/n$ in $\chi(y)$ satisfies the condition

$$b_n > -\frac{K}{n};$$

and therefore, by Theorem 11, the series Σb_n converges to the sum 0.

5. The two most interesting cases of this theorem are as follows.

(i) Suppose $a_n = \mu(n)$. Then

$$\phi(y) = \Sigma \frac{\mu(n)\, e^{-ny}}{1-e^{-ny}} = e^{-y} = O(1) = o\left(\frac{1}{y}\right)$$

and

$$a_n = O(1).$$

Hence

$$\Sigma \frac{\mu(n)}{n} = 0. \tag{5.1}$$

Naturally our argument does not involve a new proof of (5.1), for (5.1) is the same as $g(n) = o(1)$, and we used more than this, viz.,

$$g(n) = O\{(\log n)^{-2}\},$$

in our proof of the main theorem. But the example is of much interest as showing that our theorem certainly lies as deep as (5.1), and so that an elementary proof of it is hardly to be expected.

(ii) Suppose

$$a_n = \Lambda(n) - 1,$$

where $\Lambda(n)$ is zero unless n is a power of a prime p, and is then equal to $\log p$. Plainly $a_n \geqslant -1$. Also

$$f(y) = \Sigma \frac{a_n e^{-ny}}{1-e^{-ny}} = \Sigma c_n e^{-ny},$$

where $$c_n = \sum_{d|n} \{\Lambda(d)-1\} = \log n - d(n),$$

$d(n)$ being the number of divisors of n.

Now

$$c_1+c_2+\ldots+c_n = n\log n - n + o(n) - \{n\log n + (2\gamma-1)n + o(n)\} \sim -2\gamma n,$$

and so $$\Sigma c_n e^{-ny} \sim -\frac{2\gamma}{y}.$$

Hence, by Theorem A,

$$\Sigma \frac{\Lambda(n)-1}{n} = -2\gamma. \tag{5 . 2}$$

The convergence of the series (5 . 2) involves

$$\sum_{n\leqslant x} \{\Lambda(n)-1\} = o(x),$$

or

$$\psi(x) = \sum_{n\leqslant x} \Lambda(n) \sim x, \tag{5 . 3}$$

a result which has long been known to be equivalent to the Prime-number Theorem.*

6. The examples which precede suggest some remarks as to the logical relations of some of these fundamental theorems in the Analytic Theory of Numbers.

The two most famous of these are those expressed by the equations (5 . 1) and (5 . 3). Neither of these has ever been proved by elementary methods; but it is now known that the two are "equivalent", in the sense that it is possible, by elementary methods, to pass from either to the other. Each deduction is due to Landau. The deduction of (5 . 1) from (5 . 3) was made by him in 1899,† and is reproduced on pp. 591–593 of his *Handbuch*. That of (5 . 3) from (5 . 1) was not made until 1911,‡ the *Handbuch* containing only a deduction of (5 . 3) from the more drastic assumption §

$$\Sigma \frac{\mu(n)\log n}{n} = -1. \tag{6 . 1}$$

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* See, *e.g.*, Landau, *Handbuch*, pp. 79, 83–85.

† "Neuer Beweis der Gleichung $\Sigma \frac{\mu(k)}{k} = 0$", *Dissertation*, Berlin, 1899.

‡ "Über die Äquivalenz zweier Hauptsatze der analytischen Zahlentheorie", *Wiener Sitzungsberichte*, Vol. 120, Part 2*a*, 1911, pp. 973–988.

§ Pp. 597–604. This deduction was first made in 1906: see "Über den Zusammenhang einiger neuerer Sätze der analytischen Zahlentheorie", *Wiener Sitzungsberichte*, Vol. 115, Part 2*a*, 1906, pp. 589–632

It is natural to suppose that these deductions may be made also by means of some "Tauberian" theorem. Theorem A, however, does not, as it stands, suffice for either deduction.

Let us consider the second first. It is true that, in § 5 (ii), we found a proof of (5 . 3), but in this proof we assumed more than (5 . 1) or even (6 . 1). It is, however, easy to see that, if we modify our theorem by adopting a rather more stringent hypothesis concerning $f(y)$, viz. that

$$f(y) = O\left(\frac{1}{y^{1-\delta}}\right) \quad (\delta > 0), \tag{6 . 2}$$

we can arrive at the conclusion without assuming, about $g(n)$, more than

$$g(n) = o(1),$$

i.e. (5 . 1), instead of (3 . 1). We have, in fact, using practically the same notation as in § 4,

$$S = \Sigma g(q)\, h(q) = \sum_{1}^{1/y} + \sum_{1/y}^{\infty} = S_1 + S_2,$$

$$S_1 = y \sum_{1}^{1/y} o(1)\, O\left(\frac{1}{qy}\right)^{1-\delta} = y^{\delta} \sum_{1}^{1/y} o\left(\frac{1}{q^{1-\delta}}\right) = o(1),$$

and

$$S_2 = o\left(y \sum_{1/y}^{\infty} e^{-qy}\right) = o(1).$$

This modification of our theorem enables us at once to make the desired deduction, for the more stringent hypothesis (6 . 2) is in fact satisfied by the $f(y)$ of § 5 (ii).

It may be added that the deduction of (5 . 3) from (6 . 1)—the most that had been effected in this direction before Landau's paper of 1911—may be made without the use of any new theorem. For it is easily verified that

$$\Sigma \frac{\mu(n) \log n\, e^{-ny}}{1 - e^{-ny}} = -\Sigma \Lambda(n) e^{-ny}. \tag{6 . 3}$$

Hence, by (6 . 1) and the "Abelian" theorem quoted at the beginning of § 1,

$$\Sigma \Lambda(n) e^{-ny} \sim \frac{1}{y};$$

and therefore

$$\sum_{n \leqslant x} \Lambda(n) \sim x,$$

by Theorem 9.

7. If we are to deduce (5 . 1) from (5 . 3), our theorem must be modified

in a different manner ; for it is essential that (5.1) should not be used in the proof.

A theorem effective for the purpose is the following :—

THEOREM B.—*If* (i) $a_n > -K\log n$,

$$\text{(ii)}\quad f(y) \sim \frac{A}{y},$$

then $$\sum_1^n \frac{a_\nu}{\nu} = o(\log n).$$

This theorem requires no assumption concerning $g(n)$ beyond

$$g(n) = O(1),$$

which is trivial.* In fact we have, in the notation of § 6,

$$S_1 = y\sum_1^{1/y} O(1)\, o\left(\frac{1}{qy}\right) = o\left(\sum_1^{1/y}\frac{1}{q}\right) = o\left(\log\frac{1}{y}\right),$$

and $$S_3 = O(1).$$

Hence $$\chi(y) = o\left(\log\frac{1}{y}\right),$$

and the conclusion then follows from our former theorem already quoted.

We have now, by (6.3),

$$f(y) = \Sigma\frac{\mu(n)\log n e^{-ny}}{1-e^{-ny}} = -\Sigma\Lambda(n)e^{-ny} \sim -\frac{1}{y}.$$

Applying Theorem B, we obtain

$$\sum_1^n \frac{\mu(\nu)\log\nu}{\nu} = o(\log n);$$

and so, by partial summation,

$$g(n) = \sum_1^n \frac{\mu(\nu)}{\nu} = o(1).$$

8. We add a final remark suggested by Ananda Rau's paper which pre-

* See Landau, *Handbuch*, p. 582.

cedes ours. Ananda Rau says that the series $\Sigma \frac{a_n}{n}$ is summable (L), to sum A, if

$$\sum_1^\infty \frac{a_n e^{-ny}}{1-e^{-ny}} \sim \frac{A}{y} \tag{8.1}$$

when $y \to 0$, and raises the question of the relations between summability (L) and summability (A): the series being summable (A), to sum A, if

$$\lim \Sigma \frac{a_n}{n} e^{-ny} = A \tag{8.2}$$

when $y \to 0$. Now a reference to our proof of Theorem A shows that it falls into two distinct parts. In the first part of our argument we make no use of the Tauberian condition (i), but prove that (irrespective of any such condition) (8.1) implies the existence of the Abelian limit (8.2). That is to say we prove that *if a series is summable* (L), *to sum* A, *it is summable* (A) *to the same sum*; or that what Ananda Rau calls "Lambert's method" of summation is included in "Abel's", and so occupies an intermediate position between the latter and the various methods of summation by Cesàro's means, all of which it in its turn includes.

THE ABERRATIONS OF A SYMMETRICAL OPTICAL SYSTEM

By T. W. Chaundy.

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In the *Philosophical Magazine* for December, 1917,* I described a method of investigating by means of line-coordinates the aberrations of an optical system, possessing rotational symmetry about an axis. In this method the coordinates defining the position of the emergent ray were obtained in terms of the corresponding coordinates of the incident ray and of certain optical constants of the system. It is the purpose of the present paper to consider the evaluation of these optical constants for any given optical system.

1. As usual, Cartesian axes are adopted in which the x-axis is along the axis of symmetry of the system, x increasing in the sense of ongoing light, the axes of y and z being in any two directions which complete a right-handed rectangular frame of reference. The origin of reference is at our disposal. I find it convenient to separate the origins of incident and emergent light, taking the origin of incident light at the vertex of the first surface and that of emergent light at the vertex of the last surface. The direction cosines, with respect to such a frame of reference, of a typical ray are (l, m, n). I take as my three additional line-coordinates a, u, v, defined as follows:

$$a \equiv ny-mz, \quad u \equiv ly-mx, \quad v \equiv lz-nx. \tag{1}$$

These are, of course, the customary line-coordinates of three-dimensional geometry, modified however in respect of sign and notation. The reason for this modification lies in the fact that the rotational symmetry of the optical system differs from the tri-axial symmetry of the ordinary analytical three-dimensional geometry, the rotational symmetry being better suited by the forms written above.

* "A Method of Line Coordinates for Investigating the Aberrations of a Symmetrical Optical System." References will be made to this paper in the form [P.M. 6, (16)], such a symbol implying Section 6 a equation (16) of the paper.

I shall further restrict myself in this paper to the lowest aberrations, *i.e.* to those given by the second approximation, so that the aperture and the field will enter to a degree not exceeding the third. With such conventions and notation it is shown [P.M. 6, (16) *et seq.*] that the coordinates m', n', u', v' of a ray on emergence from an optical system are given in terms of the coordinates m, n, u, v of the incident ray by equations of the type*

$$\left.\begin{aligned} m' &= Pm+Qu+\tfrac{1}{2}\{(Am+Bu)(m^2+n^2)+2(Cm+Du)(mu+nv) \\ &\qquad +(Em+Fu)(u^2+v^2)\} \\ u' &= Rm+Su+\tfrac{1}{2}\{(Gm+Hu)(m^2+n^2)+2(Im+Ju)(mu+nv) \\ &\qquad +(Km+Lu)(u^2+v^2)\} \\ n' &= Pn+Qv+\tfrac{1}{2}\{(An+Bv)(m^2+n^2)+2(Cn+Dv)(mu+nv) \\ &\qquad +(En+Fv)(u^2+v^2)\} \\ v' &= Rn+Sv+\tfrac{1}{2}\{(Gn+Hv)(m^2+n^2)+2(In+Jv)(mu+nv) \\ &\qquad +(Kn+Lv)(u^2+v^2)\} \end{aligned}\right\}. \quad (2)$$

The equations for n', v' are evidently similar to those for m', u', as we should expect in view of the rotational symmetry of the system. The coefficients of the functions of m, n, u, v on the right of these equations, namely P, Q, R, S; A, B, C, D, E, F, G, H, I, J, K, L are optical constants of the system, and it is with the method of evaluating these optical constants for any given system that the present paper is concerned. It is evident that the behaviour of the optical system (within the limits of the second approximation) is completely ascertainable once these sixteen quantities have been evaluated, and that the aberrations (spherical aberration, error against the sine condition, etc.) are expressible in terms of these constants, and, of course, of the position of the object and the stop, as is done for an infinitely distant object in [P.M. 7].

2. If for a moment we limit attention to the first approximation only, the preceding equations reduce to merely

$$m' = Pm+Qu,$$
$$u' = Rm+Su,$$

with similar equations for n', v'.

* It should be noted that the equations (2) differ in the sign of C, D, I, J from the corresponding equations in the paper in the *Philosophical Magazine*. Experience has shown me that the symmetry of the results obtained is greater if the coefficients in the expressions on the right of (2) have the signs given them above.

Suppose that the ray under consideration is a ray which meets the axis, then to the first approximation m may be replaced by α the inclination of the ray to the axis, and in view of equations (1), u by y the height at which the ray crosses the plane of reference, *i.e.* with the conventions of § 1, the tangent plane to the initial or final refracting surface respectively.

The first order equations are thus

$$\alpha' = P\alpha + Qy,$$

$$y' = R\alpha + Sy.$$

But it is a standard result that

$$\mu'\alpha' = (\partial K/\partial\kappa)\,\mu\alpha - Ky,$$

$$y' = -(\partial^2 K/\partial\kappa\,\partial\kappa')\,\mu\alpha + (\partial K/\partial\kappa')\,y,$$

where K denotes the power of the system, κ, κ' the powers of the first and last surfaces, and μ, μ' the refractive indices of the first and last media.

Hence always

$$\left.\begin{aligned} P &= \mu(\partial K/\partial\kappa)/\mu' \\ Q &= -K/\mu' \\ R &= -\mu\partial^2 K/\partial\kappa\,\partial\kappa' \\ S &= \partial K/\partial\kappa' \end{aligned}\right\}, \tag{3}$$

and the fundamental identity

$$(\partial K/\partial\kappa)(\partial K/\partial\kappa') - K(\partial^2 K/\partial\kappa\,\partial\kappa') = 1,$$

gives the corresponding identity

$$PS - QR = \mu/\mu'. \tag{4}$$

The calculation and properties of the power K being well known, I pass at once to consider the evaluation of the "aberration coefficients" A, B, C, D, E, F, G, H, I, J, K, L. I shall effect this evaluation by establishing formulæ which express these twelve quantities in terms of the first order constants P, Q, R, S of certain subsidiary optical systems which can be formed out of the given system by appropriate selection and suppression of its elements. These subsidiary systems are constituted as follows.

Suppose the optical system to be composed of p refracting surfaces, separating $p+1$ media. Referring to the media by their refractive indices μ_n and to the surfaces by their powers κ_n, I shall call the medium in which lies the source of light μ_0, and subsequently employ equal suffixes for a medium and the surface by which the light enters that medium. We may

therefore represent the complete system symbolically as

$$\mu_0\kappa_1\mu_1\kappa_2\mu_2 \dots \mu_{p-1}\kappa_p\mu_p,$$

light being supposed to travel from left to right.

In this notation the subsidiary systems I desire to use are denoted by

$$\mu_0\kappa_1\mu_1,\ \mu_0\kappa_1\mu_1\kappa_2\mu_2,\ \dots,\ \mu_0\kappa_1\mu_2 \dots \mu_{n-1}\kappa_n\mu_n,\ \dots,\ \mu_0\kappa_1\mu_2 \dots \mu_{p-1}\kappa_p\mu_p,$$

i.e. the system into which the given system successively decomposes, if we peel off the media one by one, beginning at the last. The optical constants of the subsidiary system composed of the first n surfaces, *i.e.* the system $\kappa_0\kappa_1\mu_2 \dots \mu_{n-1}\kappa_n\mu_n$ will bear the suffix n, *e.g.* its first order constants will be written P_n, Q_n, R_n, S_n. In conformity with § 1, the origin of coordinates u_n, v_n in medium μ_n will be at the vertex of the surface κ_n by which the ray enters that medium.

I shall proceed to show that the twelve aberration coefficients

$$A,\ B,\ \dots,\ K,\ L,$$

are determined by the twelve equations

$$\mu(AR-PG) = \Sigma R_n(P_n/\mu_n - P_{n-1}/\mu_{n-1})(P_n-P_{n-1})^2(1/\mu_n - 1/\mu_{n-1})^{-2},$$

$$\mu(AS-QG) = \Sigma R_n(P_n/\mu_n - P_{n-1}/\mu_{n-1})(P_n-P_{n-1})(Q_n-Q_{n-1})(1/\mu_n - 1/\mu_{n-1})^{-2},$$

$$\mu(BR-PH) = \Sigma S_n(P_n/\mu_n - P_{n-1}/\mu_{n-1})(P_n-P_{n-1})^2(1/\mu_n - 1/\mu_{n-1})^{-2} + \mu_0\Sigma(P_n-P_{n-1})(P_n\mu_n - P_{n-1}\mu_{n-1})/(\mu_n-\mu_{n-1}),$$

$$\mu(BS-QH) = \Sigma S_n(P_n/\mu_n - P_{n-1}/\mu_{n-1})(P_n-P_{n-1})(Q_n-Q_{n-1})(1/\mu_n - 1/\mu_{n-1})^{-2} + \mu_0\Sigma(Q_n-Q_{n-1})(P_n\mu_n - P_{n-1}\mu_{n-1})/(\mu_n-\mu_{n-1}),$$

$$\mu(CR-PI) = \Sigma R_n(P_n/\mu_n - P_{n-1}/\mu_{n-1})(P_n-P_{n-1})(Q_n-Q_{n-1})(1/\mu_n - 1/\mu_{n-1})^{-2},$$

$$\mu(CS-QI) = \Sigma R_n(P_n/\mu_n - P_{n-1}/\mu_{n-1})(Q_n-Q_{n-1})^2(1/\mu_n - 1/\mu_{n-1})^{-2},$$

$$\mu(DR-PJ) = \Sigma S_n(Q_n/\mu_n - Q_{n-1}/\mu_{n-1})(P_n-P_{n-1})^2(1/\mu_n - 1/\mu_{n-1})^{-2},$$

$$\mu(DS-QJ) = \Sigma S_n(Q_n/\mu_n - Q_{n-1}/\mu_{n-1})(P_n-P_{n-1})(Q_n-Q_{n-1})(1/\mu_n - 1/\mu_{n-1})^{-2},$$

$$\mu(ER-PK) = \Sigma R_n(Q_n/\mu_n - Q_{n-1}/\mu_{n-1})(P_n-P_{n-1})(Q_n-Q_{n-1})(1/\mu_n - 1/\mu_{n-1})^{-2} - \mu_0\Sigma(P_n-P_{n-1})(Q_n\mu_n - Q_{n-1}\mu_{n-1})/(\mu_n-\mu_{n-1}),$$

$$\mu(ES-QK) = \Sigma R_n(Q_n/\mu_n - Q_{n-1}/\mu_{n-1})(Q_n-Q_{n-1})^2(1/\mu_n - 1/\mu_{n-1})^{-2} - \mu_0\Sigma(Q_n-Q_{n-1})(Q_n\mu_n - Q_{n-1}\mu_{n-1})/(\mu_n-\mu_{n-1}),$$

$$\mu(FR-PL) = \Sigma S_n(Q_n/\mu_n - Q_{n-1}/\mu_{n-1})(P_n-P_{n-1})(Q_n-Q_{n-1})(1/\mu_n - 1/\mu_{n-1})^{-2},$$

$$\mu(FS-QL) = \Sigma S_n(Q_n/\mu_n - Q_{n-1}/\mu_{n-1})(Q_n-Q_{n-1})^2(1/\mu_n - 1/\mu_{n-1})^{-2}.$$

3. The exact equations (*i.e.* without approximation) for refraction at a single surface are [P.M. 4, (11)]*

$$\mu' m' = \mu m - (\mu' \cos\theta' - \mu \cos\theta)\, y/r, \tag{5}$$

$$\mu' u' = \mu u + (\mu' \cos\theta' - \mu \cos\theta)\, y, \tag{6}$$

where θ, θ' are the angles of incidence and refraction, y a coordinate of the point at which the ray crosses the surface, and r the radius of this surface, reckoned positive if convex to oncoming light. The vertex of the surface is here the origin of incident and refracted rays alike.

We have also connecting these quantities the equations [P.M. 3, (10)]

$$\sin^2\theta = \{a^2 + (u^2+v^2) + 2r(mu+nv) + r^2(m^2+n^2)\}/r^2, \tag{7}$$

$$\mu' \sin\theta' = \mu \sin\theta, \tag{8}$$

the identities $u = ly - mx, \quad v = lz - nx, \quad a = ny - mz,$ (9)

and $$x^2 + y^2 + z^2 = 2rx, \tag{10}$$

the equation to the spherical refracting surface.

These equations are enough to define the coordinates of the refracted ray.

It is convenient first of all to establish that for a non-real ray whose coordinates satisfy the two conditions

$$m = in, \qquad u = iv,$$

where $$i = \sqrt{-1}, \tag{11}$$

the first order equations are *exactly* true without approximation.

For the first order equations are obtained from equations (5)–(10) by setting $\theta = 0 = \theta'$ and $y = u$ [P.M. 4].

We have therefore to show that for the non-real ray defined by equations (11), $\theta = 0 = \theta'$ and $y = u$ exactly.

But for such a ray $m^2 + n^2 = 0$, *i.e.* $l^2 = 1$. The root of this equation with which we are concerned is $l = 1$. This gives in equations (9),

$$u = y - mx, \qquad v = z - nx,$$

i.e. since $u = iv$ and $m = in$, we have also $y = iz$, whence $y^2 + z^2 = 0$. Equation (10) thus becomes $x^2 = 2rx$. Here again we are concerned

* I have amended these equations by taking μ, μ' as refractive indices in place of 1, μ.

only with the root $x = 0$, so that $u = y$. Similarly $v = z$. Hence a which is $ny - mz = nu - mv = 0$, since $u = iv$ and $m = in$. Hence also $mu + nv = 0$.

Since then both $u^2 + v^2$ and $m^2 + n^2$ also vanish, we see from (7) that $\sin^2\theta = 0$. The root we require is $\theta = 0$, giving also $\theta' = 0$.

These two equations with $u = y$ already obtained, as we have said, show that for the ray under consideration the first order equations are true without approximation. I shall distinguish the coordinates of a ray having this property by capital letters M, N, U, V, so that without approximation

$$\left.\begin{aligned} \mu' M' &= \mu M - (\mu' - \mu) U/r \\ U' &= U \end{aligned}\right\}, \tag{12}$$

these being the first order equations of a single refraction [P.M. 4, (12), (13)].

Let M_0, N_0, U_0, V_0 be the coordinates of such a ray before incidence on the optical system; and M, N, U, V its coordinates in medium μ_{n-1} before refraction at the surface κ_n, and M', N', U', V' its coordinates after refraction at this surface into medium μ_n. For any other typical ray the coordinates m_0, n_0, u_0, v_0; m, n, u, v; m', n', u', v' will have similar meanings. The origin of U_0, V_0, u_0, v_0 is the vertex of the first surface; that of both U, V, u, v, and U', V', u', v' is at the vertex of κ_n. For brevity I shall write μ, κ, μ' for μ_{n-1}, κ_n, μ_n. Also $\kappa = (\mu' - \mu)\rho$, if ρ is the *curvature* of this n-th refracting surface κ_n. Finally, I use the symbol Δ to denote excess of any function of the coordinates of the ray in medium μ' over the corresponding function for medium μ.

In particular therefore

$$\begin{aligned} \Delta\mu(Mu - Um) &= \mu'(M'u' - U'm') - \mu(Mu - Um) \\ &= \mu M(u' - u) - U\{(\mu' - \mu)\rho u' + \mu' m' - \mu m\}, \end{aligned}$$

on substitution for M', U' from equations (12), since $\rho = 1/r$.

But substitution for m and u from equations (5), (6) gives

$$\begin{aligned} -\{(\mu' - \mu)\rho u' + \mu' m' - \mu m)\} \\ &= \rho\mu(u' - u) \\ &= \rho\{(\mu - \mu')u' + y(\mu' \cos\theta' - \mu\cos\theta)\} \\ &= -\rho y\{\mu'(1 - \cos\theta') - \mu(1 - \cos\theta)\} + \rho(\mu' - \mu)(y - u'), \end{aligned}$$

Hence

$$\Delta\mu(Mu-Um)$$
$$=-(M+U\rho)\{y[\mu'(1-\cos\theta')-\mu(1-\cos\theta)]-(\mu'-\mu)(y-u')\}. \quad (13)$$

Now in the expression on the right of this equation

$$\left.\begin{array}{l} M+U\rho \text{ and } y \text{ are small quantities of the first order} \\ 1-\cos\theta \text{ and } 1-\cos\theta' \text{ are small quantities of the second order} \\ y-u' \text{ is a small quantity of the third order} \end{array}\right\}. \quad (14)$$

Hence this expression is a small quantity of the fourth order; since, however, $\Delta\mu(Mu-Um)$ is only of the second degree, we have evidently eliminated the first order terms in $\Delta\mu(Mu-Um)$, and consequently substitution in it of the first approximations to the quantities (14) will leave (13) as an equation giving $\Delta\mu(Mu-Um)$ as far as the second approximation.

Now equation (5) gives, to the first approximation,

$$\left.\begin{array}{rc} & (m'+u'\rho)/\mu=(m+u\rho)/\mu'=-(m'-m)/(\mu'-\mu) \\ \text{so} & (n'+v'\rho)/\mu=(n+v\rho)/\mu'=-(n'-n)/(\mu'-\mu) \\ \text{similarly} & (M+U\rho)/\mu'=-(M'-M)/(\mu'-\mu) \end{array}\right\}. \quad (15)$$

Again, since the incident and refracted rays alike pass through (x, y, z),

$$u'=l'y-m'x,$$

and thus

$$y-u'=y(1-l')+m'x.$$

But $x^2+y^2+z^2=2rx$, so that to lowest order

$$x=\tfrac{1}{2}\rho(y^2+z^2)=\tfrac{1}{2}\rho(u'^2+v'^2),$$

and also $1-l'=\tfrac{1}{2}(m'^2+n'^2)$ and $y=u'$ to lowest order.

Hence

$$\begin{aligned} y-u' &= \tfrac{1}{2}\{u'(m'^2+n'^2)+m'\rho(u'^2+v'^2)\} \\ &= \tfrac{1}{2}u'\{m'(m'+u'\rho)+n'(n'+v'\rho)\}+\tfrac{1}{2}v'\rho(m'v'-n'u') \\ &= -\tfrac{1}{2}\mu(\mu'-\mu)^{-1}u'\{m'(m'-m)+n'(n'-n)\} \\ &\qquad -\tfrac{1}{2}(\mu'-\mu)^{-1}(\mu'n'-\mu n)(m'v'-n'u'), \end{aligned}$$

by equations (15).

Again, since θ is the angle between (l, m, n) and the radius of the refracting surface whose direction cosines are $(1-x\rho, -y\rho, -z\rho)$, we have

$$\begin{aligned}1-\cos\theta &= 1-l(1-x\rho)+my\rho+nz\rho\\ &= \tfrac{1}{2}(m^2+n^2)+(my+nz)\rho+\tfrac{1}{2}(y^2+z^2)\rho^2 \quad \text{(to least order)}\\ &= \tfrac{1}{2}(m+y\rho)^2+\tfrac{1}{2}(n+z\rho)^2\\ &= \tfrac{1}{2}\mu'^2(\mu'-\mu)^{-2}\{(m'-m)^2+(n'-n)^2\}\end{aligned}$$

[from (15) since $y = u$ to least order].

Similarly,

$$1-\cos\theta' = \tfrac{1}{2}\mu^2(\mu'-\mu)^{-2}\{(m'-m)^2+(n'-n)^2\},$$

and thus

$$\mu'(1-\cos\theta')-\mu(1-\cos\theta) = -\tfrac{1}{2}\mu'\mu(\mu'-\mu)^{-1}\{(m'-m)^2+(n'-n)^2\}.$$

Thus, since to least order $y = u'$,

$$\begin{aligned}&y\{\mu'(1-\cos\theta')-\mu(1-\cos\theta)\}-(\mu'-\mu)(y-u')\\ &= -\tfrac{1}{2}\mu(\mu'-\mu)^{-1}u'\{(m'-m)[\mu'(m'-m)-(\mu'-\mu)m']\\ &\qquad +(n'-n)[\mu'(n'-n)-(\mu'-\mu)n']\}+\tfrac{1}{2}(\mu'n'-\mu n)(m'v'-n'u')\\ &= -\tfrac{1}{2}\mu(\mu'-\mu)^{-1}u'\{(m'-m)(\mu m'-\mu' m)+(n'-n)(\mu n'-\mu' n)\}\\ &\qquad +\tfrac{1}{2}(\mu'n'-\mu n)(m'v'-n'u').\end{aligned}$$

Hence, changing a sign and using the third equation of (15),

$$\begin{aligned}&\Delta\mu(Um-Mu)\\ &= \tfrac{1}{2}(M'-M)\{\mu'\mu(\mu'-\mu)^{-2}u'[(m'-m)(\mu m'-\mu' m)+(n'-n)(\mu n'-\mu' n)]\\ &\qquad -\mu'(\mu'-\mu)^{-1}(\mu'n'-\mu n)(m'v'-n'u')\}. \quad (17)\end{aligned}$$

Now we have so far taken the origin of u, v, U, V to be at the vertex of the surface κ_n. If this origin is removed to the vertex of the preceding surface κ_{n-1}, *i.e.* is shifted a distance, say d, to the left, the direction coordinates m, n, M, N are unaffected, but we must replace u, v, U, V by $u-md$, $v-nd$, $U-Md$, $V-Nd$. This obviously leaves $Um-Mu$ unaltered on the left; on the right u, v, U, V do not appear and so equation (17) still holds. This changed origin is in harmony with the conventions of § 1, namely that the origin in any medium is at the vertex of the surface by which the ray enters that medium, and it will be adhered to consistently from this on.

We may therefore write on the right of equation (17),

$$m' = P'm_0+Q'u_0, \quad u' = R'm_0+S'u_0,$$

$$n' = P'n_0+Q'v_0, \quad v' = R'n_0+S'v_0,$$

and on the left $M = PM_0+QU_0, \quad U = RM_0+SU_0,$

$$m = Pm_0+Qu_0+\tfrac{1}{2}\{(Am_0+Bu_0)(m_0^2+n_0^2)+2(Cm_0+Du_0)(m_0u_0+n_0v_0) + (Em_0+Fu_0)(u_0^2+v_0^2)\},$$

$$u = Rm_0+Su_0+\tfrac{1}{2}\{(Gm_0+Hu_0)(m_0^2+n_0^2)+2(Im_0+Ju_0)(m_0u_0+n_0v_0) + (Km_0+Lu_0)(u_0^2+v_0^2)\}.$$

On the left the terms linear in m_0, u_0 are

$$\Delta\mu(M_0u_0-U_0m_0)(PS-QR)$$
$$= \Delta\mu_0(M_0u_0-U_0m_0) \quad [\text{since } PS-QR = \mu_0/\mu \text{ by equation (4)}]$$
$$= 0 \quad (\text{from the definition of } \Delta).$$

For the terms of the third degree in m_0, n_0, u_0, v_0 we need to remark that the coefficient

of n_0^2 is $\tfrac{1}{2}m_0\Delta\mu\{M_0(AR-PG)+U_0(AS-QG)\} + \tfrac{1}{2}u_0\Delta\mu\{M_0(BR-PH)+U_0(BS-QH)\},$

of n_0v_0 is $\tfrac{1}{2}m_0\Delta\mu\{M_0(CR-PI)+U_0(CS-QI)\} + \tfrac{1}{2}u_0\Delta\mu\{M_0(DR-PJ)+U_0(DS-QJ)\},$

of v_0^2 is $\tfrac{1}{2}m_0\Delta\mu\{M_0(ER-PK)+U_0(ES-QK)\} + \tfrac{1}{2}u_0\Delta\mu\{M_0(FR-PL)+U_0(FS-QL)\}.$

Now, on the right of (17),

$$(m'-m)(\mu m'-\mu' m)$$
$$= \{(P'-P)m_0+(Q'-Q)u_0\}\{(\mu P'-\mu' P)m_0+(\mu Q'-\mu' Q)u_0\},$$

with a similar result for $(n'-n)(\mu n'-\mu' n)$. Hence

$$(m'-m)(\mu m'-\mu' m)+(n'-n)(\mu n'-\mu' n)$$
$$= (P'-P)(\mu P'-\mu' P)(m_0^2+n_0^2) + \{(P'-P)(\mu Q'-\mu' Q)+(Q'-Q)(\mu P'-\mu' P)\}(m_0u_0+n_0v_0) + (Q'-Q)(\mu Q'-\mu' Q)(u_0^2+v_0^2).$$

Also $m'v'-n'u' = (m_0v_0-n_0u_0)(P'S'-Q'R') = (\mu_0/\mu')(m_0v_0-n_0u_0)$,

by equation (4), and

$$\mu'n'-\mu n = (\mu'P'-\mu P)n_0+(\mu'Q'-\mu Q)\,v_0.$$

Thus on the right-hand side the coefficient

of n_0^2 is $\quad \frac{1}{2}(M'-M)\{\mu'\mu(\mu'-\mu)^{-2}u'(P'-P)(\mu P'-\mu' P)$

$$+\mu_0(\mu'-\mu)^{-1}(\mu'P'-\mu P)\,\mu_0\};$$

of n_0v_0 is

$$\tfrac{1}{2}(M'-M)\{\mu'\mu(\mu'-\mu)^{-2}u'[(P'-P)(\mu Q'-\mu' Q)+(Q'-Q)(\mu P'-\mu' P)]$$
$$+\mu_0(\mu'-\mu)^{-1}[-(\mu'P'-\mu P)\,m_0+(\mu'Q'-\mu Q)u_0]\};$$

of v_0^2 is $\quad \frac{1}{2}(M'-M)\{\mu'\mu(\mu'-\mu)^{-2}\,u'(Q'-Q)(\mu Q'-\mu' Q)$

$$-\mu_0(\mu'-\mu)^{-1}(\mu'Q'-\mu Q)\,m_0\}.$$

Now of equations (15) there still holds with the changed origin

$$(m'+u'\rho)/\mu = -(m'-m)/(\mu'-\mu),$$

i.e. equating therein coefficients of m_0 and u_0,

$$(\mu'-\mu)(P'+R'\rho)+\mu(P'-P) = 0,$$
$$(\mu'-\mu)(Q'+S'\rho)+\mu(Q'-Q) = 0,$$

i.e. $\qquad (\mu'P'-\mu P)+(\mu'-\mu)R'\rho = 0,$

and $\qquad (\mu'Q'-\mu Q)+(\mu'-\mu)S'\rho = 0.$

It follows that

$$\mu(PQ'-P'Q)+(\mu'-\mu)(P'S'-Q'R')\,\rho = 0,$$

i.e.
$$\mu\mu'(PQ'-P'Q)+(\mu'-\mu)\,\mu_0\rho = 0. \tag{18}$$

Hence
$$\mu_0(\mu'P'-\mu P) = \mu'\mu(PQ'-P'Q)\,R'$$

and
$$\mu_0(\mu'Q'-\mu Q) = \mu'\mu(PQ'-P'Q)\,S' \Big\}. \tag{19}$$

But $\quad (\mu'-\mu)(PQ'-P'Q) = (P'-P)(\mu Q'-\mu' Q)-(Q'-Q)(\mu P'-\mu' P).$

The coefficient of n_0v_0 on the right may therefore be written in the form

$$\tfrac{1}{2}(M'-M)\,\mu'\mu(\mu'-\mu)^{-2}\{u'[(P'-P)(\mu Q'-\mu' Q)+(Q'-Q)(\mu P'-\mu' P)]$$
$$+(-R'm_0+S'u_0)[(P'-P)(\mu Q'-\mu' Q)-(Q'-Q)(\mu P'-\mu' P)]\}.$$

But, since $w' = R'm_0 + S'u_0$, this is equal to

$$(M'-M)\mu'\mu(\mu'-\mu)^{-2}\{m_0R'(Q'-Q)(\mu P'-\mu'P)+u_0S'(P'-P)(\mu Q'-\mu'Q)\}.$$

Replacing $M'-M$ by its equivalent $(P'-P)M_0+(Q'-Q)U_0$ and equating, in the coefficients of n_0^2, n_0v_0, v_0^2 on the two sides, the coefficients of M_0m_0, U_0m_0, M_0u_0, U_0u_0, we have twelve equations expressing

$$\Delta\mu(AR-PG),\ \ldots,\ \Delta\mu(FS-QL)$$

in terms of P, P', Q, Q', R', S'.

If now we add these results for all the subsidiary systems, the Δ is removed from the left and on the right Σ is inserted, the summation being taken for all the subsidiary systems. The twelve equations are found to be precisely those stated at the end of § 2, when we remember that μ, μ' have been written for μ_{n-1}, μ_n, and make the purely algebraical change which turns $\mu'\mu(\mu'-\mu)^{-2}(\mu P'-\mu'P)$ into

$$(P'/\mu'-P/\mu)(1/\mu'-1/\mu)^{-2},\ \&c.$$

They are, in fact,

$$\left.\begin{aligned}
\mu(AR-PG) &= \Sigma R'(P'/\mu'-P/\mu)(P'-P)^2(1/\mu'-1/\mu)^{-2}\\
\mu(AS-QG) &= \Sigma R'(P'/\mu'-P/\mu)(P'-P)(Q'-Q)(1/\mu'-1/\mu)^{-2}\\
\mu(BR-PH) &= \Sigma S'(P'/\mu'-P/\mu)(P'-P)^2(1/\mu'-1/\mu)^{-2}\\
&\qquad +\mu_0\Sigma(P'-P)(P'\mu'-P\mu)(\mu'-\mu)^{-1}\\
\mu(BS-QH) &= \Sigma S'(P'/\mu'-P/\mu)(P'-P)(Q'-Q)(1/\mu'-1/\mu)^{-2}\\
&\qquad +\mu_0\Sigma(Q'-Q)(P'\mu'-P\mu)(\mu'-\mu)^{-1}\\
\mu(CR-PI) &= \Sigma R'(P'/\mu'-P/\mu)(P'-P)(Q'-Q)(1/\mu'-1/\mu)^{-2}\\
\mu(CS-QI) &= \Sigma R'(P'/\mu'-P/\mu)(Q'-Q)^2(1/\mu'-1/\mu)^{-2}\\
\mu(DR-PJ) &= \Sigma S'(Q'/\mu'-Q/\mu)(P'-P)^2(1/\mu'-1/\mu)^{-2}\\
\mu(DS-QJ) &= \Sigma S'(Q'/\mu'-Q/\mu)(P'-P)(Q'-Q)(1/\mu'-1/\mu)^{-2}\\
\mu(ER-PK) &= \Sigma R'(Q'/\mu'-Q/\mu)(P'-P)(Q'-Q)(1/\mu'-1/\mu)^{-2}\\
&\qquad -\mu_0\Sigma(P'-P)(Q'\mu'-Q\mu)(\mu'-\mu)^{-1}\\
\mu(ES-QK) &= \Sigma R'(Q'/\mu'-Q/\mu)(Q'-Q)^2(1/\mu'-1/\mu)^{-2}\\
&\qquad -\mu_0\Sigma(Q'-Q)(Q'\mu'-Q\mu)(\mu'-\mu)^{-1}\\
\mu(FR-PL) &= \Sigma S'(Q'/\mu'-Q/\mu)(P'-P)(Q'-Q)(1/\mu'-1/\mu)^{-2}\\
\mu(FS-QL) &= \Sigma S'(Q'/\mu'-Q/\mu)(Q'-Q)^2(1/\mu'-1/\mu)^{-2}
\end{aligned}\right\}.\quad(19a)$$

I find it convenient here and on to the end of § 4 to use unaccented letters $P, Q, A, B, \ldots$ *outside* the sign of summation Σ to refer to the complete system and accented and unaccented letters *within* the sign of summation Σ to denote respectively $P_n, Q_n, A_n, B_n, \ldots$, and $P_{n-1}, Q_{n-1}, A_{n-1}, B_{n-1}, \ldots$. I think it need not lead to any confusion and considerable simplicity in setting is thereby secured.

4. From these equations the five fundamental identities stated without proof in [P.M. 6, (17), (18)] can be readily determined.

It is at once evident that

$$AS-QG = CR-PI,$$

and

$$DS-QJ = FR-PL.$$

Again, from equations (19), we see that

$$S'(\mu'P'-\mu P) = R'(\mu'Q'-\mu Q),$$

i.e.

$$\mu(PS'-QR') = \mu'(P'S'-Q'R') = \mu_0. \tag{20}$$

Then

$$\begin{aligned}
&\mu(AS-QG)-\mu(BR-PH)\\
&\quad = \Sigma(1/\mu'-1/\mu)^{-2}(P'/\mu'-P/\mu)(P'-P)\{R'(Q'-Q)-S'(P'-P)\}\\
&\qquad\qquad -\mu_0\Sigma(\mu'-\mu)^{-1}(P'-P)(P'\mu'-P\mu)\\
&\quad = \mu_0\Sigma(P'-P)\{(1/\mu-1/\mu')^{-1}(P/\mu'-P/\mu)-(\mu'-\mu)^{-1}(P'\mu'-P\mu)\}\\
&\quad = -\mu_0\Sigma(P'-P)(P'+P) = -\mu_0\Sigma(P'^2-P^2) = -\mu_0(P^2-P_0^2),
\end{aligned}$$

the alternate terms cancelling; since also $P_0 = 1$, we have

$$AS-QG-BR+PH = -(P^2-1)\mu_0/\mu.$$

So $\mu(ES-QK)-\mu(FR-PL)$

$$\begin{aligned}
&= \Sigma(1/\mu'-1/\mu)^{-2}(Q'/\mu'-Q/\mu)(Q'-Q)\{R'(Q'-Q)-S'(P'-P)\}\\
&\qquad\qquad -\mu_0\Sigma(\mu'-\mu)^{-1}(Q'-Q)(Q'\mu'-Q\mu)\\
&= \mu_0\Sigma(Q'-Q)\{1/\mu-1/\mu')^{-1}(Q'/\mu'-Q/\mu)-(\mu'-\mu)^{-1}(Q'\mu'-Q\mu)\}\\
&= -\mu_0\Sigma(Q'^2-Q^2)\\
&= -\mu_0 Q^2,
\end{aligned}$$

since evidently $Q_0 = 0.$

Again,

$$\mu(BS-QH)-\mu(DR-PJ)$$
$$=\Sigma(1/\mu'-1/\mu)^{-2}S'(P'-P)\{(Q'-Q)(P'/\mu'-P/\mu)-(P'-P)(Q'/\mu'-Q/\mu)\}$$
$$+\mu_0\Sigma(\mu'-\mu)^{-1}(Q'-Q)(P'\mu'-P\mu).$$

But $(Q'-Q)(P'/\mu'-P/\mu)-(P'-P)(Q'/\mu'-Q/\mu)=(1/\mu'-1/\mu)(PQ'-P'Q)$,

and, by (19),

$$(\mu'-\mu)^{-1}\mu_0(P'\mu'-P\mu)=-(1/\mu'-1/\mu)R'(PQ'-P'Q).$$

Hence $\mu(BS-QH)-\mu(DR-PJ)$

$$=\Sigma(1/\mu'-1/\mu)^{-1}(PQ'-P'Q)\{S'(P'-P)-R'(Q'-Q)\}$$
$$=\mu_0\Sigma(PQ'-P'Q),\text{ by (20).}$$

But, by (18), $$PQ'-P'Q=-(\mu'-\mu)\mu_0\rho/\mu'\mu$$
$$=-\mu_0\kappa/\mu'\mu.$$

Now $\Sigma\kappa/\mu'\mu$, *i.e.* $\Sigma\kappa_n/\mu_{n-1}\mu_n$ is the Petzval Sum, usually denoted by the symbol π.

We have therefore that

$$BS-QH-DR+PJ=-\pi\mu_0^2/\mu.$$

Further

$$\mu(CS-QI)-\mu(ER-PK)$$
$$=\Sigma(1/\mu'-1/\mu)^{-2}R'(Q'-Q)\{(P'/\mu'-P/\mu)(Q'-Q)-(Q'/\mu'-Q/\mu)(P'-P)\}$$
$$+\mu_0\Sigma(\mu'-\mu)^{-1}(P'-P)(Q'\mu'-Q\mu)$$
$$=-\Sigma(1/\mu'-1/\mu)^{-1}(PQ'-P'Q)\{R'(Q'-Q)-S'(P'-P)\}$$
$$=\mu_0\Sigma(PQ'-P'Q)=\mu_0^2\pi.$$

Hence also

$$CS-QI-ER+PK+BS-QH-DR+PJ=0.$$

Then

$$\mu(CS-QI)-\mu(DR-PJ)$$
$$=\mu(ER-PK)-\mu(BS-QH)$$
$$=\Sigma(1/\mu'-1/\mu)^{-2}(P'-P)(Q'-Q)\{R'(Q'/\mu'-Q/\mu)-S'(P'/\mu'-P/\mu)\}$$
$$-\mu_0\Sigma(\mu'-\mu)^{-1}\{(P'-P)(Q'\mu'-Q\mu)+(Q'-Q)(P'\mu'-P\mu)\}$$

But $S'(P'/\mu'-P/\mu)-R'(Q'/\mu'-Q/\mu)$

$$= (P'S'-Q'R')/\mu'-(PS'-QR')/\mu$$

$$= \mu_0/\mu'^2-\mu_0/\mu^2, \text{ by (20).}$$

Thus

$$\mu(CS-QI)-\mu(DR-PJ)$$

$$= \mu_0\Sigma(\mu'-\mu)^{-1}\{(\mu'+\mu)(P'-P)(Q'-Q)-(P'-P)(Q'\mu'-Q\mu)$$
$$-(Q'-Q)(P'\mu'-P\mu)\}$$

$$= -\mu_0\Sigma(P'Q'-PQ) \text{ (on reduction)}$$

$$= -\mu_0 PQ,$$

since $$P_0Q_0 = 0.$$

Collecting these results we can write the seven fundamental identities as

$$\left.\begin{aligned} AS-QG-BR+PH &= -(P^2-1)\mu_0/\mu \\ CS-QI-DR+PJ &= -PQ\mu_0/\mu \\ ES-QK-FR+PL &= -Q^2\mu_0/\mu \\ AS-QG-CR+PI &= 0 \\ BS-QH-DR+PJ &= -\pi\mu_0^2/\mu \\ CS-QI-ER+PK &= +\pi\mu_0^2/\mu \\ DS-QJ-FR+PL &= 0 \end{aligned}\right\}. \qquad (21)$$

Now, from the first and fourth of these identities,

$$-(B-C)R+P(H-I) = -(P^2-1)\mu_0/\mu;$$

from the second and fifth,

$$(B-C)S-Q(H-I) = -\pi\mu_0^2/\mu+PQ\mu_0/\mu.$$

If we multiply these equations by Q and P, and add, we get,

since $$PS-QR = \mu_0/\mu,$$

that $$B-C = Q-P\pi\mu_0;$$

so, if we multiply by S and R and add, we get

$$H-I = S-P\mu_0/\mu-R\pi\mu_0.$$

Similarly, from the third and seventh identities,

$$(D-E)\,S-Q\,(J-K) = Q^2\mu_0/\mu\,;$$

and from the second and sixth,

$$-(D-E)\,R+P(J-K) = -PQ\mu_0/\mu-\pi\mu_0^2/\mu.$$

Thus

$$D-E = -Q\pi\mu_0,$$

$$J-K = -Q\mu_0/\mu-S\pi\mu_0.$$

We may therefore replace any four identities of (21) by the more convenient forms

$$\left.\begin{aligned} B-C &= Q-P\pi\mu_0 \\ D-E &= -Q\pi\mu_0 \\ H-I &= S-P\mu_0/\mu-R\pi\mu_0 \\ J-K &= -Q\mu_0/\mu-S\pi\mu_0 \end{aligned}\right\}. \tag{22}$$

5. I wish to conclude this paper by establishing certain results relative to the effect of differentiating the aberration coefficients $A, B, \ldots, K, L$, with respect to the power of either end-surface of the system, *i.e.* with respect to κ_1 and κ_n respectively. I suppose now for convenience that there are n surfaces in all. The corresponding relations for the first order coefficients are of course standard results of optical theory. They are, in fact,

$$\left.\begin{aligned} \partial P/\partial\kappa_1 &= 0, & \partial Q/\partial\kappa_1 &= -P/\mu_0 \\ \partial R/\partial\kappa_1 &= 0, & \partial S/\partial\kappa_1 &= -R/\mu_0 \\ \partial P/\partial\kappa_n &= -R/\mu_n, & \partial Q/\partial\kappa_n &= -S/\mu_n \\ \partial R/\partial\kappa_n &= 0, & \partial S/\partial\kappa_n &= 0 \end{aligned}\right\}, \tag{23}$$

and

as is evident in virtue of equations (3), and the fact that K, the power of the system, is linear in the power of every constituent surface.

Now in the summation formulæ (19a) it is evident that κ_n enters the

twelve expressions on the right only through the last term of the summation, and there only through the quantities P_n and Q_n.

Thus $$\mu'(A'R'-P'G')-(1/\mu'-1/\mu)^{-2}R'(P'-P)(P'/\mu'-P/\mu), \tag{24}$$

and $$\mu'(A'S'-Q'G')-(1/\mu'-1/\mu)^{-2}R'(Q'-Q)(P'-P)(P'/\mu'-P/\mu)$$

are free of κ_n.

Since S', R' do not involve κ_n, as appears from equations (23), we may multiply the expressions (24) by S' and R' and take the difference, obtaining that

$$\mu'(P'S'-Q'R')G'$$
$$+(1/\mu'-1/\mu)^{-2}R'(P'-P)(P'/\mu'-P/\mu)\{P'S'-Q'R'-PS'+QR'\},$$

is free of κ_n.

Using equations (20) this gives that

$$G'+(1/\mu'-1/\mu)^{-1}R'(P'-P)(P'/\mu'-P/\mu)$$

is free of κ_n.

More conveniently we may write that

$$G'-(\mu'-\mu)^{-1}R'(P'-P)(P'\mu-P\mu') \tag{25}$$

is free of κ_n.

Similarly the other summation formulæ enable it to be shown that

$$\left.\begin{aligned} &H'-(\mu'-\mu)^{-1}S'(P'-P)(P'\mu-P\mu')+\mu_0\kappa_n R'/\mu'\mu \\ &I'-(\mu'-\mu)^{-1}R'(Q'-Q)(P'\mu-P\mu') \\ &J'-(\mu'-\mu)^{-1}S'(P'-P)(Q'\mu-Q\mu') \\ &K'-(\mu'-\mu)^{-1}R'(Q'-Q)(Q'\mu-Q\mu')-\mu_0\kappa_n S'/\mu'\mu \\ &L'-(\mu'-\mu)^{-1}S'(Q'-Q)(Q'\mu-Q\mu') \end{aligned}\right\} \tag{26}$$

are all free of κ_n.

Again, if we multiply the expressions (24) by Q and P, which do not involve κ_n, and take the difference, we see that

$$\mu'(PS'-QR')A'-\mu'(PQ'-P'Q)G'$$
$$-(1/\mu'-1/\mu)^{-2}R'(P'-P)(P'/\mu'-P/\mu)(PQ'-P'Q)$$

is free of κ_n.

That is to say, in virtue of equations (18) and (20),

$$\mu'A'+\kappa_n G'+(1/\mu'-1/\mu)^{-2}R'(P'-P)(P'/\mu'-P/\mu)\kappa_n/\mu'$$

is free of κ_n, *i.e.*

$$A'+G'\kappa_n/\mu'+(\mu'-\mu)^{-2}\,R'(P'-P)(P'\mu-P\mu')\,\mu\kappa_n/\mu' \qquad (27)$$

is free of κ_n.

Similarly it can be shown that

$$\left.\begin{aligned}
&B'+H'\kappa_n/\mu'+(\mu'-\mu)^{-2}\,S'(P'-P)(P'\mu-P\mu')\,\mu\kappa_n/\mu'-\mu_0^2(\mu'-\mu)^{-1}\,\kappa_n R'/\mu'\mu\\
&C'+I'\kappa_n/\mu'+(\mu'-\mu)^{-2}\,R'(Q'-Q)(P'\mu-P\mu')\,\mu\kappa_n/\mu'\\
&D'+J'\kappa_n/\mu'+(\mu'-\mu)^{-2}\,S'(P'-P)(Q'\mu-Q\mu')\,\mu\kappa_n/\mu'\\
&E'+K'\kappa_n/\mu'+(\mu'-\mu)^{-2}\,R'(Q'-Q)(Q'\mu-Q\mu')\,\mu\kappa_n/\mu'+\mu_0^2(\mu'-\mu)^{-1}\,\kappa_n S'/\mu'\mu\\
&F'+L'\kappa_n/\mu'+(\mu'-\mu)^{-2}\,S'(Q'-Q)(Q'\mu-Q\mu')\,\mu\kappa_n/\mu'
\end{aligned}\right\} \qquad (28)$$

are all free of κ_n.

Since the expressions (25), (26), (27), (28) do not involve κ_n, the derivatives of the aberration coefficients $A, B, \ldots, K, L$ are easily found to be the following.

[I drop the accents, since the coefficients now refer only to the complete system.]

$$\left.\begin{aligned}
\partial A/\partial\kappa_n &= -G/\mu_n-P^2R\theta_n-3PR^2\phi_n-R^3\psi_n\\
\partial B/\partial\kappa_n &= -H/\mu_n-P^2S\theta_n-(2PRS+QR^2)\phi_n-R^2S\psi_n\\
\partial C/\partial\kappa_n &= -I/\mu_n-PQR\theta_n-(2PRS+QR^2)\phi_n-R^2S\psi_n\\
\partial D/\partial\kappa_n &= -J/\mu_n-PQS\theta_n-(2QRS+PS^2)\phi_n-RS^2\psi_n\\
\partial E/\partial\kappa_n &= -K/\mu_n-Q^2R\theta_n-(2QRS+PS^2)\phi_n-RS^2\psi_n\\
\partial F/\partial\kappa_n &= -L/\mu_n-Q^2S\theta_n-3QS^2\phi_n-S^3\psi_n\\
\partial G/\partial\kappa_n &= PR^2/\mu_n+PR^2\theta_n+R^3\phi_n\\
\partial H/\partial\kappa_n &= (2PRS-QR^2)/\mu_n+QR^2\theta_n+R^2S\phi_n\\
\partial I/\partial\kappa_n &= QR^2/\mu_n+PRS\theta_n+R^2S\phi_n\\
\partial J/\partial\kappa_n &= PS^2/\mu_n+QRS\theta_n+RS^2\phi_n\\
\partial K/\partial\kappa_n &= (2QRS-PS^2)/\mu_n+PS^2\theta_n+RS^2\phi_n\\
\partial L/\partial\kappa_n &= QS^2/\mu_n+QS^2\theta_n+S^3\phi_n
\end{aligned}\right\}, \qquad (29)$$

where

$$\left.\begin{aligned}\theta_n &\equiv 1/\mu_n+1/\mu_{n-1}\\ \phi_n &\equiv \kappa_n(\mu_n+\mu_{n-1})/\mu_n\mu_{n-1}(\mu_n-\mu_{n-1})\\ \psi_n &\equiv \kappa_n^2(2\mu_n+\mu_{n-1})/\mu_n\mu_{n-1}(\mu_n-\mu_{n-1})^2\end{aligned}\right\}. \qquad (30)$$

If we multiply the expression (25) by $\mu\rho_n/\mu'$, *i.e.* by $\mu\kappa_n/\mu'(\mu'-\mu)$, and add it to expression (27), we see that $A'+G'\rho_n$ is *linear* in κ_n, for

$$\begin{aligned}G'\kappa_n/\mu'+G'\mu\rho_n/\mu' &= G'\{(\mu'-\mu)\rho_n+\mu\rho_n\}/\mu'\\ &= G'\rho_n.\end{aligned}$$

Similarly (25) and (28) give that

$$B'+H'\rho_n, \quad C'+I'\rho_n, \quad D'+J'\rho_n, \quad E'+K'\rho_n, \quad \text{and} \quad F+L\rho_n,$$

are linear in the power of the last surface. This result is, of course, bound up with the fact that the centre of a sphere is an aplanatic point for refraction at the surface of the sphere.

For differentiation with respect to κ_1 we need to remember that κ_1 is the first surface of every subsidiary system, and that hence the first four equations of (19), namely

$$\left.\begin{aligned}\partial P/\partial\kappa_1 = 0, \quad \partial Q/\partial\kappa_1 &= -P/\mu_0\\ \partial R/\partial\kappa_1 = 0, \quad \partial S/\partial\kappa_1 &= -R/\mu_0\end{aligned}\right\},$$

hold for every subsidiary system. We must however make an exception in the case of the coefficients P_0, Q_0, R_0, S_0 which occur in the first terms of the summation formulæ, for since, evidently,

$$m_0 = P_0m_0+Q_0u_0 \quad \text{and} \quad u_0 = R_0m_0+S_0u_0,$$

it is seen that $P_0 = 1 = S_0$,

and that $Q_0 = 0 = R_0$.

The equation $\partial Q/\partial\kappa_1 = -P/\mu_0$

is thus not satisfied for Q_0, P_0.

I prefer therefore to write the summation formulæ for the present in the form

$$\begin{aligned}\mu(FS-QL) = \Sigma_2 S_n(Q_n/\mu_n-Q_{n-1}/\mu_{n-1})(Q_n-Q_{n-1})^2(1/\mu_n-1/\mu_{n-1})^{-2}\\ +S_1(Q_1/\mu_1-Q_0/\mu_0)(Q_1-Q_0)^2(1/\mu_1-1/\mu_0)^{-2},\end{aligned}$$

with similar equations for the other eleven formulæ, the summation extending as indicated by the suffix from the second subsidiary system to the last.

Since $Q_1 = -\kappa_1/\mu_1$ and $S_1 = 1$, the term on the right outside the sign of summation equals $-\kappa_1^3\mu_0^2/(\mu_1-\mu_0)^2\mu_1^2$.

Returning to the notation $Q' \equiv Q_n$, $Q \equiv Q_{n-1}$, etc., we have

$$\begin{aligned}\mu_0\mu_n\partial(FS-QL)/\partial\kappa_1 = &-\Sigma_2 R'(Q'/\mu'-Q/\mu)(Q'-Q)^2/(1/\mu'-1/\mu)^2\\ &-2\Sigma_2 S'(Q'/\mu'-Q/\mu)(Q'-Q)(P'-P)/(1/\mu'-1/\mu)^2\\ &-\Sigma_2 S'(P'/\mu'-P/\mu)(Q'-Q)^2/(1/\mu'-1/\mu)^2\\ &-3\kappa_1^2\mu_0^3/(\mu_1-\mu_0)^2\mu_1^2.\end{aligned}$$

Now

$$\begin{aligned}&R'(Q'/\mu'-Q/\mu)(Q'-Q)^2+S'(P'/\mu'-P/\mu)(Q'-Q)^2\\ &\quad=2S'(Q'/\mu'-Q/\mu)(Q'-Q)(P'-P)\\ &\qquad+(Q'-Q)\{(Q'/\mu'-Q/\mu)(R'Q'-P'S'-R'Q+PS')\\ &\qquad\qquad+S'(P'Q-PQ')(1/\mu-1/\mu')\}\\ &\quad=2S'(Q'/\mu'-Q/\mu)(Q'-Q)(P'-P)\\ &\qquad+(Q'-Q)(1/\mu-1/\mu')\mu_0\{(Q'/\mu'-Q/\mu)-(Q'/\mu-Q/\mu')\}\\ &\quad=2S'(Q'/\mu'-Q/\mu)(Q'-Q)(P'-P)-\mu_0(1/\mu-1/\mu')^2(Q'-Q)(Q'+Q).\end{aligned}$$

Thus $\mu_0\mu_n\partial(FS-QL)/\partial\kappa_1$

$$\begin{aligned}&=-4\Sigma_2 S'(Q'/\mu'-Q/\mu)(Q'-Q)(P'-P)/(1/\mu'-1/\mu)^2\\ &\quad+\mu_0\Sigma_2(Q'^2-Q^2)-3\kappa_1^2\mu_0^3/(\mu_1-\mu_0)^2\mu_1^2\\ &=-4\mu_n(FR-PL)+\mu_0(Q^2-Q_1^2)\\ &\quad+4S_1(Q_1/\mu_1-Q_0/\mu_0)(Q_1-Q_0)(P_1-P_0)/(1/\mu_1-1/\mu_0)^2\\ &\quad-3\kappa_1^2\mu_0^3/(\mu_1-\mu_0)^2\mu_1^2\\ &=-4\mu_n(FR-PL)+\mu_0 Q^2-\mu_0(2\mu_0+\mu_1)\kappa_1^2/\mu_1(\mu_1-\mu_0)^2.\end{aligned}$$

In analogy with θ_n, ϕ_n, ψ_n defined in equations (30), it is convenient to define

$$\theta_0 \equiv 1/\mu_0+1/\mu_1,$$

$$\phi_0 \equiv \kappa_1(\mu_0+\mu_1)/\mu_0\mu_1(\mu_0-\mu_1),$$

$$\psi_0 \equiv \kappa_1^2(2\mu_0+\mu_1)/\mu_0\mu_1(\mu_0-\mu_1)^2.$$

Then we may write

$$\left.\begin{aligned}
&\partial(FS-QL)/\partial\kappa_1 = -4(FR-PL)/\mu_0+Q^2/\mu_n-\mu_0\psi_0/\mu_n.\\
&\text{In similar fashion it can be proved that}\\
&\partial(FR-PL)/\partial\kappa_1 = -2(CS-QI)/\mu_0-(ER-PK)/\mu_0-2PQ/\mu_n+\mu_0\phi_0/\mu_n\\
&\partial(ES-QK)/\partial\kappa_1 = -2(CS-QI)/\mu_0-(ER-PK)/\mu_0+\mu_0\phi_0/\mu_n\\
&\partial(ER-PK)/\partial\kappa_1 = -2(CR-PI)/\mu_0-\mu_0/\mu_1\mu_n\\
&\partial(DS-QJ)/\partial\kappa_1 = -2(CS-QI)/\mu_0-(ER-PK)/\mu_0-2PQ/\mu_n+\mu_0\phi_0/\mu_n\\
&\partial(DR-PJ)/\partial\kappa_1 = -2(CR-PI)/\mu_0-P^2/\mu_n\\
&\partial(CS-QI)/\partial\kappa_1 = -2(AS-QG)/\mu_0\\
&\partial(CR-PI)/\partial\kappa_1 = -(AR-PG)/\mu_0\\
&\partial(BS-QH)/\partial\kappa_1 = -2(AS-QG)/\mu_0-\mu_0/\mu_1\mu_n-P^2/\mu_n\\
&\partial(BR-PH)/\partial\kappa_1 = -(AR-PG)/\mu_0\\
&\partial(AS-QG)/\partial\kappa_1 = -(AR-PG)/\mu_0\\
&\partial(AR-PG)/\partial\kappa_1 = 0
\end{aligned}\right\} \qquad (31)$$

It is to be observed that these equations show that a knowledge of $FS-QL$ and of its first, second, third, and fourth derivatives with respect to κ_1 is sufficient in conjunction with the seven fundamental equations (21) to determine the twelve aberration coefficients $A, B, \ldots, K, L$. It is, however, easier (as I hope to have subsequent occasion to explain) to determine the coefficients by means of the quantity $FP-DQ$ and its derivatives since these possess a certain symmetry absent in $FS-QL$.

From equations (31) we at once deduce that

$$\left.\begin{aligned}
\partial A/\partial\kappa_1 &= 0\\
\partial B/\partial\kappa_1 &= -A/\mu_0 - P\theta_0\\
\partial C/\partial\kappa_1 &= -A/\mu_0\\
\partial D/\partial\kappa_1 &= -2C/\mu_0 + P\pi + P\phi_0\\
\partial E/\partial\kappa_1 &= -2C/\mu_0 + Q/\mu_1 + P\phi_0\\
\partial F/\partial\kappa_1 &= -3E/\mu_0 + 2Q\pi - Q\phi_0 - P\psi_0\\
\partial G/\partial\kappa_1 &= 0\\
\partial H/\partial\kappa_1 &= -G/\mu_0 - R\theta_0\\
\partial I/\partial\kappa_1 &= -G/\mu_0\\
\partial J/\partial\kappa_1 &= -2I/\mu_0 + P/\mu_n + R\pi + R\phi_0\\
\partial K/\partial\kappa_1 &= -2I/\mu_0 + S/\mu_1 + R\phi_0\\
\partial L/\partial\kappa_1 &= -3K/\mu_0 + 2Q/\mu_n + 2S\pi - S\phi_0 - R\psi_0
\end{aligned}\right\} \qquad (32)$$

The results obtained in this concluding § 5 are not without their practical application, since it is at times desirable to ascertain the effect on certain aberrations of varying the curvatures of the end-surfaces.

GROUPS INVOLVING THREE AND ONLY THREE OPERATORS WHICH ARE SQUARE

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1. *Introduction.*

If a group G involves only one operator which is a square of operators of G, then G is evidently the Abelian group of order 2^m and of type $(1, 1, 1, \ldots)$. A determination of all the groups characterized by the fact that two and only two of the operators of each group are squares of other operators of the group, was recently communicated by the present writer to the National Academy of Sciences for publication in its *Proceedings*. In the present paper a similar determination is made of the finite groups characterized by the fact that three and only three of the operators of each group are squares of operators of the group. In what follows the symbol G stands for such a group.

When the order g of G is divisible by an odd prime number, this number must be 3, since otherwise G would involve a cyclic group containing more than three operators which are squares. When g is divisible by an odd prime number, g must therefore be of the form $3 \cdot 2^m$. Moreover, G can involve only one sub-group of order 3, and its sub-group or sub-groups of order 2^m must be Abelian and of type $(1, 1, 1, \ldots)$. If G contains only one sub-group of order 2^m, it is the direct product of the group of order 3 and the Abelian group of order 2^m and of type $(1, 1, 1, \ldots)$.*

In the case when G contains three Sylow sub-groups of order 2^m, these sub-groups are transformed under G according to the symmetric group of degree 3. Hence the cross-cut of these sub-groups is of order 2^{m-1}, and this cross-cut is also the central of G. Exactly one-third of the operators of G have therefore orders which exceed 2, and G is known to be the direct product of the symmetric group of order 6 and the Abelian

* Cf. W. Burnside, *Theory of Groups of Finite Order*, 1911, p 44.

group of order 2^{m-1} and of type (1, 1, 1, ...).* Hence it has been proved that, *if a group whose order is not of the form* 2^m *involves three and only three operators which are squares under the group, then it is the direct product of either the group of order* 3 *or the symmetric group of order* 6, *and an Abelian group of order* 2^a *and of type* (1, 1, 1, ...).

2. *Groups of Order* 2^m.

A group G of order 2^m involving three and only three operators which are squares cannot involve an operator of order 8. Moreover, G cannot be Abelian, since an Abelian group of order 2^m which contains more than two square operators must involve at least four such operators. Hence it will be assumed in what follows that G is non-Abelian.

Suppose that the operators of order 2 contained in G generate a non-Abelian sub-group H of G, and let s_1^2 and s_2^2 represent the two distinct operators of order 2 contained in G which are squares of its operators of order 4. Since H is non-Abelian and generated by its operators of order 2, it must involve an operator t_1 of order 2, which is transformed by some other operator of order 2 in H into either $s_1^2 t_1$ or $s_2^2 t_1$, s_1^2 and s_2^2 being necessarily contained in the central of G.

On the other hand, every operator of G which is not found in the invariant sub-group H, *i.e.* every operator in $G-H$, transforms t_1 either into itself or into itself multiplied by $s_1^2 s_2^2$. As this is not in accord with the last sentence of the preceding paragraph, it has been proved that *the totality of the operators of order* 2 *contained in* G *generate either* G *itself or an Abelian sub-group of type* (1, 1, 1, ...).

All the Operators of Order 2 *contained in* G *are Commutative.*

If the sub-group H generated by all the operators of order 2 contained in G is the central of G, the commutator whose elements are two non-commutative operators of G must be either s_1^2 or s_2^2. As an operator of order 4 contained in G is commutative only with the operators found in its own co-set with respect to H, and cannot have more than two conjugates under G, it results that the quotient group G/H is of order 4, and hence the commutator sub-group of G is of order 2.

It may therefore be assumed without loss of generality that G is ob-

* G. A. Miller, *Bulletin of the American Mathematical Society*, Vol. 13 (1907), p. 235.

tained by extending the Abelian group of type (2, 1, 1, ...) by means of an operator of order 4, which transforms each operator of this Abelian group into its inverse, but whose square is not the same as the square of an operator of order 4 found in this Abelian sub-group. The order of G must therefore be at least 16, and when $g > 16$, G is the direct product of this group of order 16 and an Abelian group of order 2^{m-4} and of type (1, 1, 1, ...). In particular, there is one and only one such group of order 2^a, $a \geqslant 4$.

When H is not the central of G it must include the central, since no operator of order 4 can be invariant under G. If this central is of index 2 under H, and the commutator sub-group of G is of order 2, then this sub-group must be 1, $s_1^2 s_2^2$. Hence the order of G is twice the order of H. There is one and only one such group of order 2^m, $m \geqslant 4$, and when $m > 4$ this group must again be the direct product of the group of order 16 which satisfies the given condition and the Abelian group of order 2^{m-4} and of type (1, 1, 1, ...).

If the central of G is of index 2 under H, but the order of the commutator sub-group exceeds 2, this order must be 4. As an operator of order 4 contained in G cannot be commutative with any other operator of this order unless it is found in the same co-set with respect to H, and as G contains operators of order 4 which are commutative with only half of the operators of H, it results that the order of G is four times that of H in the present case. Moreover, G contains an Abelian sub-group of type (2, 1, 1, ...) including H, since all the operators of order 4 in G transform its operators of order 2 either into themselves or into themselves multiplied by $s_1^2 s_2^2$.

As the transformation of this Abelian sub-group by one of the remaining operators of order 4 is completely determined by the given conditions, there is one and only one such group of order 2^m, $m \geqslant 5$. This simply infinite system is again obtained by forming the direct products with the smallest group of the system and an Abelian group of type (1, 1, 1, ...).

Having considered all the possible groups when the central of G is either H or a sub-group of index 2 under H, it remains to see what groups are possible when the central has a larger index under H. It is easy to see that this index can not exceed 4, since the number of co-sets of G with respect to H must be 4, and each operator of order 4 in G must be commutative with one-half of the operators of H. As all the operators of order 4 in G transform each operator of H either into itself or into itself multiplied by $s_1^2 s_2^2$, and it may be assumed without loss of generality that $s_2^{-1} s_1 s_2 = s_1^{-1}$, it results that there is one and only one such group of

order 2^m, $m \geqslant 6$. One-half of the operators of order 4 have s_1^2 for their squares while the squares of the others are equal to s_2^2.

Hence there are four infinite systems of groups of order 2^m composed separately of groups containing three and only three operators which are squares of operators of the group and satisfying the condition that all their operators of order 2 are commutative. In each of these systems there is only one group of lowest order, and all the other groups of the system are the direct products of this minimal group and an Abelian group of type (1, 1, 1, ...). The systems are therefore completely determined by these four minimal groups of orders 16, 16, 32, and 64 respectively. Two of these groups have commutator sub-groups of order 2, while the commutator sub-groups of the other two are of order 4.

The Operators of Order 2 contained in G generate it.

It will first be proved that at least one of the operators of order 2 contained in G has four conjugates under G. If this were not so G would contain two pairs of non-commutative operators of order 2 generating separately two operators of order 4, such that the square of one of these operators would be s_1^2 while that of the other would be s_2^2. These two operators of order 4 would have to be commutative because the given pair of operators which generate the one would have to be separately commutative with each of the given pair generating the other. As this is impossible it results that at least one t_1 of the operators of order 2 contained in G has four conjugates under G.

Let H_1 be the sub-group composed of all the operators of G which are commutative with t_1. The co-set of G with respect to H_1 composed of all the operators which transform t_1 into itself multiplied by s_1^2, must have half of its operators of order 2, while the square of each of the remaining operators is s_1^2. Similarly, half the operators of the co-set which transforms t_1 into $s_2^2 t_1$ are of order 2, while the common square of the rest is s_2^2. The co-set composed of the operators of G which transforms t_1 into $s_1^2 s_2^2 t_1$ contains only operators of order 4, half of which have s_1^2 for their common square.

From these facts it results that when H_1 is generated by its operators of order 2 it must be Abelian and of type (1, 1, 1, ...). In fact, the operators of order 2 in H_1 must transform every operator in the co-set which transforms t_1 into $s_1^2 t_1$ into itself or into itself multiplied by s_1^2. Similar remarks apply to the co-set composed of the operators which transform t_1

into $s_2^2 t_1$. From this it results directly that the operators of order 2 contained in H_1 must generate an Abelian group, since this group would have both the commutator sub-group 1, s_1^2 and 1, s_2^2 if it were non-Abelian.

It will now be proved that H_1 must always be Abelian. If H_1 were non-Abelian the operators of order 4 in H_1 could not transform some of its operators of order 2 into themselves multiplied by $s_1^2 s_2^2$, since all the operators in the co-set which transforms t_1 into $s_1^2 t_1$ could not then transform every operator of order 2 in H_1 either into itself or into itself multiplied by s_1^2. Moreover, the index under H_1 of the sub-group generated by its operators of order 2 cannot exceed 2, since an operator in the co-set of G with respect to H_1 composed of operators of order 4 cannot be commutative with any operator of order 4 in H_1, and is non-commutative with t_1. From these facts it follows that H_1 is Abelian and either of type (1, 1, 1, ...) or of type (2, 1, 1, ...).

If H_1 is of type (1, 1, 1, ...), the central of G must be of index 2 under H_1. Hence the transformation of H_1 by the remaining operators of the group is completely determined. Let t_2 and t_3 represent two operators of order 2 contained in G, and transforming t_1 into $s_1^2 t_1$ and $s_2^2 t_1$ respectively. From the conditions noted above it results that t_3 must transform t_2 either into $s_1^2 t_2$ or into $s_2^2 t_2$. As simply isomorphic groups result under these two transformations, it results that there is one and only one group of order 2^m, $m \geqslant 5$, in which H_1 is of type (1, 1, 1, ...).

When H_1 is of type (2, 1, 1, ...), the central of G is of index 4 under H_1. If we assume that the common square of the operators of order 4 in H_1 is s_1^2, then t_2 is commutative with one-half of the operators of H_1, while t_3 is commutative with only one-fourth of these operators, since t_2 and t_3 must transform into their inverses one-half of the operators of order 4 in H_1. Hence the transformations are again completely determined by the conditions which t_2 and t_3 are required to satisfy. It results therefore that there is one and only one such group of order 2^m $m \geqslant 6$. Each of the two systems just determined contains one and only one group of lowest order, and all of the other groups of the system are the direct products of this minimal group and an Abelian group of type (1, 1, 1, ...).

By combining these results we have the following theorem:—*There are six infinite systems of groups of order* 2^m, *such that each group involves three and only three operators which are squares of operators of*

the group. Each of these systems contains only one group of smallest order, and every other group of the system is the direct product of this smallest non-Abelian group and an Abelian group of type (1, 1, 1, ...). The order of this smallest group varies with the system, ranging from 16 to 64. For every integral value of $m \geqslant 6$, there are therefore exactly six groups of order 2^m which have the property that each group contains exactly three operators which are squares of other operators of the group.

3. *Defining equations of the smallest group in each of the eight possible infinite systems.*

Since all of the groups of each of the possible systems can be constructed by forming the direct products of the smallest group of the system and an Abelian group of order 2^k and of type (1, 1, 1, ...), defining equations of each of the possible groups can be obtained directly from the defining equations of the smallest group of each system. The latter equations are as follows:—

$$s_1^3 = 1,$$

$$s_1^3 = s_2^2 = 1, \quad s_2 s_1 s_2 = s_1^2,$$

$$s_1^4 = s_2^4 = 1, \quad s_2^3 s_1 s_2 = s_1^3,$$

$$s_1^4 = s_2^4 = 1, \quad s_2^3 s_1 s_2 = s_2^2 s_1^3,$$

$$s_1^4 = s_2^4 = s_3^2 = 1, \quad s_2^3 s_1 s_2 = s_1^3, \quad s_2^3 s_3 s_2 = s_1^2 s_2^2 s_3, \quad s_1 s_3 = s_3 s_1,$$

$$\left\{ \begin{array}{l} s_1^4 = s_2^4 = s_3^2 = s_4^2 = 1, \quad s_2^3 s_1 s_2 = s_1^3, \quad s_2^3 s_3 s_2 = s_1^2 s_2^2 s_3, \quad s_2 s_4 = s_4 s_2, \\ s_1 s_3 = s_3 s_1, \quad s_1^3 s_3 s_1 = s_1^2 s_2^2 s_3, \quad s_3 s_4 = s_4 s_3, \end{array} \right.$$

$$s_1^4 = s_2^4 = s_3^2 = 1, \quad s_2^3 s_1 s_2 = s_1^3, \quad s_1^3 s_3 s_1 = s_1^2 s_3, \quad s_2^3 s_3 s_2 = s_2^2 s_3,$$

$$\left\{ \begin{array}{l} s_1^4 = s_2^4 = s_3^2 = s_4^2 = 1, \quad s_2^3 s_1 s_2 = s_1^3, \quad s_2^3 s_3 s_2 = s_2^2 s_3, \quad s_1 s_3 = s_3 s_1, \\ s_1 s_4 = s_4 s_1, \quad s_4 s_3 s_4 = s_1^2 s_3, \quad s_2 s_4 = s_4 s_2. \end{array} \right.$$

ON THE PRODUCT OF SEMI-CONVERGENT SERIES

By T. S. BRODERICK.

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1. It is well known that the product of any number of absolutely convergent series converges absolutely to the product of their sums. But the product series formed from semi-convergent series does not necessarily converge. We know, however (from a proof exactly similar to Abel's proof for the case of two series), that if the product series formed from several semi-convergent series converges, it converges to the product of the sums of these series.*

Conditions for the convergence of the product of *two* semi-convergent series have been deduced by a number of writers,† but there has been no general investigation into the conditions of convergence of the product of *several* semi-convergent series. Cajori has investigated the convergence of the q-th power of some special series, but he does so by the repeated application of a rule applicable to two series.‡ Cajori does not give any general formula for the convergence of the product of several semi-convergent series.

The object of the following investigation is to establish conditions for the convergence of the product of *several* semi-convergent series. In the case of the most general type of series sufficient conditions only are established. In the case where each series is of the type $\Sigma(-)^{n+1}a_n$, a_n being positive, and tending to zero steadily, as n tends to infinity, the necessary and sufficient conditions are deduced. It will be seen that in the case of the product of two series these latter conditions are identical with conditions previously given by Pringsheim.§

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* This result also follows from a general theorem due to Cesàro. See Bromwich's *Infinite Series*, pp. 314–316.

† However, the conditions (14) given below are new.

‡ See *American Journal of Mathematics*, Vol. 18 (1896), pp. 195–209. See also p. 69 of the present paper (§ 9).

§ See *Transactions of the American Mathematical Society*, Vol. 2 (1896), pp. 404–412.

2. The following definitions will be required.

DEFINITION (i). — The product series formed from p infinite series Σa_r, Σb_r, ..., Σh_r, Σk_r is defined to be the infinite series in which the r-th term is the sum of all the products of the type $a_A b_B \dots h_H k_K$ for which

$$A+B+\dots+H+K = r+p-1.$$

Denote this term by $w_r\,(ab \dots hk)$.

It easily follows from this definition that

$$w_r\,(a \dots k) = k_1 w_r\,(a \dots h) + k_2 w_{r-1}\,(a \dots h) + \dots + k_r w_1\,(a \dots h).$$

There are $(p-1)$ similar equations.

DEFINITION (ii).—If Σa_r be any convergent series, we shall denote by $\rho_n(a)$ the upper bound of $\left| \sum_p^\infty a_r \right|$, where p takes all integer values greater than n.

Evidently $\rho_n\,(a)$ tends to zero steadily as n tends to infinity.

DEFINITION (iii).—Let $F_1(n)$, $F_2(n)$ be two positive functions of n, both tending to infinity as n tends to infinity.* We shall write

$$F_1(n) \asymp F_2(n),$$

if two positive numbers A and B can be found such that

$$A < F_1\,(n)/F_2\,(n) < B$$

for all values of n.

In the language of Du Bois Reymond $F_1(n)$ and $F_2(n)$ are said to be of the same order of infinity. We shall write (n) to denote a function of n such that $(n) \asymp n$.

3. The following lemma will prove useful.

LEMMA.—*Let $a_1, a_2, \dots, a_n$ be an infinite sequence of positive numbers such that $a_{n+p}/a_n \leqslant K$ for all positive integer values of n and p, where K is a fixed positive number. Then, if $m \asymp n$,*

$$\sum_1^m a_r \asymp \sum_1^n a_r,$$

provided that $\sum_1^\infty a_r$ diverges.

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* The notation is due to G. H. Hardy. See his *Orders of Infinity : The Infinitarcalcül of Paul Du Bois-Reymond*, p. 2.

Since $m \asymp n$, two positive integers λ and μ can be found such that $1/\lambda < m/n < \mu$ for all values of n. Thus $m < n\mu$. Therefore

$$\sum_1^m a_r < \sum_1^{n\mu} a_r = \sum_1^n a_r + \sum_{n+1}^{2n} a_r + \ldots + \sum_{(\mu-1)n+1}^{\mu n} a_r.$$

But $a_{n+p} \leqslant K a_n$ for all positive integer values of n and p. Hence

$$\sum_1^m a_r < \sum_1^n a_r + (\mu-1) K \sum_1^n a_r.$$

Therefore

$$\sum_1^m a_r \Big/ \sum_1^n a_r < 1 + (\mu-1) K.$$

In a similar manner it can be shown that

$$\sum_1^m a_r \Big/ \sum_1^n a_r > 1/\{1 + K(\lambda-1)\}$$

for all values of n. Hence we have

$$\sum_1^m a_r \asymp \sum_1^n a_r.$$

COROLLARY.—*If the sequence* $a_1, a_2, \ldots, a_r, \ldots$ *tends to the limit* a *steadily,* $a_{n+p}/a_n \leqslant 1$ *when the sequence is a non-increasing one, and* $a_{n+p}/a_n \leqslant a/a_1$ *when the sequence is non-decreasing. Hence in either case*

$$\sum_1^m a_r \asymp \sum_1^n a_r,$$

when $m \asymp n$.

4. THEOREM A.—*If* Σa_r, Σb_r *be two infinite series,*

$$\sum_1^n a_r \sum_1^n b_r = \sum_1^n w_r(ab) + \sum_{r=2}^{n'} \{a_r(b_n + b_{n-1} + \ldots + b_{n+2-r}\}$$

$$+ \sum_{r=2}^{n'} \{b_r(a_n + a_{n-1} + \ldots + a_{n+2-r})\} + \sum_{r=n'+1}^{n} a_r \sum_{r=n'+1}^{n} b_r, \tag{1}$$

n' being the greatest integer less than $\frac{1}{2}(n+2)$.

Let $a_p b_q$ be a typical term of the product $\sum_1^n a_r \sum_1^n b_r$. It is clear from Definition (i) that $\sum_1^n w_r(ab)$ is the sum of these terms of $\sum_1^n a_r \sum_1^n b_r$ in which

$p+q \leqslant n+1$. It is readily seen that

$$\sum_{=}^{n'} \{a_r(b_n+b_{n-1}+\ldots+b_{n+2-r})\}$$

is the sum of those terms in which $p \leqslant n'$, $p+q > n+1$,

$$\sum_{r=2}^{n'} \{b_r(a_n+a_{n-1}+\ldots+a_{n+2-r})\}$$

is the sum of those terms in which $q \leqslant n'$, $p+q > n+1$, while $\sum_{r=n'+1}^{n} a_r \sum_{r=n'+1}^{n} b_r$ is the sum of terms in which $q > n'$, $p > n'$.

From this it follows that every term of $\sum_1^n a_r \sum_1^n b_r$ occurs at least once on the right-hand side of (1). Moreover, it is easy to prove, when n' has the value assigned to it above, that no term of $\sum_1^n a_r \sum_1^n b_r$ occurs more than once on the right-hand side of equation (1). Hence Theorem A is established.

5. THEOREM B.—*If* $\Sigma a_r, \Sigma b_r, \ldots, \Sigma h_r, \Sigma k_r$ *be* p *convergent series, a positive number* A, *independent of* n, *can be found such that*

$$\left| \sum_1^\infty a_r \sum_1^\infty b_r \ldots \sum_1^\infty h_r \sum_1^\infty k_r - \sum_1^n w_r(a \ldots k) \right|$$

$$< A \left[\rho_{(n)}(a) \sum_1^{(n)} |b_r| \ldots \sum_1^{(n)} |k_r| + \rho_{(n)}(b) \sum_1^{(n)} |c_r| \ldots \sum_1^{(n)} |a_r| + \ldots \right.$$

$$\left. + \rho_{(n)}(k) \sum_1^{(n)} |a_r| \ldots \sum_1^{(n)} |h_r| \right], \quad (2)$$

for all values of n.

In this relation the upper limits of summation on the right-hand side are not necessarily all equal. It is merely asserted that each upper limit is of the same order of infinity as n. Similarly for the suffixes of the ρ's.

In order to establish (B), we shall prove that it holds in the case of two series, and also that if it hold for p series it holds for $p+1$ series.

(1) *To prove that* (B) *holds for two series*, we observe that, from

Theorem (A),

$$\sum_1^\infty a_r \sum_1^\infty b_r - \sum_1^n w_r(ab)$$

$$= \sum_1^\infty a_r \sum_{n+1}^\infty b_r + \sum_1^\infty b_r \sum_{n+1}^\infty a_r - \sum_{n+1}^\infty a_r \sum_{n+1}^\infty b_r + \sum_{n'+1}^n a_r \sum_{n'+1}^n b_r$$

$$+ \sum_{r=2}^{n'} \{a_r(b_n + b_{n-1} + \ldots + b_{n+2-r})\} + \sum_{r=2}^{n'} \{b_r(a_n + a_{n-1} + \ldots + a_{n+2-r})\}, \quad (3)$$

where $\frac{1}{2}n \leqslant n' < \frac{1}{2}(n+2)$. If $q \geqslant p+1$,

$$\sum_{p+1}^q a_r = \sum_{p+1}^\infty a_r - \sum_{q+1}^\infty a_r.$$

Hence
$$\left| \sum_{p+1}^q a_r \right| \leqslant \left| \sum_{p+1}^\infty a_r \right| + \left| \sum_{q+1}^\infty a_r \right| \leqslant 2\rho_p(a). \quad (4)$$

Similarly
$$\left| \sum_{p+1}^q b_r \right| \leqslant 2\rho_p(b). \quad (5)$$

Hence, from equation (3), we have

$$\left| \sum_1^\infty a_r \sum_1^\infty b_r - \sum_1^n w_r(ab) \right|$$

$$\leqslant \rho_n(b) \left| \sum_1^\infty a_r \right| + \rho_n(a) \left| \sum_1^\infty b_r \right| + \rho_n(a)\,\rho_n(b) + 4\rho_{n'}(a)\,\rho_{n'}(b)$$

$$+ 2\rho_{n'}(b) \sum_2^{n'} |a_r| + 2\rho_{n'}(a) \sum_2^{n'} |b_r|$$

$$< \rho_{n'}(b) \left[\left| \sum_1^z a_r \right| + 5\rho_{n'}(a) + 2 \sum_1^{n'} |a_r| \right] + \rho_{n'}(a) \left[\left| \sum_1^z b_r \right| + 2 \sum_1^{n'} |b_r| \right]; \quad (6)$$

for evidently $\rho_n(a) \leqslant \rho_{n'}(a)$ and $\rho_n(b) \leqslant \rho_{n'}(b)$.

Since $\left| \sum_1^\infty a_r \right|$ and $\left| \sum_1^\infty b_r \right|$ are constant numbers and $5\rho_{n'}(a)$ is bounded for all values of n, a positive number A_2 can be found such that

$$\left| \sum_1^\infty a_r \sum_1^\infty b_r - \sum_1^n w_r(ab) \right| < A_2 \left[\rho_{n'}(a) \sum_1^{n'} |b_r| + \rho_{n'}(b) \sum_1^{n'} |a_r| \right] \quad (7)$$

for all values of n.

Since $\frac{1}{2}n \leqslant n' < \frac{1}{2}(n+2)$, $n' \asymp n$. Hence (B) is proved in the case of two series.

(2) *To prove that if* (B) *hold for* p *series it holds for* $(p+1)$ *series,* we suppose that $\Sigma a_r, \Sigma b_r, \ldots, \Sigma h_r, \Sigma k_r, \Sigma l_r$ are $p+1$ convergent series. From Definition (i) and Theorem (A), we get

$$\sum_1 w_r(a \ldots k) \sum_1^n l_r$$

$$= \sum_1^n w_r(a \ldots l) + \sum_{n'+1}^n w_r(a \ldots k) \sum_{n'+1}^n l_r + \sum_{r=2}^{n'} \{w_r(a \ldots k)(l_n + l_{n-1} + \ldots + l_{n+2-r})\}$$

$$+ \sum_{r=2}^{n'} [l_r \{w_n(a \ldots k) + w_{n-1}(a \ldots k) + \ldots + w_{n+2-r}(a \ldots k)\}], \quad (8)$$

where $\frac{1}{2}n \leqslant n' < \frac{1}{2}(n+2)$.

Assume that (B) holds for p series. We can then determine a positive number A such that

$$\sum_1^n w_r(a \ldots k) = \sum_1^\infty a_r \sum_1^\infty b_r \ldots \sum_1^\infty h_r \sum_1^\infty k_r + \theta_n, \quad (9)$$

where

$$|\theta_n| < A \Big[\rho_{(n)}(a) \sum_1^{(n)} |b_r| \ldots \sum_1^{(n)} |k_r| + \rho_{(n)}(b) \sum_1^{(n)} |c_r| \ldots \sum_1^{(n)} |a_r| + \ldots$$

$$+ \rho_n(k) \sum_1^{(n)} |a_r| \ldots \sum_1^{(n)} |h_r| \Big], \quad (10)$$

for all values of n.

Substituting from (9) in (8) and replacing $\sum_1^n l_r$ by $\sum_1^\infty l_r - \sum_{n+1}^\infty l_r$, we get

$$\sum_1^\infty a_r \sum_1^\infty b_r \ldots \sum_1^\infty k_r \sum_1^\infty l_r - \sum^n w_r(a \ldots l)$$

$$= -\theta_n \sum_1^\infty l_r + \theta_n \sum_{n+1}^\infty l_r + \sum_{n+1}^\infty l_r \sum_1^\infty a_r \ldots \sum_1^\infty k_r + \sum_{r=2}^{n'} l_r(\theta_n - \theta_{n+1-r})$$

$$+ \sum_{r=2}^{n'} w_r(a \ldots k) \{l_n + l_{n-1} + \ldots + l_{n+2-r}\} + (\theta_n - \theta_{n'}) \sum_{n'+1}^n l_r.$$

Since

$$|w_2(a \ldots k)| + |w_3(a \ldots k)| + \ldots + |w_{n'}(a \ldots k)| < \sum_1^{n'} |a_r| \ldots \sum_1^{n'} |k_r|,$$

$$\Big| \sum_{r=2}^{n'} [w_r(a \ldots k) \{l_n + l_{n-1} + \ldots + l_{n+2-r}\}] \Big| < 2\rho_{n'}(l) \sum_1^{n'} |a_r| \ldots \sum_1^{n'} |k_r|.$$

Hence

$$\left|\sum_1^\infty a_r \dots \sum_1^\infty l_r - \sum_1^n w_r(a \dots l)\right|$$

$$< |\theta_n| \left|\sum_1^\infty l_r\right| + |\theta_n| \left|\sum_{n+1}^\infty l_r\right| + \left|\sum_{n+1}^\infty l_r\right| \left|\sum_1^\infty a_r \dots \sum_1^\infty k_r\right|$$

$$+ \sum_{r=2}^n |l_r| \{|\theta_n| + |\theta_{n+1-r}|\} + 2\rho_{n'}(l) \sum_1^{n'} |a_r| \dots \sum_1^{n'} |k_r|$$

$$+ [|\theta_n| + |\theta_{n'}|]\, 2\rho_{n'}(l).$$

Now $n-1, n-2, \dots, n+1-n'$, n' are all of the same order of infinity as n. Hence, from (10),

$$\left|\sum_1^\infty a_r \dots \sum_1^\infty l_r - \sum_1^n w_r(a \dots l)\right|$$

$$< A\left[\rho_{(n)}(a) \sum_1^{(n)} |b_r| \dots \sum_1^{(n)} |k_r| + \dots + \rho_{(n)}(k) \sum_1^{(n)} |a_r| \dots \sum_1^{(n)} |h_r|\right]$$

$$\left[\left|\sum_1^\infty l_r\right| + \left|\sum_{n+1}^\infty l_r\right| + 4\rho_{n'}(l) + 2\sum_2^n |l_r|\right]$$

$$+ 2\rho_{n'}(l)\left[\sum_1^{n'} |a_r| \dots \sum_1^{n'} |k_r| + \left|\sum_1^\infty a_r \dots \sum_1^\infty k_r\right|\right].$$

Since $\sum_1^\infty l_r$, $\sum_{n+1}^\infty l_r$, $\sum_1^\infty a_r \dots \sum_1^\infty k_r$, $4\rho_{n'}(l)$ are all bounded, a positive number K can be determined such that

$$\left|\sum_1^\infty a_r \dots \sum_1^\infty l_r - \sum_1^n w_r(a \dots l)\right|$$

$$< K\left[\rho_{(n)}(a) \sum_1^{(n)} |b_r| \dots \sum_1^{(n)} |l_r| + \dots + \rho_{(n)}(l) \sum_1^{(n)} |a_r| \dots \sum_1^{(n)} |k_r|\right],$$

for all values of n. Hence, if (B) hold for p series, it holds for $p+1$ series. Thus the truth of (B) is established.

Hence the product series formed from the p convergent series $\Sigma a_r, \Sigma b_r, \dots, \Sigma h_r, \Sigma k_r$ converges if

$$\left.\begin{aligned} \rho_n(a) \sum_1^m |b_r| \dots \sum_1^m |k_r| &\to 0, \\ \rho_n(b) \sum_1^m |c_r| \dots \sum_1^m |a_r| &\to 0, \\ \dots \quad \dots \quad \dots \quad \dots \\ \rho_n(k) \sum_1^m |a_r| \dots \sum_1^m |h_r| &\to 0, \end{aligned}\right\} \qquad (11)$$

as $n \to \infty$, for all values of m such that $m \asymp n$.

By exactly similar methods we can deduce conditions of convergence in which the limits of summation are more precisely assigned. These sufficient conditions of convergence are

$$\left.\begin{aligned} &\rho_n(a)\sum_1^{n_1}|b_r|\dots\sum_1^{n_1}|k_r|\to 0,\\ &\rho_n(b)\sum_1^{n_2}|c_r|\dots\sum_1^{n_3}|a_r|\to 0,\\ &\dots\quad\dots\quad\dots\quad\dots\\ &\rho_n(k)\sum_1^{n_p}|a_r|\dots\sum_1^{n_p}|h_r|\to 0, \end{aligned}\right\} \tag{12}$$

as $n\to\infty$, where $n_1=n_2$ and $n_s=2^{p-s}n$ when $2\leqslant s\leqslant p$.*

However the lack of symmetry in this result detracts from its interest.

6. DEFINITION (iv).—If an infinite series Σa_r be such that

$$\sum_1^{m}|a_r|\asymp\sum_1^{n}|a_r|$$

whenever $m\asymp n$, we shall say that the series satisfies the condition (K).

It is clear from the above discussion that, if the condition (K) is satisfied by each of the p series Σa_r, Σb_r, ..., Σh_r, Σk_r, their product will converge if

$$\left.\begin{aligned} &\rho_n(a)\sum_1^{n}|b_r|\dots\sum_1^{n}|k_r|\to 0,\\ &\rho_n(b)\sum_1^{n}|c_r|\dots\sum_1^{n}|a_r|\to 0,\\ &\dots\quad\dots\quad\dots\quad\dots\\ &\rho_n(k)\sum_1^{n}|a_r|\dots\sum_1^{n}|h_r|\to 0, \end{aligned}\right\} \tag{13}$$

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* These conditions can be deduced from the following theorem, the proof of which is very similar to that of Theorem B.

A positive number A can be found such that

$$\left|\sum_1^{\infty}a_r\sum_1^{\infty}b_r\dots\sum_1^{\infty}h_r\sum_1^{\infty}k_r-\sum_1^{n}w_r(ab\dots hk)\right|$$
$$<A\left[\rho_{m_1}(a)\sum_1^{n'}|b_r|\dots\sum_1^{n'}|k_r|+\rho_{m_2}(b)\sum_1^{n'}|l_r|\dots\sum_1^{n'}|a_r|+\dots+\rho_{m_p}(h)\sum_1^{n'}|a_r|\dots\sum_1^{n'}|h_r|\right]$$

for all values of n, where $\frac{1}{2}n\leqslant n'\leqslant\frac{1}{2}(n+1)$, $m_1=m_2$ and $|m_s-n'.2^{s-p}|$ is bounded for $2\leqslant s\leqslant p$.

as $n \to \infty$. It is to be noted that in the case of two series Σa_r and Σb_r,

$$\rho_n(a) \sum_1^n |b_r| \to 0, \qquad \rho_n(b) \sum_1^n |a_r| \to 0, \tag{14}$$

are sufficient conditions for the convergence of the product series even when the condition (K) is not satisfied by either of the given series. This follows immediately from (7).

7. THEOREM C.—*Let* $\Sigma(-)^{r+1}a_r$, $\Sigma(-)^{r+1}b_r$, ..., $\Sigma(-)^{r+1}h_r$, $\Sigma(-)^{r+1}k_r$ *be* p *infinite series, where* a_r, b_r, ..., h_r, k_r *are positive real numbers for all values of* r, *and each tends to zero steadily as* n *tends to infinity. The necessary and sufficient conditions that their product series should converge are*

$$\left.\begin{aligned} a_n \sum_1^n b_r \dots \sum_1^n k_r &\to 0, \\ b_n \sum_1^n c_r \dots \sum_1^n a_r &\to 0, \\ \dots \qquad \dots \qquad &\dots \\ k_n \sum_1^n a_r \dots \sum_1^n h_r &\to 0, \end{aligned}\right\} \tag{15}$$

as $n \to \infty$.

It is readily seen that

$$\rho_n(a) < a_n, \ \rho_n(b) < b_n, \ \dots, \ \rho_n(k) < k_n.$$

Moreover it easily follows from the lemma proved above that the condition (K) is satisfied by each of the p series.

Hence the conditions (15) are *sufficient* to ensure the convergence of the product series. In order to prove that the conditions (15) are *necessary* conditions, we shall prove a general theorem from which this can be deduced.

8. THEOREM D.—*Let* $\Sigma(-)^{r+1}a_r$, $\Sigma(-)^{r+1}b_r$, ..., $\Sigma(-)^{r+1}h_r$, $\Sigma(-)^{r+1}k_r$ *be* p *infinite series, where* a_r, b_r, ..., h_r, k_r *are all positive for all values of* r, *and such that* a_{n+r}/a_n, b_{n+r}/b_n, ..., h_{n+r}/h_n, k_{n+r}/k_n *are all less than a positive number* K *for all positive integer values of* n *and* r. *The necessary and sufficient conditions that the* n*-th term of the product series formed from* $\Sigma(-)^{r+1}a_r$, ..., $\Sigma(-)^{r+1}k_r$ *should tend to zero as* n *tends to infinity are the conditions* (15) *written above.*

In proving this theorem we shall use a notation slightly different from

that previously employed. We shall denote by $(-)^{r+1}w_r(a \dots k)$ the r-th term of the product series of $\Sigma(-)^{r+1}a_r, \dots, \Sigma(-)^{r+1}k_r$. Then clearly $w_r(a \dots k)$ is positive for all values of r.

In order to prove that the conditions (15) are *necessary* that $w_n(a \dots k) \to 0$ as $n \to \infty$, we shall prove (D_1) that a positive number R can be found such that

$$w_n(a \dots k) > Ra_n \sum_1^n b_r \dots \sum_1^n k_r$$

for all values of n, with $(p-1)$ similar relations. To prove (D_1) we shall prove that (D_1) holds for two series, and that if it hold for p series it holds for $p+1$ series.

(1) *To prove that* (D_1) *holds for two series*, we observe that

$$w_n(ab) = a_1b_n + \dots + a_nb_1.$$

Therefore $$w_n(ab) > a_1b_n + a_2b_{n-1} + \dots + a_{n'}b_{n-n'+1},$$

if $n' < n$. Let $\frac{1}{2}n \leqslant n' \leqslant \frac{1}{2}(n+1)$, and let $b_{n''}$ be the least of the numbers

$$b_n,\ b_{n-1},\ \dots,\ b_{n-n'+1}.$$

Then $$w_n(ab) > b_{n''} \sum_1^{n'} a_r.$$

But $n'' \leqslant n$. Hence $Kb_{n''} \geqslant b_n$ for all values of n. Also $n' \asymp n$. Hence from the lemma proved above

$$\sum_1^{n'} a_r \asymp \sum_1^n a_r.$$

Hence a positive number R_2 can be found such that

$$w_n(ab) > R_2b_n \sum_1^n a_r.$$

Hence (D_1) holds for two series.

(2) *To prove that if* (D_1) *hold for p series it holds for* $(p+1)$ *series*, we suppose that $\Sigma(-)^{r+1}a_r$, $\Sigma(-)^{r+1}b_r$, ..., $\Sigma(-)^{r+1}k_r$, $\Sigma(-)^{r+1}l_r$ are $p+1$ series all of the type under consideration. We have

$$w_n(a \dots l) = l_1w_n(a \dots k) + l_2w_{n-1}(a \dots k) + \dots + l_nw_1(a \dots k).$$

Therefore

$$w_n(a \dots l) > l_1 w_n(a \dots k) + l_2 w_{n-1}(a \dots k) + \dots + l_{n'} w_{n-n'+1}(a \dots k),$$

if $n' < n$. Take n' so that $\frac{1}{2}(n+1) \geqslant n' \geqslant \frac{1}{2}n$. Let $w_m(a \dots k)$ be the least of the numbers

$$w_r(a \dots k) \quad (r = n, n-1, \dots, n-n'+1).$$

Then

$$w_m(a \dots l) > w_m(a \dots k) \sum_1^n l_r.$$

But, if (D_1) hold for p series, we can find a positive number R so that

$$w_m(a \dots k) > R a_m \sum_1^m b_r \dots \sum_1^m k_r$$

for all values of m. Hence

$$w_n(a \dots l) > R a_m \sum_1^m b_r \dots \sum_1^m k_r \sum_1^{n'} l_r$$

for all values of n (m and n' being in every case determined as above).

But clearly $n' \asymp n$, and, since $n \geqslant m \geqslant \frac{1}{2}n+1$, $m \asymp n$. Hence by the lemma

$$\sum_1^m b_r \asymp \sum_1^n b_r, \quad \dots, \quad \sum_1^m k_r \asymp \sum_1^n k_r, \quad \sum_1^{n'} l_r \asymp \sum_1^n l_r.$$

Also $a_m \geqslant a_n/K$. Hence a number R' can be found such that

$$w_n(a \dots l) > R' a_n \sum_1^n b_r \dots \sum_1^n l_r$$

for all values of n. Hence, if (D_1) hold for p series it holds for $(p+1)$ series. Thus (D_1) holds for any number of series.

It is evident now that the conditions (15) are *necessary* conditions that

$$w_n(a \dots k) \to 0$$

as $n \to \infty$. As the series contemplated in Theorem (C) are particular cases of the series under consideration here, it is clear that we have now completed the proof of Theorem (C).

In order to prove the second part of (D), namely that the conditions (15) are sufficient to ensure that

$$w_n(a \dots k) \to 0$$

as $n \to \infty$, we shall prove (D_2) that a positive number Q can be found

such that

$$w_n(a \dots k) < Q\Big[a_m \sum_1^m b_r \dots \sum_1^m k_r + b_m \sum_1^m l_r \dots \sum_1^m a_r + \dots + k_m \sum_1^m a_r \dots \sum_1^m h_r\Big]$$

for all values of n, m being an integer depending on n and such that $m \asymp n$. This theorem is also proved by induction.

(1) *To prove that* (D_2) *holds for two series,* we observe that

$$w_n(ab) = a_1 b_n + \dots + a_n b_1.$$

Therefore

$$w_n(ab) < a_1 b_n + a_2 b_{n-1} + \dots + a_{n'} b_{n-n'+1} + b_1 a_n + b_2 a_{n-1} + \dots + b_{n'} a_{n-n'+1},$$

if $n' > \frac{1}{2}(n+1)$. Let $\frac{1}{2}(n+3) \geqslant n' > \frac{1}{2}(n+1)$. Let a_{m_1} be the greatest of the numbers $a_n, a_{n-1}, \dots, a_{n-n'+1}$; and let b_{m_1} be the greatest of the numbers $b_n, b_{n-1}, \dots, b_{n-n'+1}$. Then

$$w_n(ab) < a_{m_1} \sum_1^{n'} b_r + b_{m_2} \sum_1^{n'} a_r.$$

Suppose m is the lesser of the integers m_1 and m_2. We have $a_{m_1} \leqslant K a_m$ and $b_{m_2} \leqslant K b_m$. Moreover $m \asymp n \asymp n'$.

Hence a positive number Q_2 can be found such that

$$w_n(ab) < Q_2 \Big(a_m \sum_1^n b_r + b_m \sum_1^m a_r\Big)$$

for all values of n. Hence (D_2) is established for the case of two series.

(2) *To prove that if* (D_2) *hold for p series it holds for* $(p+1)$ *series,* we suppose that $\Sigma(-)^{r+1} a_r, \dots, \Sigma(-)^{r+1} k_r, \Sigma(-)^{r+1} l_r$ are $p+1$ series of the type under consideration, It is not difficult to prove that

$$\begin{aligned} w_n(a \dots l) < {} & a_1 w_n(b \dots l) + a_2 w_{n-1}(b \dots l) + \dots + a_{n'} w_{n-n'+1}(b \dots l) \\ & + b_1 w_n(c \dots a) + b_2 w_{n-1}(c \dots a) + \dots + b_{n'} w_{n-n'+1}(c \dots a) \\ & \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \\ & + l_1 w_n(a \dots k) + l_2 w_{n-1}(a \dots k) + \dots + l_{n'} w_{n-n'+1}(a \dots k), \end{aligned} \tag{16}$$

provided that $n' > (n+p)/(p+1)$. Let, in fact, $a_A b_B \dots k_K l_L$ be any term of $w_n(a \dots l)$. Then $A+B+\dots+K+L = n+p$. Hence at least one of the positive integers $A, B, \dots, K, L$ must be less than or equal to

$(n+p)/(p+1)$. Hence each term of $w_n(a \dots l)$ must occur at least once on the right-hand side of (16). Therefore the inequality is established.

Let n' be such that

$$(n+p)/(p+1) < n' \leqslant (n+p)/(p+1)+1.$$

Let

$w_\alpha(b \dots l)$ be the greatest of the numbers $w_r(b \dots l)$, $r = n, n-1, \dots, n-n'+1$,

$w_\beta(c \dots a)$. „ „ „ $w_r(c \dots a)$, $r = n, n-1, \dots, n-n'+1$,

... ...

$w_\lambda(a \dots k)$ be the greatest of the numbers $w_r(a \dots k)$, $r = n, n-1, \dots, n-n'+1$.

Hence, from (16),

$$w_n(a \dots l) < w_\alpha(b \dots l) \sum_1^{n'} a_r + w_\beta(c \dots a) \sum_1^{n'} b_r + \dots + w_\lambda(a \dots k) \sum_1^{n'} l_r.$$

Suppose that (D_2) holds for p series. Then a number Q can be found such that

$$w_n(a \dots l) < Q\Big[\Big\{b_{\alpha'} \sum_1^{\alpha'} c_r \dots \sum_1^{\alpha'} l_r + \dots + l_{\alpha'} \sum_1^{\alpha'} b_r \dots \sum_1^{\alpha'} k\Big\} \sum_1^{n'} a_r + \dots$$

$$+ \Big\{a_{\lambda'} \sum_1^{\lambda'} b_r \dots \sum_1^{\lambda'} k_r + \dots + k_{\lambda'} \sum_1^{\lambda'} a_r \dots \sum_1^{\lambda'} h_r\Big\} \sum_1^{n'} l_r\Big]$$

for all values of n, α', β', ..., λ' being integer functions of n and such that

$$\alpha' \asymp \alpha \asymp n, \quad \beta' \asymp \beta \asymp n, \quad \dots, \quad \lambda' \asymp \lambda \asymp n.$$

Let m be the least of the numbers α', β', ..., λ'. Then, since a_μ/a_m, b_μ/b_m, ..., l_μ/l_m are all less than or equal to K, when $\mu \geqslant m$, and since

$$\alpha' \asymp \beta' \asymp \gamma' \asymp \dots \asymp \lambda' \asymp n \asymp n' \asymp m,$$

a number Q' can be found such that

$$w_n(a \dots l) < Q'\Big[a_m \sum_1^m b_r \dots \sum_1^m l_r + \dots + l_m \sum_1^m a_r \dots \sum_1^m k_r\Big]$$

for all values of n.

Hence (D_2) is proved. (D_1) being already established, the proof of (D) is complete.

9. Cajori has shown* that if in each of two series the terms be alter-

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* *American Journal of Mathematics*, Vol. 18 (1896), pp. 195–209.

nately positive and negative, and if one of them, on associating its terms into groups of two terms each, becomes absolutely convergent, the necessary and sufficient condition for the convergence of their product is that the n-th term of that product should tend to zero as n tends to infinity. It follows immediately from this that if there be p series, each with its terms alternately positive and negative, and such that $(p-1)$ of them are rendered absolutely convergent on associating terms into groups of two terms each, the necessary and sufficient condition for convergence of the product series is that the n-th term of this series tends to zero as n tends to infinity. If moreover the p series be such that a_{n+r}/a_n, b_{n+r}/b_n, ..., k_{n+r}/k_n are all bounded when n and r are integers, the necessary and sufficient conditions of the convergence of the product series are the conditions (15). This latter result follows from Cajori's theorem and (C).

10. *Application of the above tests.*—The following three well known theorems will often simplify the application of the rules of convergence deduced above. We shall denote these theorems by (E), (F), (G) respectively.

THEOREM (E).—*If $\mu(x)$ be a positive function of x such that $\mu(x) \to 0$ steadily as $x \to \infty$, and if moreover $\sum_1^\infty \mu(r)$ diverges, then*

$$\sum_1^n \mu(r) \asymp \int_1^n \mu(x)\,dx.$$

THEOREM (F).—*If $\mu(x)$ be a function of (x) such that $\mu(x) \to 0$ steadily as $n \to \infty$, and if moreover $\sum_1^\infty \mu(r)$ converges, then**

$$\sum_n^\infty \mu(r) = O\left[\int_n^\infty \mu(x)\,dx\right].$$

THEOREM (G).—*If $a_1, a_2, \ldots, w_1, w_2, \ldots$ be any numbers real or complex,*

$$\left|\sum_{n=1}^{m} a_n w_n\right| \leqslant A\left\{\sum_{n=1}^{m-1} |w_{n+1}-w_n| + |w_m|\right\},$$

where A is the greatest of the sums $\left|\sum_1^p a_n\right|$, $(p = 1, 2, \ldots, m)$.

* This notation is due to Landau, see Hardy's *Orders of Infinity*, p. 5.

11. EXAMPLE.—As an illustration of the above results let us consider the convergence of the q-th power of the series

$$\Sigma a_n \equiv \Sigma n^{-s} \sin nx,$$

where x is real and the real part of s is positive, n^s being understood to mean $\exp(s \log n)$.

The q-th power will converge if

$$\rho_n(a) \left\{ \sum_1^m |a_r| \right\}^{q-1} \to 0,$$

as $n \to \infty$, when $m \asymp n$. Now $|a_r| \leqslant r^{-a}$, where a is the real part of s. By hypothesis a is positive. Hence

$$\sum_1^m |a_r| \leqslant \sum_1^m r^{-a}.$$

Also $r^{-a} \to 0$ steadily as $r \to \infty$. Hence, by the corollary to the lemma proved above, we have, if $m \asymp n$,

$$\sum_1^m r^{-a} \asymp \sum_1^n r^{-a}.$$

Therefore, by (E),

$$\sum_1^m |a_r| \leqslant \sum_1^m r^{-a} \asymp \sum_1^n r^{-a} \asymp n^{1-a}/(1-a). \tag{17}$$

Again, from (G),

$$\left| \sum_u^v r^{-s} \sin rx \right| \leqslant A \left\{ \sum_u^{v-1} |(r+1)^{-s} - r^{-s}| + |v^{-s}| \right\},$$

where A is the greatest of the sums

$$\left| \sum_u^t \sin tx \right| \quad (t = u,\ u+1,\ \ldots,\ v).$$

But $\left| \sum_u^t \sin tx \right|$ is bounded whatever integer values p and t have. Let the upper bound be B.

Then $$\left| \sum_u^v r^{-s} \sin rx \right| \leqslant B \left\{ \sum_u^{v-1} |(r+1)^{-s} - r^{-s}| + |v^{-s}| \right\}. \tag{18}$$

But $$r^{-s} - (r+1)^{-s} = s \int_r^{r+1} \xi^{-s-1} d\xi.$$

Hence $$|(r+1)^{-s} - r^{-s}| \leqslant |s| \left[\int_r^{r+1} |\xi^{-s-} | d\xi \right] < |s| r^{-a-1}.$$

Therefore $$\left|\sum_{u}^{v} a_r\right| < B\left\{|s|\sum_{u}^{v-1} r^{-\alpha-1}+v^{-\alpha}\right\}.$$

By (F), $$\sum_{r=n}^{\infty} r^{-\alpha-1} = O(n^{-\alpha}).$$

Hence $$\rho_n(a) = O(n^{-\alpha}).$$

Therefore, by (17), if $m \asymp n$,

$$\rho_n(a)\left\{\sum_{1}^{m}|a_r|\right\}^{q-1} \to 0,$$

as $n \to \infty$, when

$$n^{-\alpha}\{n^{1-\alpha}/(1-\alpha)\}^{q-1} \to 0,$$

as $n \to \infty$. This latter condition is satisfied if

$$q(1-\alpha) < 1.$$

This reduces to one of Cajori's results if s be real.

12. *Uniformity of Convergence of the Product Series.*

Let the terms of the p convergent series $\Sigma a_r, \Sigma b_r, \ldots, \Sigma h_r, \Sigma k_r$ depend on a set of variables capable of variation in a given domain D. Assume that the two following conditions are satisfied.

(i) A positive number K exists such that

$$\left|\sum_{1}^{v} a_r\right|, \quad \left|\sum_{1}^{v} b_r\right|, \quad \ldots, \quad \left|\sum_{1}^{v} h_r\right|, \quad \left|\sum_{1}^{v} k_r\right|$$

are each less than K for all values of v and for every point of D.

(ii) A positive number η and a positive integer N can be found such that

$$\sum_{1}^{N}|a_r|, \quad \sum_{1}^{N}|b_r|, \quad \ldots, \quad \sum_{1}^{N}|h_r|, \quad \sum_{1}^{N}|k_r|$$

are each greater than η for all points of D.

If the conditions (i) and (ii) hold, numbers A and M, independent of n, and of the position of the "point" of D under consideration, can be found such that

$$\left|\sum_{1}^{\infty} a_r \ldots \sum_{1}^{\infty} k_r - \sum_{1}^{n} w_r(a \ldots k)\right|$$

$$< A\left[\rho_{(n)}(a)\sum_{1}^{(n)}|b_r| \ldots \sum_{1}^{(n)}|k_r| + \ldots + \rho_{(n)}(k)\sum_{1}^{(n)}|a_r| \ldots \sum_{1}^{(n)}|h_r|\right].$$

provided only that $n \geqslant M$.

The proof of this proposition is very similar to that of Theorem (B). It readily follows that the product series converges uniformly, if for every assigned integer function m of n such that $m \asymp n$,

$$\left.\begin{aligned} &\rho_n(a) \sum_1^m |b_r| \dots \sum_1^m |k_r| \to 0, \\ &\rho_n(b) \sum_1^m |c_r| \dots \sum_1^m |a_r| \to 0, \\ &\dots \quad \dots \quad \dots \quad \dots \\ &\rho_n(k) \sum_1^m |a_r| \dots \sum_1^m |h_r| \to 0, \end{aligned}\right\} \tag{19}$$

all these conditions being satisfied uniformly, as $n \to \infty$.

It may be of some interest to note that the product series of series which are absolutely and uniformly convergent is not necessarily uniformly convergent.

Thus the series Σa_r, where

$$a_r = z^{r-1} \text{ (when } r \geqslant 2),$$

$$a_1 = \exp(z^{-1}) \text{ (when } z \neq 0),$$

$$a_1 = 1 \text{ (when } z = 0),$$

converges absolutely and uniformly when $|z| < \delta$, δ being a positive number less than unity. Still the series $\Sigma w_r(aa)$ does not converge uniformly in the domain $(0, \delta)$.

If $z \neq 0$, $\qquad w_n(aa) = \exp(z^{-1})\, z^{n-1} + (n-1)\, z^{n-1}$.

Let z be a positive number, then

$$w_n(aa) > z^{-1}/n!.$$

Hence $w_n(aa)$ does not tend to zero uniformly in the domain $(0, \delta)$. Hence the series $\Sigma w_r(aa)$ does not converge uniformly in this domain.

[*Added October*, 1919.—We can sometimes establish the convergence of the product series by merely considering the "order of smallness" of $\rho_n(a)$, $\rho_n(b)$, ..., $\rho_n(k)$.

Let

$$\rho_n(a) = O\{A(n)\},\ \rho_n(b) = O\{B(n)\},\ \dots,\ \rho_n(k) = O\{K(n)\},$$

where $A(x)$, $B(x)$, ..., $K(x)$ are monotone decreasing functions of x.

Putting $q = p+1$ in equation (4), we get

$$|a_{p+1}| \leqslant 2\rho_p(a).$$

Hence $$\sum_0^{n-1} |a_{p+1}| \leqslant 2 \sum_0^{n-1} \rho_p(a) = O\left\{\int_1^n A(x)\,dx\right\}.$$

Also, from lemma proved above,

$$\int_1^n A(x)\,dx \asymp \int_1^m A(x)\,dx,$$

when $m \asymp n$. Hence the conditions (11) are satisfied if

$$\left.\begin{aligned} &A(n)\int_1^n B(x)dx \int_1^n C(x)dx \ldots \int_1^n K(x)dx \to 0,\\ &B(n)\int_1^n C(x)dx \int_1^n D(x)dx \ldots \int_1^n A(x)dx \to\\ &\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots\\ &K(n)\int_1^n A(x)dx \int_1^n B(x)dx \ldots \int_1^n H(x)dx \to 0, \end{aligned}\right\} \tag{20}$$

as $n \to \infty$. In particular, the q-th power of Σa_r will converge if

$$A(n)\left\{\int_1^n A(x)dx\right\}^{q-1} \to 0, \tag{21}$$

as $n \to \infty$. Thus, if $\rho_n(a) = O(n^{-\alpha})$, where α is a positive number, the q-th power converges, if

$$\frac{1}{n^\alpha}\left(\frac{n^{1-\alpha}}{1-\alpha}\right)^{q-1} \to 0,$$

as $n \to \infty$, *i.e.* if $$\alpha > \frac{q-1}{q}.]$$

DIVISORS OF NUMBERS AND THEIR CONTINUATIONS IN THE THEORY OF PARTITIONS

By P. A. MacMahon.

[Read March 13th, 1919.]

In this paper I regard divisors of numbers from the point of view of partitions of numbers and introduce into analysis certain new arithmetical functions. In important instances these occur in the theory of elliptic functions which has thus an application in a new department of the Theory of Numbers. The divisors of numbers have been studied in regard to their number, sum, sum of powers, &c., both when the divisors are unrestricted and when they satisfy certain congruences, by Euler, Jacobi, Gauss, Glaisher, and others. I diverge from the simple arithmetical functions, which involve divisors, along another track. I first define the principal new functions and deduce some of their properties from elementary considerations. Later I introduce elliptic functions into the theory.

The paper considers certain infinite series, in powers of q, of which the typical one is

$$A_k = \Sigma a_{n,k} q^n.$$

In this the coefficients $a_{n,k}$ have a definite meaning in the theory of the partition of numbers, $a_{n,1}$ being the sum of the divisors of n, and $a_{n,k}$ standing for $\Sigma s_1 s_2 \dots s_k$, where $s_1 s_2 \dots s_k$ are such that

$$s_1 m_1 + s_2 m_2 + \dots + s_k m_k = n,$$

and the summation is for all such k-partitions of n.

The main theorem obtained with regard to the A's is represented symbolically by the equation

$$2^{2k}(2k+1)!\, A_k = (--)^k \frac{1}{J_1} J(J^2-1^2)(J^2-3^2)\dots[J^2-(2k-1)^2], \qquad (1)$$

in which, after expansion of the right-hand side, J^r is to be replaced by

$$J_r = 1-3^r q+5^r q^3-7^r q^6+\ldots,$$

the exponents of q being the triangular numbers 1, 3, 6, 10, ...

By comparing coefficients of powers of q on the two sides of the equation the partition constants $a_{n,k}$ may be successively determined for $n = k+1, k+2, \ldots$ as far as may be required. This is carried out for $n = 1, 2, 3, \ldots, 16$, $k = 1, 2, 3, 4, 5$.

In addition to the A series several analogous series are discussed. Thus

$$B_k = \Sigma b_{n,k}\, q^n,$$

when

$$b_{n,k} = \Sigma(-)^{s_1+s_2+\ldots+s_k+k}\, s_1 s_2 \ldots s_k,$$

and the s's are defined as before. It is proved that, symbolically,

$$2^{2k}(2k)!\, B_k = \frac{1}{J_0}(J^2-1^2)(J^2-3^2)\ldots[J^2-(2k-1)^2], \qquad (2)$$

and the values of the b's are deduced for certain values of n, k.

Similar theorems are proved for C and D series, in which, while $s_1 m_1+\ldots+s_k m_k = n$ as before, only those partitions are taken into account in which the m's are all odd numbers.

In the E and F series the m's take only the values $5r \pm 1$, and in the G and H series only the values $5r \pm 2$.

1. Taking as my starting point the identity

$$1-2q^m\cos 2x+q^{2m} = (1-q^m)^2\left\{1+4\frac{q^m}{(1-q^m)^2}\sin^2 x\right\},$$

I form an infinite product on each side by taking m from 1 to ∞. Thus

$$\prod_1^\infty(1-2q^m\cos 2x+q^{2m}) = \prod_1^\infty(1-q^m)^2\prod_1^\infty\left\{1+4\frac{q^m}{(1-q^m)^2}\sin^2 x\right\},$$

and I observe that

$$\prod_1^\infty\left\{1+4\frac{q^m}{(1-q^m)^2}\sin^2 x\right\}$$

$$= 1+4\sum_1^\infty\frac{q^m}{(1-q^m)^2}\sin^2 x+4^2\sum_{m_1<m_2}\frac{q^{m_1+m_2}}{(1-q^{m_1})^2(1-q^{m_2})^2}\sin^4 x$$

$$+4^3\sum_{m_1<m_2<m_3}\frac{q^{m_1+m_2+m_3}}{(1-q^{m_1})^2(1-q^{m_2})^2(1-q^{m_3})^2}\sin^6 x+\ldots$$

$$= 1+4A_1\sin^2 x+4^2A_2\sin^4 x+4^3A_3\sin^6 x+\ldots,$$

where $A_1, A_2, A_3, \ldots$ are functions of q to be studied.

$$A_1 = \sum_1^\infty \frac{q^m}{(1-q^m)^2} = \Sigma a_{n,1} q^n,$$

is the well known function which generates the sums of the divisors of all numbers. Glaisher has denoted it by $\Sigma\sigma(n)q^n$, but for my present purpose I prefer to employ the notation $a_{n,1}$ in lieu of $\sigma(n)$.

In the theory of partitions, a number n possesses a certain number of partitions which involve but one magnitude of part. Thus the number 6 has four such partitions which, denoting repetitions of part by exponent numbers, are

$$6^1,\ 3^2,\ 2^3,\ 1^6.$$

It will be seen that the parts involved are the divisors of 6, and also that the exponent (repetitional) numbers are also collectively the divisors of 6. In each partition the part magnitude and the exponent are conjugate divisors of 6. For convenience of generalisation I regard the exponent numbers as being the divisors, and the part magnitude numbers as the conjugate divisors respectively.

In general the typical term in A_1 is

$$sq^{sm},$$

so that $a_{n,1}$ is equal to the sum of the numbers s such that $sm = n$; in other words $a_{n,1}$ is equal to the sum of the divisors of n.

We have the associated partition of n, viz. m^s. Also

$$A_2 = \sum_{m_1<m_2}\sum \frac{q^{m_1+m_2}}{(1-q^{m_1})^2\,(1-q^{m_2})^2} = \Sigma s_1 s_2 q^{s_1 m_1 + s_2 m_2} = \Sigma a_{n,2}\, q^n,$$

where if

$$s_1 m_1 + s_2 m_2 = n,$$

and the corresponding partition of n be m_1 repeated s_1 times, and m_2, s_2 times, we have the set of partitions of n, each of which involves parts of two different magnitudes which I denote by

$$m_1^{s_1}\, m_2^{s_2},$$

and then

$$a_{n,2} = \Sigma s_1 s_2,$$

where

$$s_1 m_1 + s_2 m_2 = n.$$

Thus for the number 6 we have the partitions

$$5^1 1^1 \quad 4^1 2^1 \quad 4^1 1^2 \quad 3^1 1^3 \quad 2^2 1^2 \quad 2^1 1^4,$$

and

$$a_{62} = 1.1+1.1+1.2+1.3+2.2+1.4 = 15.$$

Similarly for A_3 we have partitions of the type

$$m_1^{s_1} m_2^{s_2} m_3^{s_3}$$

and $$a_{n,3} = \Sigma s_1 s_2 s_3,$$

where $$s_1 m_1 + s_2 m_2 + s_3 m_3 = n.$$

Thus for the number 8, we have the partitions

$$5^1 2^1 1^1 \quad 4^1 3^1 1^1 \quad 4^1 2^1 1^2 \quad 3^1 2^2 1^1 \quad 3^1 2^1 1^3$$

and $$a_{8,3} = 1.1.1+1.1.1+1.1.2+1.2.1+1.1.3 = 9.$$

In general for A_k we have partitions of the type

$$m_1^{s_1} m_2^{s_2} \ldots m_k^{s_k}$$

and $$a_{n,k} = \Sigma s_1 s_2 \ldots s_k,$$

where $$s_1 m_1 + s_2 m_2 + \ldots + s_k m_k = n.$$

The functions A_k $(k > 1)$ I regard as the partition continuations of the function A_1 which exhibits the sums of the divisors of numbers, and writing

$$A_k = \Sigma a_{n,k} q^k,$$

a definite meaning has been assigned to the number $a_{n,k}$ in the theory of the partitions of numbers.

2. In the next place consider the identity

$$1 + 2q^m \cos 2x + q^{2m} = (1+q^m)^2 \left\{ 1 - 4 \frac{q^m}{(1+q^m)^2} \sin^2 x \right\},$$

and proceed to the identity

$$\prod_1^\infty (1 + 2q^m \cos 2x + q^{2m}) = \prod_1^\infty (1+q^m)^2 \prod_1^\infty \left\{ 1 - 4 \frac{q^m}{(1+q^m)^2} \sin^2 x \right\}.$$

Observe that we may write

$$\prod_1^\infty \left\{ 1 - 4 \frac{q^m}{(1+q^m)^2} \sin^2 x \right\}$$

$$= 1 - 4 \sum_1^\infty \frac{q^m}{(1+q^m)^2} \sin^2 x + 4^2 \sum_{m_1 < m_2} \sum \frac{q^{m_1+m_2}}{(1+q^{m_1})^2 (1+q^{m_2})^2} \sin^4 x$$

$$- 4^3 \sum_{m_1 < m_2 < m_3} \sum \sum \frac{q^{m_1+m_2+m_3}}{(1+q^{m_1})^2 (1+q^{m_2})^2 (1+q^{m_3})^2} \sin^6 x + \ldots$$

$$= 1 - 4B_1 \sin^2 x + 4^2 B_2 \sin^4 x - 4^3 B_3 \sin^6 x + \ldots,$$

where again B_1, B_2, B_3, ... are functions of q to be studied.

Now $$B_1 = \sum_1^\infty \frac{q^m}{(1+q^m)^2} = \Sigma b_{n,1} q^n$$

is a well known arithmetical function since $b_{n,1}$ is equal to the excess of the sum of the uneven divisors over the sum of the even divisors of n. It has been denoted by Glaisher by $\xi(n)$.

Writing $B_k = \Sigma b_{n,k} q^n$, it will be seen that

$$b_{n,k} = \Sigma(-)^{s_1+s_2+\ldots+s_k+k} s_1 s_2 \ldots s_k,$$

where $$s_1 m_1 + s_2 m_2 + \ldots + s_k m_k = n,$$

and we refer to the partition of n of type

$$m_1^{s_1} m_2^{s_2} \ldots m_k^{s_k}.$$

Thus when $n = 8$, we have for $k = 3$, the partitions

$$5^1 2^1 1^1 \quad 4^1 3^1 1^1 \quad 4^1 2^1 1^2 \quad 3^1 2^2 1^1 \quad 3^1 2^1 1^3,$$

and $$b_{8,3} = +1 \quad +1 \quad -2 \quad -2 \quad +3 \quad = +1.$$

The function B_k I regard as the partition continuation of B_1 which is Glaisher's $\Sigma \xi(n) q^n$, and a definite meaning has been assigned to the number $b_{n,k}$ in the theory of the partitions of numbers.

If we write $$\prod_1^\infty (1-q^{tm}) = P_t, \quad \prod_1^\infty (1+q^{tm}) = Q_t,$$

we have now the two identities

$$\Pi(1-2q^m \cos 2x + q^{2m}) = P_1^2(1+4A_1 \sin^2 x + 4^2 A_2 \sin^4 x + 4^3 A_3 \sin^6 x + \ldots),$$

$$\Pi(1+2q^m \cos 2x + q^{2m}) = Q_1^2(1-4B_1 \sin^2 x + 4^2 B_2 \sin^4 x - 4^3 B_3 \sin^6 x + \ldots),$$

and two more obtained by writing $\frac{1}{2}\pi - x$ for x, viz.,

$$\Pi(1+2q^m \cos 2x + q^{2m}) = P_1^2(1+4A_1 \cos^2 x + 4^2 A_2 \cos^4 x + 4^3 A_3 \cos^6 x + \ldots),$$

$$\Pi(1-2q^m \cos 2x + q^{2m}) = Q_1^2(1-4B_1 \cos^2 x + 4^2 B_2 \cos^4 x - 4^3 B_3 \cos^6 x + \ldots).$$

Equating the right-hand sides of the first and fourth of these, there results

$$P_1^2(1+4A_1 \sin^2 x + 4^2 A_2 \sin^4 x + \ldots) = Q_1^2(1-4B_1 \cos^2 x + 4^2 B_2 \cos^4 x - \ldots).$$

Putting in succession $x = \frac{1}{2}\pi,\ \frac{1}{3}\pi,\ \frac{1}{4}\pi,\ \frac{1}{6}\pi,\ 0$, we find that

$$1+4A_1+4^2A_2+4^3A_3+\ldots$$

$$= \frac{1+3A_1+3^2A_2+3^3A_3+\ldots}{1-B_1+B_2-B_3+\ldots}$$

$$= \frac{1+2A_1+2^2A_2+2^3A_3+\ldots}{1-2B_1+2^2B_2-2^3B_3+\ldots}$$

$$= \frac{1+A_1+A_2+A_3+\ldots}{1-3B_1+3^2B_2-3^3B_3+\ldots}$$

$$= \frac{1}{1-4B_1+4^2B_2-4^3B_3+\ldots}$$

$$= \frac{Q_1^2}{P_1^2}.$$

Expanding in powers of x and equating coefficients we obtain a series of relations from which are readily deduced the formulæ

$$A_k = \frac{Q_1^2}{P_1^2} \sum_k^\infty (-)^{s+k} \binom{s}{k} 4^{s-k} B_s,$$

$$B_k = \frac{P_1^2}{Q_1^2} \sum_k^\infty \binom{s}{k} 4^{s-k} A_s.$$

In the two identities putting $x = \frac{1}{3}\pi,\ \frac{1}{4}\pi,\ \frac{1}{6}\pi$, we find

$$1+3A_1+3^2A_2+3^3A_3+\ldots = \frac{P_3}{P_1^3},$$

$$1+2A_1+2^2A_2+2^3A_3+\ldots = \frac{Q_2}{P_1^2},$$

$$1+\ A_1+\ \ A_2+\ \ A_3+\ldots = \frac{Q_3}{P_1P_2},$$

$$1-3B_1+3^2B_2-3^3B_3+\ldots = \frac{Q_3}{Q_1^3},$$

$$1-2B_1+2^2B_2-2^3B_3+\ldots = \frac{Q_2}{Q_1^2},$$

$$1-\ B_1+\ \ B_2-\ \ B_3+\ldots = \frac{P_3}{P_2Q_1}.$$

Multiplying together the first and second identities, we find

$$\Pi(1-2q^{2m}\cos 4x+q^{4m})$$
$$= P_2^2\{1+4(A_1-B_1)\sin^2 x+4^2(A_2-A_1B_1+B_2)\sin^4 x$$
$$+4^3(A_3-A_2B_1+A_1B_2-B_3)\sin^6 x+\ldots\},$$

and comparing this with the first identity, when we write q^2 for q and $2x$ for x therein, we find

$$(A_1-B_1)\sin^2 x+4(A_2-A_1B_1+B_2)\sin^4 x$$
$$+4^2(A_3-A_2B_1+A_1B_2-B_3)\sin^6 x+\ldots$$
$$= A_1'\sin^2 2x+4A_2'\sin^4 2x+4^2A_3'\sin^6 2x+\ldots,$$

where A_k' denotes that q^2 is written for q in A_k.

Writing $2\sin x\cos x$ for $\sin 2x$, dividing out by $\sin^2 x$ and then putting $x=0$, we find

$$A_1-B_1=4A_1' \quad \text{or} \quad B_1=A_1-4A_1',$$

a relation which is otherwise obvious.

By comparing the coefficients of x^4 we find after reduction

$$B_2=A_1^2-(A_2+A_1')-4A_1A_1'+16A_2',$$

and so on; we find in general that B_k is expressible in terms of the $2k$ functions

$$A_1,\ A_2,\ \ldots,\ A_k, \qquad A_1',\ A_2',\ \ldots,\ A_k'.$$

3. To proceed to another set of arithmetical functions, consider the identity

$$1-2q^{2m-1}\cos 2x+q^{4m-2}=(1-q^{2m-1})^2\left\{1+4\frac{q^{2m-1}}{(1-q^{2m-1})^2}\sin^2 x\right\}$$

leading to

$$\Pi(1-2q^{2m-1}\cos 2x+q^{4m-2})=\Pi(1-q^{2m-1})^2\,\Pi\left\{1+4\frac{q^{2m-1}}{(1-q^{2m-1})^2}\sin^2 x\right\},$$

which may be written

$$\Pi(1-2q^{2m-1}\cos 2x+q^{4m-2})$$
$$=\frac{1}{Q_1^2}(1+4C_1\sin^2 x+4^2C_2\sin^4 x+4^3C_3\sin^6 x+\ldots),$$

where $C_1 = \sum_1^\infty \frac{q^{2m-1}}{(1-q^{2m-1})^2} = \Sigma c_{n,1}\, q^n,$

$$C_2 = \sum_{m_1<m_2}\sum \frac{q^{2m_1+2m_2-2}}{(1-q^{2m_1-1})^2(1-q^{2m_2-1})^2} = \Sigma c_{n,2}\, q^n,$$

$$C_3 = \sum_{m_1<m_2<m_3}\sum\sum \frac{q^{2m_1+2m_2+2m_3-3}}{(1-q^{2m_1-1})^2(1-q^{2m_2-1})^2(1-q^{2m_3-1})^2} = \Sigma c_{n,3}\, q^n, \text{ \&c.,}$$

are functions to be studied.

C_1 is the well known function which in regard to a number n exhibits the sum of those divisors which have uneven conjugates.

In the theory of partitions it is concerned with partitions of type

$$(2m-1)^s,$$

and generates the sum $\Sigma s = c_{n,1},$

where $s(2m-1) = n.$

C_2 is concerned with partitions of n of type

$$(2m_1-1)^{s_1}(2m_2-1)^{s_2},$$

and generates the sum $\Sigma s_1 s_2 = c_{n,2},$

where $s_1(2m_1-1)+s_2(2m_2-1) = n,$

and generally C_k has to do with partitions of n of type

$$(2m_1-1)^{s_1}(2m_2-1)^{s_2}\ldots(2m_k-1)^{s_k},$$

and generates the sum $\Sigma s_1 s_2 \ldots s_k = c_{n,k},$

where $s_1(2m_1-1)+s_2(2m_2-1)+\ldots+s_k(2m_k-1) = n.$

For instance, when $n = 6$, we take the partitions which involve uneven parts, viz.,

$$3^2 \quad 1^6 \qquad 5^1 1^1 \quad 3^1 1^3,$$

and find $c_{6,1} = 8, \qquad c_{6,2} = 4.$

4. Similarly we write

$$\Pi(1+2q^{2m-1}\cos 2x+q^{4m-2}) = \frac{Q_1^2}{Q_2^2}(1-4D_1\sin^2 x+4^2D_2\sin^4 x-4^3D_3\sin^6 x+\ldots),$$

where $D_1 = \sum_1^\infty \frac{q^{2m-1}}{(1+q^{2m-1})^2} = \Sigma d_{n,1} q^n$,

$$D_2 = \sum_{m_1<m_2} \frac{q^{2m_1+2m_2-2}}{(1+q^{2m_1-1})^2 (1+q^{2m_2-1})^2} = \Sigma d_{n,2} q^n,$$

$$D_3 = \sum_{m_1<m_2<m_3} \frac{q^{2m_1+2m_2+2m_3-3}}{(1+q^{2m_1-1})^2 (1+q^{2m_2-1})^2 (1+q^{2m_3-1})^2} = \Sigma d_{n,3} q^n, \text{ \&c.}$$

D_1 exhibits the excess of the sum of the uneven divisors which have uneven conjugates over the sum of the even divisors which have uneven conjugates, so that $d_{n,1} = (-)^{n+1}$ (sum of the divisors of n which have uneven conjugates). Glaisher denotes $c_{n,1}$ by $\Delta'(n)$ and $d_{n,1}$ by $(-)^{n+1}\Delta'(n)$. We have

$$d_{n,1} = (-)^{n+1} c_{n,1}.$$

In the theory of partitions D_k has to do with partitions of n of type

$$(2m_1-1)^{s_1} (2m_2-1)^{s_2} \dots (2m_k-1)^{s_k},$$

and generates the sum

$$\Sigma (-)^{s_1+s_2+\dots+s_k+k} s_1 s_2 \dots s_k = d_{n,k},$$

where $s_1(2m_1-1)+s_2(2m_2-1)+\dots+s_k(2m_k-1) = n.$

Since $\Sigma s \equiv n \bmod 2$, we have

$$\Sigma (-)^{n+k} s_1 s_2 \dots s_k = d_{n,k},$$

so that $d_{n,k} = (-)^{n+k} c_{n,k}$, because n and k are both constant throughout the summation.

C_k and D_k are the partition continuations of

$$\Sigma \Delta'(n) q^n.$$

To the two identities

$$\Pi (1-2q^{2m-1} \cos 2x + q^{4m-2}) = \frac{1}{Q_1^2} (1+4C_1 \sin^2 x + 4^2 C_2 \sin^4 x + 4^3 C_3 \sin^6 x + \dots),$$

$$\Pi (1+2q^{2m-1} \cos 2x + q^{4m-2}) = \frac{Q_1^2}{Q_2^2} (1-4D_1 \sin^2 x + 4^2 D_2 \sin^4 x - 4^3 D_3 \sin^6 x + \dots),$$

we add two more by writing $\frac{1}{2}\pi - x$ for x

$$\Pi(1+2q^{2m-1}\cos 2x+q^{4m-2}) = \frac{1}{Q_1^2}(1+4C_1\cos^2 x+4^2C_2\cos^4 x+4^3C_3\cos^6 x+\ldots),$$

$$\Pi(1-2q^{2m-1}\cos 2x+q^{4m-2}) = \frac{Q_1^2}{Q_2^2}(1-4D_1\cos^2 x+4^2D_2\cos^4 x-4^3D_3\cos^6 x+\ldots),$$

and proceeding, as in § 2, we are led to the relations

$$\begin{aligned}
&1+4C_1+4^2C_2+4^3C_3+\ldots \\
&= \frac{1+3C_1+3^2C_2+3^3C_3+\ldots}{1-D_1+D_2-D_3+\ldots} \\
&= \frac{1+2C_1+2^2C_2+2^3C_3+\ldots}{1-2D_1+2^2D_2-2^3D_3+\ldots} \\
&= \frac{1+C_1+C_2+C_3+\ldots}{1-3D_1+3^2D_2-3^3D_3+\ldots} \\
&= \frac{1}{1-4D_1+4^2D_2-4^3D_3+\ldots} \\
&= \frac{Q_1^4}{Q_2^2}.
\end{aligned}$$

Expanding in powers of x and equating coefficients, we obtain the relations

$$C_k = \frac{Q_1^4}{Q_2^2}\sum_k^\infty (-)^{s+k}\binom{s}{k}4^{s-k}D_s, \qquad D_k = \frac{Q_2^2}{Q_1^4}\sum_k^\infty \binom{s}{k}4^{s-k}C_s.$$

In the two identities putting $x = \frac{1}{3}\pi, \frac{1}{4}\pi, \frac{1}{6}\pi$ in succession, we find

$$1+3C_1+3^2C_2+3^3C_3+\ldots = \frac{Q_1^3}{Q_3}$$

$$1+2C_1+2^2C_2+2^3C_3+\ldots = \frac{Q_1^2Q_2}{Q_4}$$

$$1+C_1+C_2+C_3+\ldots = \frac{Q_1Q_2Q_3}{Q_6},$$

$$1-3D_1+3^2D_2-3^3D_3+\ldots=\frac{Q_2^3Q_3}{Q_1^3Q_6},$$

$$1-2D_1+2^2D_2-2^3D_3+\ldots=\frac{Q_2^3}{Q_1^2Q_4},$$

$$1-\ D_1+\ \ D_2-\ \ D_3+\ldots=\frac{Q_2^2}{Q_1Q_3}.$$

Multiplying together the first and second identities, we find

$$\Pi\,(1-2q^{4m-2}\cos 4x+q^{8m-4})$$

$$=\frac{1}{Q_2^2}\{1+4(C_1-D_1)\sin^2 x+4^2(C_2-C_1D_1+D_2)\sin^4 x+\ldots\},$$

leading to

$$(C_1-D_1)\sin^2 x+4\,(C_2-C_1D_1+D_2)\sin^4 x-\ldots$$

$$=C_1'\sin^2 2x+4C_2'\sin^4 2x+4^2C_3'\sin^6 2x+\ldots,$$

showing that the functions

$$C_1,\ C_2,\ C_3,\ \ldots,\ D_1,\ D_2,\ D_3,\ \ldots,\ C_1',\ C_2',\ C_3',\ \ldots$$

are connected by this relation in the same manner as

$$A_1,\ A_2,\ A_3,\ \ldots,\ B_1,\ B_2,\ B_3,\ \ldots,\ A_1',\ A_2',\ A_3',\ \ldots;$$

so that, *ex. gr.*, $D_1=C_1-4C_1'$, or $d_{2n,1}=C_{2n,1}-4C_{n,1}$;

and since $d_{n,k}=(-)^{n+k}C_{n,k}$, we find that

$$C_{2n,1}=2C_{n,1},$$

which is, of course, a known property of the numbers.

5. The next sets of arithmetical functions that I consider will prove to be of importance from the point of view of elliptic functions. They are connected with the numbers of the forms ± 1 mod 5 and ± 2 mod 5, respectively.

(i) I consider the infinite product

$$(1-2q\cos 2x+q^2)(1-2q^4\cos 2x+q^8)(1-2q^6\cos 2x+q^{12})\ldots$$

$$=\prod_1^\infty(1-2q^{5m-4}\cos 2x+q^{10m-8})\prod_1^\infty(1-2q^{5m-1}\cos 2x+q^{10m-2})$$

$$=\prod_1^\infty(1-q^{5m-4})^2(1-q^{5m-1})^2\prod_1^\infty\left\{1+4\frac{q^{5m-4}}{(1-q^{5m-4})^2}\sin^2 x\right\}$$

$$\times\prod_1^\infty\left\{1+4\frac{q^{5m-1}}{(1-q^{5m-1})^2}\sin^2 x\right\}$$

$$=R_1^2(1+4E_1\sin^2 x+4^2E_2\sin^4 x+4^3E_3\sin^6 x+\ldots),$$

where $$R_1=(1-q)(1-q^4)(1-q^6)(1-q^9)(1-q^{11})\ldots,$$

$$E_1=\sum_1^\infty\left\{\frac{q^{5m-4}}{(1-q^{5m-4})^2}+\frac{q^{5m-1}}{(1-q^{5m-1})^2}\right\}=\Sigma e_{n,1}q^n,$$

$$E_2=\sum_{m_1<m_2}\sum\frac{q^{m_1+m_2}}{(1-q^{m_1})^2(1-q^{m_2})^2}=\Sigma e_{n,2}q^n,\ \&c.,$$

where the summations are in respect of values of m_1, m_2, which are of the forms

$$5m-4,\quad 5m-1.$$

E_1 clearly exhibits the sum of those divisors of a number n, which have conjugates of the form

$$5m\pm 1.$$

E_k is concerned with partitions of the type

$$m_1^{s_1}m_2^{s_2}\ldots m_k^{s_k},$$

wherein $m_1, m_2, \ldots, m_k$ are numbers of form

$$5m\pm 1,$$

and $$e_{n,k}=\Sigma s_1s_2\ldots s_k,$$

where $$s_1m_1+s_2m_2+\ldots+s_km_k=n.$$

I am not aware that any of these arithmetical functions have previously been specially considered.

6. Also

$$\prod_1^\infty (1+2q^{5m-4}\cos 2x+q^{10m-8}) \prod_1^\infty (1+2q^{5m-1}\cos 2x+q^{10m-2})$$

$$= S_1^2(1-4F_1\sin^2 x+4^2F_2\sin^4 x-4^3F_3\sin^6 x+\ldots),$$

where $$S_1 = (1+q)(1+q^4)(1+q^6)(1+q^9)(1+q^{11})\ldots,$$

$$F_1 = \sum_1^\infty \frac{q^{m_1}}{(1+q^{m_1})^2} = \Sigma f_{n,1}\, q^n,$$

$$F_2 = \sum\sum_{m_1<m_2} \frac{q^{m_1+m_2}}{(1+q^{m_1})^2(1+q^{m_2})^2} = \Sigma f_{n,2}\, q^n, \text{ \&c.},$$

where the summations are in respect of values of $m_1, m_2, \ldots$, which are of the forms $5m \pm 1$.

Since $$F_1 = \Sigma\Sigma(-)^{s+1} s q^{sm},$$

it clearly exhibits the excess of the sum of the uneven divisors which have conjugates of the form $5m \pm 1$ over the sum of the even divisors which have conjugates of the form $5m \pm 1$.

Thus, for $n = 12$, the partitions 1^{12}, 4^3, 6^2 give

$$f_{12,1} = 3-12-2 = -11.$$

Also $$F_k = \Sigma\Sigma(-)^{s_1+s_2+\ldots+s_k+k} s_1 s_1 \ldots s_k\, q^{s_1m_1+s_2m_2+\ldots+s_km_k},$$

shows that for partitions of type

$$m_1^{s_1} m_2^{s_2} \ldots m_k^{s_k},$$

$m_1, m_2, \ldots, m_k$ being numbers of form $5m \pm 1$,

$$f_{n,k} = \Sigma(-)^{s_1+s_2+\ldots+s_k+k} s_1 s_2 \ldots s_k.$$

Since when n is uneven, it has only uneven divisors, it is clear that

$$f_{2n-1,1} = e_{2n-1,1}.$$

We have now the four identities

$$\Pi(1-2q^{5m-4}\cos 2x+q^{10m-8})(1-2q^{5m-1}\cos 2x+q^{10m-2})$$

$$= R_1^2(1+4E_1\sin^2 x+4^2E_2\sin^4 x+\ldots),$$

$$\Pi(1+2q^{5m-4}\cos 2x+q^{10m-8})(1+2q^{5m-1}\cos 2x+q^{10m-2})$$

$$= S_1^2(1-4F_1\sin^2 x+4^2F_2\sin^4 x-\ldots),$$

$$\Pi(1+2q^{5m-4}\cos 2x+q^{10m-8})(1+2q^{5m-1}\cos 2x+q^{10m-2})$$
$$= R_1^2(1+4E_1\cos^2 x+4^2E_2\cos^4 x+\ldots),$$

$$\Pi(1-2q^{5m-4}\cos 2x+q^{10m-8})(1-2q^{5m-1}\cos 2x+q^{10m-2})$$
$$= S_1^2(1-4F_1\cos^2 x+4^2F_2\cos^4 x-\ldots),$$

which lead, by the first and fourth of them, to the relations

$$1+4E_1+4^2E_2+\ldots = \frac{1+3E_1+3^2E_2+\ldots}{1-F_1+F_2-\ldots} = \frac{1+2E_1+2^2E_2+\ldots}{1-2F_1+2^2F_2-\ldots}$$
$$= \frac{1+E_1+E_2+\ldots}{1-3F_1+3^2F_2-\ldots} = \frac{1}{1-4F_1+4^2F_2-\ldots} = \frac{S_1^2}{R_1^2}.$$

By expanding in powers of x we find as before

$$E_1 = \Sigma(-)^{k-1}4^{k-1}\binom{k}{1}F_k, \quad F_1 = \Sigma 4^{k-1}\binom{k}{1}E_k,$$

$$E_2 = \Sigma(-)^{k}4^{k-2}\binom{k}{2}F_k, \qquad F_2 = \Sigma 4^{k-2}\binom{k}{2}E_k,$$

&c., &c.

In the first identity, putting $x = \frac{1}{3}\pi, \frac{1}{4}\pi, \frac{1}{6}\pi$ in succession, we find

$$1+3E_1+3^2E_2+\ldots = \frac{R_3}{R_1^3},$$

$$1+2E_1+2^2E_2+\ldots = \frac{S_2}{R_1^2},$$

$$1+E_1+E_2+\ldots = \frac{S_3}{R_1R_2},$$

and similarly for the second identity

$$1-3F_1+3^2F_2-\ldots = \frac{S_3}{S_1^3},$$

$$1-2F_1+2^2F_2-\ldots = \frac{S_2}{S_1^2},$$

$$1-F_1+F_2-\ldots = \frac{R_3}{R_2S_1}.$$

Multiplying together the first two identities,

$$\Pi(1-2q^{10m-8}\cos 4x+q^{20m-16})(1-2q^{10m-2}\cos 4x+q^{20m-4})$$
$$= R_2^2\{1+4(E_1-F_1)\sin^2 x+4^2(E_2-E_1F_1+F_2)\sin^4 x+\ldots\},$$

we find that the functions

$$E_1, E_2, E_3, \ldots, F_1, F_2, F_3, \ldots, E_1', E_2', E_3', \ldots,$$

where E_k' is what E_k becomes on writing q^2 for q, are connected in the same manner as

$$A_1, A_2, A_3, \ldots, B_1, B_2, B_3, \ldots, A_1', A_2', A_3', \ldots,$$

so that $F_1 = E_1-4E_1'$ or $f_{2n,1} = e_{2n,1}-4e_{n,1}$.

7. We have next to consider the functions derived from the numbers of form $5m \pm 2$. Writing

$$(1-2q^2\cos 2x+q^4)(1-2q^3\cos 2x+q^6)(1-2q^7\cos 2x+q^{14})\ldots$$
$$= \Pi(1-2q^{5m-3}\cos 2x+q^{10m-6})(1-2q^{5m-2}\cos 2x+q^{10m-4})$$
$$= T_1^2(1+4G_1\sin^2 x+4^2G_2\sin^4 x+\ldots),$$

$$\Pi(1+2q^{5m-3}\cos 2x+q^{10m-6})(1+2q^{5m-2}\cos 2x+q^{10m-4})$$
$$= U_1^2(1-4H_1\sin^2 x+4^2H_2\sin^4 x-\ldots),$$

where $T_1 = (1-q^2)(1-q^3)(1-q^7)(1-q^8)\ldots,$

$$U_1 = (1+q^2)(1+q^4)(1+q^7)(1+q^8)\ldots = \frac{T_2}{T_1}.$$

It is clear that in the foregoing analysis we may throughout substitute T, U for R, S, and also G, H for E, F, while replacing $5m \pm 1$ where it occurs by $5m \pm 2$.

8. In what has preceded we have introduced certain definite arithmetical functions because they are connected with elliptic function theory. There is no difficulty in extending the idea so as to derive continuations from the function, which exhibits the sum of those divisors which have conjugates of the form

$$\epsilon \bmod \mu,$$

which has the form $\sum_1^\infty \frac{q^{(m-1)\mu+\epsilon}}{(1-q^{(m-1)\mu+\epsilon})^2}$.

They do not appear to be in general connected with elliptic functions. A few words, however, may be said about the continuations which are derived from the function which enumerates the divisors.

From the identities

$$1-q^m(1-g) = (1-q^m)\left\{1+\frac{q^m}{1-q^m}g\right\},$$

$$1+q^m(1-g) = (1+q^m)\left\{1-\frac{q^m}{1-q^m}g\right\}.$$

We pass to the infinite products

$$\prod_1^\infty \{1-q^m(1-g)\} = \prod_1^\infty (1-q^m)\prod_1^\infty \left(1+\frac{q^m}{1-q^m}g\right),$$

$$\prod_1^\infty \{1+q^m(1-g)\} = \prod_1^\infty (1+q^m)\prod_1^\infty \left(1-\frac{q^m}{1+q^m}g\right),$$

and these relations we may write

$$1+\sum_1^\infty (-)^m \frac{q^{\frac{1}{2}m(m+1)}}{(1-q)(1-q^2)\dots(1-q^m)}(1-g)^m = P_1(1+N_1g+N_2g^2+\dots),$$

$$1+\sum_1^\infty \frac{q^{\frac{1}{2}m(m+1)}}{(1-q)(1-q^2)\dots(1-q^m)}(1-g)^m = Q_1(1-M_1g+M_2g^2-\dots),$$

where $$N_1 = \Sigma\frac{q^m}{1-q^m} = \Sigma\nu_1(n)q^n$$

is the function which enumerates the divisors, and

$$M_1 = \Sigma\frac{q^m}{1+q^m} = \Sigma\mu_1(n)q^n$$

enumerates the excess of the number of uneven divisors over the number of even divisors.

From the partition point of view, N_k enumerates the partitions of n which involve parts of k different magnitudes, and M_k the excess of those partitions involving parts of k different magnitudes for which the sum-number of parts plus k is even over the number in which that sum is uneven,

$$N_k = \sum_{m_1<m_2\dots<m_k}\Sigma\dots\Sigma\frac{q^{m_1+m_2+\dots+m_k}}{(1-q^{m_1})(1-q^{m_2})\dots(1-q^{m_k})} = \Sigma\nu_k(n)q^n,$$

$$M_k = \sum_{m_1<m_2\dots<m_k}\Sigma\dots\Sigma\frac{q^{m_1+m_2+\dots+m_k}}{(1+q^{m_1})(1+q^{m_2})\dots(1+q^{m_k})} = \Sigma\mu_k(n)q^n.$$

Write $$\frac{q^{\frac{1}{2}m(m+1)}}{(1-q)(1-q^2)\dots(1-q^m)} = S'_m,$$

so that $$P_1(1+N_1g+N_2g^2+\dots) = 1-S'_1(1-g)+S'_2(1-g)^2-\dots,$$

$$Q_1(1-M_1g+M_2g^2-\dots) = 1+S'_1(1-g)+S'_2(1-g)^2-\dots,$$

wherein g may be given any value at pleasure.

Thence we find the relations

$$P_1N_k = S'_k-\binom{k+1}{1}S'_{k+1}+\binom{k+2}{2}S'_{k+2}-\dots,$$

$$Q_1M_k = S'_k+\binom{k+1}{1}S'_{k+1}+\binom{k+2}{2}S'_{k+2}+\dots,$$

$$1+2N_1+2^2N_2+\dots = \frac{1+N_1+N_2+\dots}{1-M_1+M_2-\dots} = \frac{1}{1-2M_1+2^2M_2-\dots} = \frac{Q_1}{P_1}$$

$$= \frac{1+S'_1+S'_2+\dots}{1-S'_1+S'_2-\dots},$$

$$P_1(1+N_1+N_2+\dots) = Q_1(1-M_1+M_2-\dots) = 1,$$

$$P_1N_k = Q_1\sum_k^\infty(-)^{s+k}\binom{s}{k}2^{s-k}M_s, \qquad Q_1M_k = P_1\sum_k^\infty\binom{s}{k}2^{s-k}N_s.$$

Introduction of Elliptic Functions.

9. Jacobi in the *Fundamenta Nova*, p. 185, gives the identity

$$P_1\sin x\prod_1^\infty(1-2q^m\cos 2x+q^{2m}) = \sin x-q\sin 3x+q^3\sin 5x-q^6\sin 7x+\dots,$$

from which, by putting $\frac{1}{2}\pi-x$ for x,

$$P_1\cos x\prod_1^\infty(1+2q^m\cos 2x+q^{2m}) = \cos x+q\cos 3x+q^3\cos 5x+q^6\cos 7x+\dots.$$

In the first of these, substituting for the infinite product the expression found in § 1, and putting

$$1-3^{2m+1}q+5^{2m+1}q^3-7^{2m+1}q^6+\dots = J_{2m+1},$$

so that $$P_1^3 = J_1,$$

we find

$$J_1(\sin x+4A_1\sin^3 x+4^2A_2\sin^5 x+\ldots)=J_1x-\frac{1}{3!}x^3J_3+\frac{1}{5!}x^5J_5-\ldots.$$

Expanding the trigonometrical functions and comparing coefficients, we are led to the results

$$2^2.3!\,A_1=-\frac{1}{J_1}(J_3-J_1),$$

$$2^4.5!\,A_2=+\frac{1}{J_1}\{J_5-(1^2+3^2)J_3+1^2.3^2J_1\},$$

$$2^6.7!\,A_3=-\frac{1}{J_1}\{J_7-(1^2+3^2+5^2)J_5+(1^2.3^2+1^2.5^2+3^2.5^2)J_3-1^2.3^2.5^2J_1\},$$

&c.,

and if we write $J_{2m+1}=J^{2m+1}$ symbolically,

$$2^{2k}(2k+1)!\,A_k=(-)^k\frac{1}{J_1}J(J^2-1^2)(J^2-3^2)\ldots\{J^2-(2k-1)^2\},$$

expressing the arithmetical function A_k by means of elliptic series.

Writing $$A_k=\Sigma a_{n,k}q^n,$$

we obtain formulæ which are convenient for calculating the numbers $a_{n,k}$. Thus

$$2^2.3!\,(a_{n,1}-3a_{n-1,1}+5a_{n-3,1}-7a_{n-6,1}+\ldots)$$

= coefficient of q^n in

$$(3^3-3)q-(5^3-5)q^3+(7^3-7)q^6-\ldots,$$

the exponents of q being the triangular numbers.

Also $$2^4.5!\,(a_{n,2}-3a_{n-1,2}+5a_{n-3,2}-7a_{n-6,2}+\ldots)$$

= coefficient of q^n in the series

$$(9.5-10.5^3+5^5)q^3-(9.7-10.7^3+7^5)q^6+(9.9-10.9^3+9^5)q^{10}-\ldots$$

$$2^6.7!\,(a_{n,3}-3a_{n-1,3}+5a_{n-3,3}-7a_{n-6,3}+\ldots)$$

= coefficient of q^n in the series

$$(7^7-35.7^5+259.7^3-225.7)q^6-(9^7-35.9^5+259.9^3-225.9)q^{10}+\ldots$$

where $35=1^2+3^2+5^2$, $259=1^2.3^2+1^2.5^2+3^2.5^2$, $225=1^2.3^2.5^2$.

10. Similarly the second of Jacobi's identities may, if we put

$$1+3^{2m}q+5^{2m}q^3+7^{2m}q^6+\ldots = J_{2m},$$

be written (since $P_1 Q_1^2 = J_0$)

$$J_0(\cos x - 4B_1 \sin^2 x \cos x + 4^2 B_2 \sin^4 \cos x - \ldots) = J_0 - \frac{1}{2!}x^2 J_2 + \frac{1}{4!}x^4 J_4 - \ldots,$$

and we obtain, on expansions in powers of x, the results

$$2^2.2!\, B_1 = \frac{1}{J_0}(J_2 - J_0),$$

$$2^4.4!\, B_2 = \frac{1}{J_0}\{J_4 - (1^2+3^2)J_2 + 1^2.3^2 J_0\},$$

$$2^6.6!\, B_3 = \frac{1}{J_0}\{J_6 - (1^2+3^2+5^2)J_4 + (1^2.3^2+1^2.5^2+3^2.5^2)J_2 - 1^2.3^2.5^2 J_0\},$$

&c.,

and if we write $J_{2m} = J^{2m}$ symbolically,

$$2^{2k}(2k)!\, B_k = \frac{1}{J_0}(J^2-1^2)(J^2-3^2)\ldots\{J^2-(2k-1)^2\},$$

expressing the arithmetical function B_k by means of elliptic series.

Writing $$B_k = \Sigma b_{n,k} q^n,$$

we find $$2^2.2!\,(b_{n,1} - 3b_{n-1,1} + 5b_{n-3,1} - 7b_{n-6,1} + \ldots)$$

= coefficient of q^n in

$$(3^2-1)q + (5^2-1)q^3 + (7^2-1)q^6 + \ldots;$$

$$2^4.4!\,(b_{n,2} - 3b_{n-1,2} + 5b_{n-3,2} - 7b_{n-6,2} + \ldots)$$

= coefficient of q^n in

$$(5^4 - 10.5^2 + 9)q^3 + (7^4 - 10.7^2 + 9)q^6 + (9^4 - 10.9^2 + 9)q^{10} + \ldots;$$

$$2^6.6!\,(b_{n,3} - 3b_{n-1,3} + 5b_{n-3,3} - 7b_{n-6,3} + \ldots)$$

= coefficient of q^n in

$$(7^6 - 35.7^4 + 259.7^2 - 225)q^6 + (9^6 - 35.9^4 + 259.9^2 - 225)q^{10} + \ldots;$$

formulæ from which the numbers $b_{n,k}$ may be conveniently calculated.

In §§ 9 and 10 we have to do with the theta functions

$$\theta_2(x), \quad \theta_3(x)$$

of Jacobi, for we have merely to multiply these functions by $\frac{1}{2}q^{-\frac{1}{4}}$ and then write $\sqrt{q}$ for q to obtain the series employed.

The arithmetical functions are elegantly expressed in terms of the coefficients which present themselves when the series are expanded in powers of x.

11. Another identity of Jacobi is

$$P_2 \prod_1^\infty (1-2q^{2m-1}\cos 2x+q^{4m-2}) = 1-2q\cos 2x+2q^4\cos 4x-2q^9\cos 6x+\ldots = \theta(x),$$

which we may now write

$$\frac{P_1}{Q_1}(1+4C_1\sin^2 x+4^2C_2\sin^4 x+\ldots) = 1-2q\cos 2x+2q^4\cos 4x-2q^9\cos 6x+\ldots,$$

and putting either $\frac{1}{2}\pi-x$ for x or $-q$ for q,

$$\frac{P_1}{Q_1}\prod_1^\infty (1+2q^{2m-1}\cos 2x+q^{4m-2}) = 1+2q\cos 2x+2q^4\cos 4x+2q^9\cos 6x+\ldots = \theta_3(x),$$

or

$$\frac{P_2Q_1^2}{Q_2^2}(1-4D_1\sin^2 x+4^2D_2\sin^4 x-\ldots) = 1+2q\cos 2x+2q^4\cos 4x+2q^9\cos 6x+\ldots.$$

If we denote $2^{2s}q-4^{2s}q^4+6^{2s}q^9-\ldots$ by Kc_{2s},* we have

$$\frac{P_1}{Q_1}(1+4C_1\sin^2 x+4^2C_2\sin^4 x+\ldots) = 1-2Kc_0+2Kc_2\frac{x^2}{2!}-2Kc_4\frac{x^4}{4!}+2Kc_6\frac{x^6}{6!}-\ldots.$$

Expanding the trigonometrical functions and equating coefficients

$$2.2!\,C_1 = \frac{1}{1-2Kc_0}Kc_2,$$

$$2^3.4!\,C_2 = \frac{1}{1-2Kc_0}(4Kc_2-Kc_4),$$

$$2^5.6!\,C_3 = \frac{1}{1-2Kc_0}(64Kc_2-20Kc_4+Kc_6), \text{ \&c.,}$$

* Kc and Kd below are single symbols and have no connexion with the elliptic transcendent K.

and writing $Kc_{2m} = Kc^{2m}$ symbolically,

$$2^{2k-1}(2k)!\, C_k = \frac{1}{1-2Kc_0} Kc^2(2^2-Kc^2)(4^2-Kc^2)\ldots\{(2k-2)^2-Kc^2\}.$$

12. In a precisely similar manner, writing

$$2^{2s}q+4^{2s}q^4+6^{2s}q^9+\ldots = Kd_{2s},$$

$$\frac{P_2Q_1^2}{Q_2^2}(1-4D_1\sin^2x+4^2D_2\sin^4x-\ldots) = 1+2Kd_0-2Kd_2\frac{x^2}{2!}+2Kd_4\frac{x^4}{4!}-\ldots,$$

we find that

$$2.2!\, D_1 = \frac{1}{1+2Kd_0} Kd_2,$$

$$2^3.4!\, D_2 = \frac{1}{1+2Kd_0}(4Kd_2-Kd_4),$$

$$2^5.6!\, D_3 = \frac{1}{1+2Kd_0}(64Kd_2-20Kd_4+Kd_6), \text{ \&c.},$$

and in general putting $Kd_{2m} = Kd^{2m}$ symbolically,

$$2^{2k-1}(2k)!\, D_k = \frac{1}{1+2Kd_0} Kd^2(2^2-Kd^2)(4^2-Kd^2)\ldots\{(2k-2)^2-Kd^2\}.$$

The arithmetical functions are elegantly expressed in terms of the coefficients which arise when the theta functions $\theta(x)$, $\theta_3(x)$ are developed in powers of x.

13. There are no formulæ in the *Fundamenta Nova*, or apparently elsewhere in print, that are suitable for investigating the arithmetical functions connected with the divisors of the forms $5m\pm1$, $5m\pm2$, so that I have recourse to H. B. C. Darling's formulæ.*

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* H. B. C. Darling permits me to give his investigation which follows :—

Starting from the known identity

$$(1-a)\left(1-\frac{x}{a}\right)(1-ax)\left(1-\frac{x}{a}\right)(1-ax^2)\ldots = \frac{1+\sum_1^\infty(-)^r\{a^rx^{[r(r-1)]/2}+a^{-r}x^{[r(r+1)]/2}\}}{(1-x)(1-x^2)(1-x^3)\ldots}, \quad (1)$$

and writing q^5 for x and aq for a, we have

$$(1-aq)\left(1-\frac{q^4}{a}\right)(1-aq^6)\left(1-\frac{q^9}{a}\right)(1-aq^{11})\ldots = \frac{1+\sum_1^\infty(-)^r\{a^rq^{r+[5r(r-1)]/2}+a^{-r}q^{4r+[5r(r+1)]/2}\}}{P_5};$$

so that

$$(1-aq)\left(1-\frac{q}{a}\right)(1-aq^4)\left(1-\frac{q^4}{a}\right)(1-aq^6)\left(1-\frac{q^6}{a}\right)(1-aq^9)\ldots \quad (2)$$

The first is

$$\prod_1^\infty (1-2q^{5m-4}\cos 2x+q^{10m-8})\prod_1^\infty (1-2q^{5m-1}\cos 2x+q^{10m-2})$$

$$=\frac{Q_5}{P_5}\Big[S_2\{1+2q^5\cos 4x+2q^{20}\cos 8x+\ldots+2q^{5r^2}\cos 4rx+\ldots\}$$

$$-2q\,\frac{U_1}{S_2}\{\cos 2x+q^{10}\cos 6x+q^{30}\cos 10x+\ldots+q^{5r(r+1)}\cos(4r+2)x+\ldots\Big],$$

---

is equal to the product of the two series

$$\frac{1}{P_5}\left(1-\alpha q-\frac{1}{\alpha}q^4+\alpha^2q^7+\frac{1}{\alpha^2}q^{11}-\alpha^3q^{18}-\frac{1}{\alpha^3}q^{23}+\alpha^4q^{31}+\frac{1}{\alpha^4}q^{36}\mp\ldots\right)$$

and

$$\frac{1}{P_5}\left(1-\frac{1}{\alpha}q-\alpha q^4+\frac{1}{\alpha^2}q^7+\alpha^2q^{13}-\frac{1}{\alpha^3}q^{18}-\alpha^3q^{27}+\frac{1}{\alpha^4}q^{34}+\alpha^4q^{46}\mp\ldots\right).$$

In this product the term independent of α is

$$\frac{1}{P_5^2}(1+q^2+q^8+q^{14}+q^{26}+q^{36}+q^{54}+\ldots)$$

$$=\frac{1}{P_5^2}(1+q^2)(1+q^8)(1+q^{12})(1+q^{18})\ldots\times(1-q^{10})(1-q^{20})(1-q^{30})\ldots$$

$$=\frac{1}{P_5^2}.S_2.P_5.Q_5=\frac{Q_5}{P_5}.S_2,$$

where

$$S_1\equiv(1+q)(1+q^4)(1+q^6)(1+q^9)\ldots,$$

and S_2 signifies that in S_1, q^2 is written for q.

The coefficient of α in the product is

$$-q.P_5^{-2}(1+q^3+q^7+q^{16}+q^{24}+\ldots)=-\frac{q}{P_5^2}\cdot\frac{U_1}{S_2}.P_5.Q_5=-\frac{Q_5}{P_5}q\,\frac{U_1}{S_2},$$

where

$$U_1\equiv(1+q^2)(1+q^3)(1+q^7)(1+q^8)\ldots;$$

and the coefficient of $1/\alpha$ is evidently the same. Similarly the coefficient of α^2 and $1/\alpha^2$ is $+\frac{Q_5}{P_5}q^5S_2$; that of α^3 and $1/\alpha^3$ is $-\frac{Q_5}{P_5}q^{11}\frac{U_1}{S_2}$; that of α^4 and $1/\alpha^4$ is $+\frac{Q_5}{P_5}q^{20}S_2$; and so on. Thus writing $\alpha=e^{2xi}$ in (2), we see that

$$\prod_1^\infty (1-2q^{5m-4}\cos 2x+q^{10m-8})\prod_1^\infty (1-2q^{5m-1}\cos 2x+q^{10m-2})$$

$$=\frac{Q_5}{P_5}[S_2\{1+2q^5\cos 4x+2q^{20}\cos 8x+\ldots+2q^{5r^2}\cos 4rx+\ldots\}$$

$$-2q\,\frac{U_1}{S_2}\{\cos 2x+q^{10}\cos 6x+q^{30}\cos 10x+\ldots+q^{5r(r+1)}\cos(4r+2)x+\ldots\}].$$

In a precisely similar manner, writing q^5 for x, and αq^2 for α in (1), and subsequently making $\alpha=e^{2xi}$, we obtain

$$\prod_1^\infty (1-2q^{5m-3}\cos 2x+q^{10m-6})\prod_1^\infty (1-2q^{5m-2}\cos 2x+q^{10m-4})$$

$$=\frac{Q_5}{P}[U_2\{1+2q^5\cos 4x+2q^{20}\cos 8x+\ldots+2q^{5r^2}\cos 4rx+\ldots\}$$

$$-2q^2\,\frac{S_1}{U_1}\{\cos 2x+q^{10}\cos 6x+q^{30}\cos 10x+\ldots+q^{5r(r+1)}\cos(4r+2)x+\ldots\}].$$

where $$S_1 = (1+q)\,(1+q^4)(1+q^6)(1+q^9)\ldots(1+q^{5m\pm1})\ldots,$$

$$U_1 = (1+q^2)(1+q^3)(1+q^7)(1+q^8)\ldots(1+q^{5m\pm2})\ldots,$$

as in §§ 6, 7.

S_t, U_t signify that in S_1, U_1, q^t is written for q.

S_2 involves exponents which are twice the numbers of the forms $5m\pm1$,

$$U_1 \div S_2 = (1+q^3)(1+q^7)(1+q^{13})(1+q^{17})\ldots,$$

involving exponents which are of the forms $10m\pm3$.

Writing as in § 5

$$R_1 = (1-q)(1-q^4)(1-q^6)(1-q^9)\ldots,$$

we have $$\frac{P_5}{Q_5}R_1^2(1+4E_1\sin^2 x+4^2E_2\sin^4 x+\ldots)$$

$$= S_2(1+2q^5\cos 4x+2q^{20}\cos 8x+2q^{45}\cos 12x+\ldots)$$

$$-2q\frac{U_1}{S_2}(\cos 2x+q^{10}\cos 6x+q^{30}\cos 10x+\ldots).$$

If $$\gamma_{2s} = 4^{2s}q^5+8^{2s}q^{20}+12^{2s}q^{45}+\ldots,$$

$$\delta_{2s} = 2^{2s}\ \ +6^{2s}q^{10}+10^{2s}q^{30}+\ldots,$$

the right-hand side when expanded in powers of x may be written

$$S_2(1+2\gamma_0)-2q\frac{U_1}{S_2}\delta_0-2\left(S_2\gamma_2-q\frac{U_1}{S_2}\delta_2\right)\frac{x^2}{2!}+2\left(S_2\gamma_4-q\frac{U_1}{S_2}\delta_4\right)\frac{x^4}{4!}-\ldots.$$

Expanding similarly the left-hand side, we find after some obvious reductions

$$\frac{P_5}{Q_5}R_1^2 = S_2(1+2\gamma_0)-2q\frac{U_1}{S_2}\delta_0,$$

$$\frac{P_5}{Q_5}R_1^2.2.2!\,E_1 = -S_2\gamma_2+q\frac{U_1}{S_2}\delta_2,$$

$$\frac{P_5}{Q_5}R_1^2.2^3.4!\,E_2 = +S_2(\gamma_4-4\gamma_2)-q\frac{U_1}{S_2}(\delta_4-4\delta_2),$$

$$\frac{P_5}{Q_5}R_1^2.2^5.6!\,E_3 = -S_2(\gamma_6-20\gamma_4+64\gamma_2)+q\frac{U_1}{S_2}(\delta_6-20\delta_4+64\delta_2),$$

and writing symbolically $\gamma_{2m} = \gamma^{2m}$, $\delta_{2m} = \delta^{2m}$, we have in general

$$\frac{P_5}{Q_5} R_1^2 . 2^{2k-1}(2k)!\, E_k = (-)^k S_2 \gamma^2(\gamma^2-2^2)(\gamma^2-4^2)\ldots\{\gamma^2-(2k-2)^2\}$$
$$+(-)^{k+1} q \frac{U_1}{S_2} \delta^2(\delta^2-2^2)(\delta^2-4^4)\ldots\{\delta^2-(2k-2)^2\}.$$

We observe that γ_{2s} is derivable from Kd_{2s} by multiplying by 2^{2s} and writing q^5 for q; and δ_{2s} from J_{2s} by multiplying by 2^{2s} and writing q^{10} for q. (See §§ 10 and 12.)

14. In the formula of § 13 write $\frac{1}{2}\pi - x$ for x, obtaining

$$\prod_1^\infty (1+2q^{5m-4}\cos 2x+q^{10m-8}) \prod_1^\infty (1+2q^{5m-1}\cos 2x+q^{10m-2})$$
$$= \frac{Q_5}{P_5}\left\{ S_2(1+2q^5\cos 4x+2q^{20}\cos 8x+\ldots \right.$$
$$\left. +2q\frac{U_1}{S_2}(\cos 2x+q^{10}\cos 6x+q^{30}\cos 10x+\ldots)\right\}.$$

We may now write

$$\frac{P_5}{Q_5} S_1^2 (1-4F_1\sin^2 x+4^2F_2\sin^4 x-\ldots)$$
$$= S_2(1+2q^5\cos 4x+2q^{20}\cos 8x+\ldots)$$
$$+2q\frac{U_1}{S_2}(\cos 2x+q^{10}\cos 6x+q^{30}\cos 10x+\ldots),$$

and find that

$$\frac{P_5}{Q_5} S_1^2 = S_2(1+2\gamma_0)+2q\frac{U_1}{S_2}\delta_0,$$

$$\frac{P_5}{Q_5} S_1^2 . 2.2!\, F_1 = S_2\gamma_2+q\frac{U_1}{S_2}\delta_2,$$

$$\frac{P_5}{Q_5} S_1^2 . 2^3 . 4!\, F_2 = S_2(\gamma_4-4\gamma_2)+q\frac{U_1}{S_2}(\delta_4-4\delta_2),$$

$$\frac{P_5}{Q_5} S_1^2 . 2^5 . 6!\, F_3 = S_2(\gamma_6-20\gamma_4+64\gamma_2)+q\frac{U_1}{S_2}(\delta_6-20\delta_4+64\delta_2),$$

and, in general, in symbolic form,

$$\frac{P_5}{Q_5} S_1^2 . 2^{2k-1}(2k)!\, F_k = S_2\gamma^2(\gamma^2-2^2)(\gamma^2-4^2)\ldots\{\gamma^2-(2k-2)^2\}$$
$$+q\frac{U_1}{S_2}\delta^2(\delta^2-2^2)(\delta^2-4^2)\ldots\{\delta^2-(2k-2)^2\}.$$

15. Darling's second formula enables the consideration of the complementary case of divisors of the forms $5m \pm 2$. It is

$$\prod_1^\infty (1-2q^{5m-3}\cos 2x+q^{10m-6}) \prod_1^\infty (1-2q^{5m-2}\cos 2x+q^{10m-4})$$

$$= \frac{Q_5}{P_5}\Big[U_2\{1+2q^5\cos 4x+2q^{20}\cos 8x+\ldots+2q^{5r^2}\cos 4rx+\ldots\}$$

$$-2q^2\frac{S_1}{U_2}\{\cos 2x+q^{10}\cos 6x+q^{30}\cos 10x+\ldots+q^{5r(r+1)}\cos(4r+2)x+\ldots\}\Big].$$

Thence, from § 7,

$$\frac{P_5}{Q_5}T_1^2(1+4G_1\sin^2 x+4^2G_2\sin^4 x+\ldots)$$

$$= U_2(1+2q^5\cos 4x+2q^{20}\cos 8x+2q^{45}\cos 12x+\ldots)$$

$$-2q^2\frac{S_1}{U_2}(\cos 2x+q^{10}\cos 6x+q^{30}\cos 10x+\ldots),$$

and γ_{2s}, δ_{2s} signifying the same as in §§ 13 and 14, the right-hand side when expanded in powers of x takes the form

$$U_2(1+2\gamma_0)-2q^2\frac{S_1}{U_2}\delta_0-2\left(U_2\gamma_2-q^2\frac{S_1}{U_2}\delta_2\right)\frac{x^2}{2!}+2\left(U_2\gamma_4-q^2\frac{S_1}{U_2}\delta_4\right)\frac{x^4}{4!}-\ldots.$$

Expanding similarly the left-hand side, we find after some obvious reductions

$$\frac{P_5}{Q_5}T_1^2 = U_2(1+2\gamma_0)-2q^2\frac{S_1}{U_2}\delta_0,$$

$$\frac{P_5}{Q_5}T_1^2.2.2!\,G_1 = -U_2\gamma_2+q^2\frac{S_1}{U_2}\delta_2,$$

$$\frac{P_5}{Q_5}T_1^2.2^3.4!\,G_2 = +U_2(\gamma_4-4\gamma_2)-q^2\frac{S_1}{U_2}(\delta_4-4\delta_2),$$

$$\frac{P_5}{Q_5}T_1^2.2^5.6!\,G_3 = -U_2(\gamma_6-20\gamma_4+64\gamma_2)+q^2\frac{S_1}{U_2}(\delta_6-20\delta_4+64\delta_2),$$

and generally, in symbolic form,

$$\frac{P_5}{Q_5}T_1^2.2^{2k-1}(2k)!\,G_k = (-)^k U_2\gamma^2(\gamma^2-2^2)(\gamma^2-4^2)\ldots\{\gamma^2-(2k-2)^2\}$$

$$+(-)^{k+1}q^2\frac{S_1}{U_2}\delta^2(\delta^2-2^2)(\delta^2-4^2)\ldots\{\delta^2-(2k-2)^2\}.$$

16. In Darling's second formula write $\frac{1}{2}\pi - x$ for x, obtaining

$$\prod_1^\infty (1+2q^{5m-3}\cos 2x+q^{10m-6}) \prod_1^\infty (1+2q^{5m-2}\cos 2x+q^{10m-4})$$

$$= \frac{Q_5}{P_5}\left\{ U_2(1+2q^5\cos 4x+2q^{20}\cos 8x+2q^{45}\cos 12x+\ldots) \right.$$

$$\left. +2q^2\frac{S_1}{U_2}(\cos 2x+q^{10}\cos 6x+q^{30}\cos 10x+\ldots)\right\}.$$

Thence, from § 7,

$$\frac{P_5}{Q_5}U_1^2(1-4H_1\sin^2 x+4^2H_2\sin^4 x-\ldots)$$

$$= U_2(1+2q^5\cos 4x+2q^{20}\cos 8x+2q^{45}\cos 12x+\ldots)$$

$$+2q^2\frac{S_1}{U_2}(\cos 2x+q^{10}\cos 6x+q^{30}\cos 10x+\ldots),$$

leading as before to the general symbolic formula

$$\frac{P_5}{Q_5}U_1^2.2^{2k-1}(2k)!\,H_k = U_2\gamma^2(\gamma^2-2^2)(\gamma^2-4^2)\ldots\{\gamma^2-(2k-2)^2\}$$

$$+q^2\frac{S_1}{U_2}\delta^2(\delta^2-2^2)(\delta^2-4^2)\ldots\{\delta^2-(2k-2)^2\}.$$

17. The function $A_k = \Sigma a_{n,k}q^n$ has apparently the property that the coefficient $a_{n,k}$ is expressible as a linear function of the sum of the uneven powers of the divisors of n. I have not succeeded in reaching the general theory, but by the help of certain results of Glaisher, obtained by means of elliptic functions, I am able to give the expressions in the cases $n=2$, $n=3$, and $n=4$.

A_1 is the series in which the coefficients are the sums of the divisors of the exponents. Glaisher, in 1885 (*Messenger of Mathematics*, New Series, Nos. 166, 171), obtained expressions for the coefficients of the second, third, fourth, and fifth powers of the series.

Now $$A_1^2 = \{\Sigma\sigma_1(n)q^n\}^2 = \Sigma\frac{q^{2m}}{(1-q^m)^4}+2A_2$$

in the notation of this paper,

$$\Sigma\frac{q^{2m}}{(1-q^m)^4} = \Sigma\sum_{s=0}^{\infty}\binom{s+3}{3}q^{(s+2)m} = \tfrac{1}{6}\Sigma\{\sigma_3(n)-\sigma_1(n)\}q^n,$$

and Glaisher gives

$$\{\Sigma\sigma_1(n)\,q^n\}^2 = \tfrac{1}{12}\Sigma\,\{5\sigma_3(n)-(6n-1)\,\sigma_1(n)\}\,q^n.$$

Thence we find $A_2 = \tfrac{1}{8}\Sigma\,\{\sigma_3(n)-(2n-1)\,\sigma_1(n)\}\,q^n$.

Again,

$$A_1^3 = \{\Sigma\sigma_1(n)\,q^n\}^3 = \Sigma\frac{q^{3m}}{(1-q^m)^6}+3\Sigma\frac{q^{2m_1+m_2}}{(1-q^{m_1})^4\,(1-q^{m_2})^2}+6A_3,$$

where $$\Sigma\frac{q^{3m}}{(1-q^m)^6} = \Sigma\sum_{s=0}^{\infty}\binom{s+5}{5}q^{(s+3)m},$$

and putting $s+3=d$, a divisor of n, the coefficient of q^n is

$$\Sigma\tfrac{1}{120}(d+2)(d+1)(d)(d-1)(d-2) = \Sigma\tfrac{1}{120}(d^5-5d^3+4d).$$

Hence $$\Sigma\frac{q^{3m}}{(1-q^m)^6} = \tfrac{1}{120}\Sigma\,\{\sigma_5(n)-5\sigma_3(n)+4\sigma_1(n)\}\,q^n.$$

To calculate $\Sigma\frac{q^{2m_1+m_2}}{(1-q^{m_1})^4\,(1-q^{m_2})^2}$ we observe that it may be written

$$\left\{\Sigma\frac{q^m}{(1-q^m)^2}\right\}\left\{\Sigma\frac{q^{2m}}{(1-q^m)^4}\right\}-\Sigma\frac{q^{3m}}{(1-q^m)^6},$$

and is thus equal to

$$\{\Sigma\sigma_1(n)q^n\}\,\{\tfrac{1}{6}\Sigma[\sigma_3(n)-\sigma_1(n)]\,q^n\}-\tfrac{1}{120}\Sigma\{\sigma_5(n)-5\sigma_3(n)+4\sigma_1(n)\}\,q^n.$$

The product which is the first term of this expression is

$$\tfrac{1}{6}\sum_{r=1}^{n-1}\sigma_1(r)\,\sigma_3(n-r)q^n-\tfrac{1}{6}\{\Sigma\sigma_1(n)\,q^n\}^2,$$

and since, by a result of Glaisher obtained by elliptic functions,

$$\Sigma\sigma_1(r)\,\sigma_3(n-r)\,q^n = \tfrac{1}{240}\Sigma\,\{21\sigma_5(n)-10(3n-1)\,\sigma_3(n)-\sigma_1(n)\}\,q^n,$$

it may be written

$$\Sigma\tfrac{1}{1440}\{21\sigma_5(n)-10(3n-1)\,\sigma_3(n)-\sigma_1(n)\}\,q^n$$
$$-\tfrac{1}{72}\Sigma\,\{5\sigma_3(n)-(6n-1)\,\sigma_1(n)\}\,q^n,$$

or $$\tfrac{1}{1440}\Sigma\,\{21\sigma_5(n)-30(n+3)\,\sigma_3(n)+(120n-21)\,\sigma_1(n)\}\,q^n,$$

and thence

$$\Sigma\frac{q^{2m_1+m_2}}{(1-q^{m_1})^4\,(1-q^{m_2})^2} = \tfrac{1}{480}\Sigma\,\{3\sigma_5(n)-10\,(n+1)\,\sigma_3(n)+(40n-23)\,\sigma_1(n)\}\,q^n.$$

Moreover Glaisher gives

$$A_1^3 = \{\Sigma\sigma_1(n)\,q^n\}^3$$
$$= \tfrac{1}{192}\Sigma\{7\sigma_5(n) - 10(3n-1)\,\sigma_3(n) + (24n^2-12n+1)\,\sigma_1(n)\}\,q^n,$$

and we now obtain the result

$$A_3 = \tfrac{1}{1920}\Sigma\{3\sigma_5(n) - 10(3n-5)\,\sigma_3(n) + (40n^2-100n+37)\,\sigma_1(n)\}\,q^n.$$

Observe that the lowest exponent of q in A_3 is 6, so that the circumstance that the expression found must vanish for smaller values of n furnishes a verification.

In general we may take the following view.

Denoting by $\sigma_k(n)$ the sum of the k-th powers of the divisors of n, the function $\Sigma\sigma_1(n)\,q^n$ is

$$\frac{q}{(1-q)^2} + \frac{q^2}{(1-q^2)^2} + \frac{q^3}{(1-q^3)^2} + \ldots,$$

or writing

$$\frac{q^m}{(1-q^m)^2} = \alpha_m,$$

it is

$$\alpha_1 + \alpha_2 + \alpha_3 + \ldots = \Sigma\alpha.$$

Glaisher has found an expression for the coefficients of q^n in the case of the powers

$$(\Sigma\alpha)^2, \quad (\Sigma\alpha)^3, \quad (\Sigma\alpha)^4, \quad (\Sigma\alpha)^5,$$

by an application of elliptic functions.

We may consider other symmetric functions of the elements α.

Writing as usual

$$\Sigma\,\alpha_1^{k_1}\alpha_2^{k_2}\ldots\alpha_s^{k_s} = (k_1 k_2 \ldots k_s)_\alpha,$$

Glaisher has dealt with

$$(1)_\alpha^2, \quad (1)_\alpha^3, \quad (1)_\alpha^4, \quad (1)_\alpha^5,$$

and it is obvious that

$$A_k = (1^k)_\alpha,$$

when A_k is the arithmetical function dealt with in § 1 of this paper.

Observe that it is easy to obtain the expression of the coefficient in the symmetric function

$$(k)_\alpha = \sum_1^\infty \frac{q^{km}}{(1-q^m)^{2k}},$$

and this will lead, potentially, to the expression of the coefficient in the

symmetric function

$$(k_1 k_2 \dots k_s)_a.$$

For
$$\frac{q^{km}}{(1-q^m)^{2k}} = \sum_{s=0}^{\infty} \binom{2k+s-1}{s} q^{(k+s)m},$$

where, if $k+s = d$, a divisor of n, the coefficient of q^n in $(k)_a$ is

$$\Sigma \binom{d+k-1}{2k-1},$$

where the sum is taken for every divisor d of n.

This sum has the expression

$$\frac{1}{(2k-1)!} \Sigma\, d(d^2-1^2)(d^2-2^2) \dots \{d^2-(k-1)^2\},$$

which, if

$$(x-1^2)(x-2^2) \dots \{x-(k-1)^2\} = x^{k-1} - c_{1,k-1}\, x^{k-2} + c_{2,k-1}\, x^{k-2} - \dots,$$

may be written

$$\frac{1}{(2k-1)!} \Sigma \{d^{2k-1} - c_{1,k-1}\, d^{2k-3} + \dots + (-)^{k-1} c_{k-1,k-2}\, d\}.$$

Thence

$$(k)_a = \frac{1}{(2k-1)!} \Sigma \{\sigma_{2k-1}(n) - c_{1,k-1}\, \sigma_{2k-3}(n) + \dots + (-)^{k-1} c_{k-1,k-2}\, \sigma_1(n)\},$$

a general formula.

In obtaining the results, for the order 4, I have made much use of Glaisher's paper above referred to. In order to arrive finally at the expression of $A_4 \equiv (1^4)$ it was practically necessary to find the expression of the remaining symmetric functions. These are perhaps worth putting on record.

In the process the following formulæ which I could not find elsewhere had to be investigated, viz.,

$$\Sigma\Sigma\, r\sigma_1(r)\, \sigma_3(n-r)\, q^n = \tfrac{1}{240} \Sigma \{7n\sigma_5(n) - 6n^2\sigma_3(n) - n\sigma_1(n)\}\, q^n,$$

$$\Sigma\Sigma\, (n-r)\, \sigma_1(r)\, \sigma_3(n-r)\, q^n = \tfrac{1}{120} \Sigma \{7n\sigma_5(n) - n(12n-5)\, \sigma_3(n)\}\, q^n,$$

$$\Sigma\Sigma\, \sigma_1(r)\, \sigma_5(n-r)\, q^n = \tfrac{1}{504} \Sigma \{20\sigma_7(n) - 21(2n-1)\, \sigma_5(n) + \sigma_1(n)\}\, q^n,$$

and the complete system of symmetric functions, obtained through them

is, commencing with that due to Glaisher,

$$(1)_a^4 = \frac{1}{3!\,4!\,4!}\Sigma\{5\sigma_7(n)-21(2n-1)\sigma_5(n)+3(36n^2-30n+5)\sigma_3(n) -(72n^3-72n^2+18n-1)\sigma_1(n)\}q^n,$$

$$(2)_a(1)_a^2 = \frac{1}{3!\,4!\,5!}\Sigma\{5\sigma_7(n)-21(n+2)\sigma_5(n)+3(6n^2+70n-25)\sigma_3(n) -(180n^2-93n+8)\sigma_1(n)\}q^n,$$

$$(2)_a^2 = \frac{1}{3!\,3!\,5!}\Sigma\{\sigma_7(n)-21\sigma_5(n)+3(10n+13)\sigma_3(n) -(60n-11)\sigma_1(n)\}q^n,$$

$$(3)^a(1)_a = \frac{1}{4!\,7!}\Sigma\{40\sigma_7(n)-21(4n+19)\sigma_5(n)+210(3n+7)\sigma_3(n) -(2016n-359)\sigma_1(n)\}q^n,$$

$$(4)_a = \frac{1}{7!}\Sigma\{\sigma_7(n)-14\sigma_5(n)+49\sigma_3(n)-36\sigma_1(n)\}q^n$$

$$(31)_a = \frac{1}{4!\,7!}\Sigma\{16\sigma_7(n)-21(4n+3)\sigma_5(n)+42(15n+7)\sigma_3(n) -(2016n-1223)\sigma_1(n)\}q^n,$$

$$(2^2)_a = \frac{2}{4!\,7!}\Sigma\{\sigma_7(n)-63\sigma_5(n)+21(10n-1)\sigma_3(n) -(420n-293)\sigma_1(n)\}q^n,$$

$$(21^2)_a = \frac{1}{4!\,7!}\Sigma\{5\sigma_7(n)-63\sigma_5(n)+(126n^2+420n-1365)\sigma_3(n) -(1260n^2-3507n+1433)\sigma_1(n)\}q^n,$$

$$A_4 = (1^4)_a = \frac{1}{4!\,4!\,7!}\Sigma\{15\sigma_7(n)-189(2n-7)\sigma_5(n) +63(36n^2-21n+235)\sigma_3(n) -3(840n^3-5880n^2+9870n-3229)\sigma_1(n)\}q^n.$$

The only difficulty in continuing the series for the higher orders lies in the circumstance that it is necessary to evaluate the forms

$$\Sigma r^t\sigma_{m_1}(r)\,\sigma_{m_2}(n-r)\,q^n.$$

This soon becomes laborious.

The expression of $(k)_a$ which has been given above in terms of the co-

efficients $c_{1,k-1}, c_{2,k-1}, \ldots, c_{k-1,k-1}$ as defined, are in the early cases:

$$(1)_a = \Sigma \sigma_1(n)\, q^n,$$

$$3!\,(2)_a = \Sigma \sigma_3(n)\, q^n - \Sigma \sigma_1(n)\, q^n,$$

$$5!\,(3)_a = \Sigma \sigma_5(n)\, q^n - 5\Sigma \sigma_3(n)\, q^n + 4\Sigma \sigma_1(n)\, q^n,$$

$$7!\,(4)_a = \Sigma \sigma_7(n)\, q^n - 14\Sigma \sigma_5(n)\, q^n + 49\Sigma \sigma_3(n)\, q^n - 36\Sigma \sigma_1(n)\, q^n,$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

the coefficients being exhibited in the table

1				
1	1			
1	5	4		
1	14	49	36	
1	30	273	820	576
...	...	...	...	...

These numbers are readily calculated by means of the relation

$$c_{k,s} = c_{k,s-1} + s^2 c_{k-1,s-1},$$

from which it appears that the number in the r-th row and c-th column is obtainable by adding the number in the $(r-1)$-th row and c-th column to $(r-1)^2$ times the number in the $(r-1)$-th row and $(c-1)$-th column.

Thus, for $r = 5,\ c = 3$,

$$273 = 49 + 4^2 \cdot 14,$$

and in general we obtain a number from the number above it and the number to the left of the latter.

From the relations we deduce the set

$$\Sigma \sigma_1(n) q^n = (1)_a,$$

$$\Sigma \sigma_3(n) q^n = (1)_a + 3!\,(2)_a,$$

$$\Sigma \sigma_5(n) q^n = (1)_a + 5 \cdot 3!\,(2)_a + 5!\,(3)_a,$$

$$\Sigma \sigma_7(n) q^n = (1)_a + 21 \cdot 3!\,(2)_a + 14 \cdot 5!\,(3)_a + 7!\,(4)_a,$$

$$\Sigma \sigma_9(n) q^n = (1)_a + 85 \cdot 3!\,(2)_a + 147 \cdot 5!\,(3)_a + 30 \cdot 7!\,(4)_a + 9!\,(5)_a,$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

and we have the table of coefficients

1				
1	1			
1	5	1		
1	21	14	1	
1	85	147	30	1
...	...	...	...	...

where the number in the r-th row and c-th column is $h_{r-c,\,c}$, where $h_{k,}$ is the homogeneous product sum of the numbers

$$1^2,\ 2^2,\ 3^2,\ \ldots,\ s^2,$$

taken k together. Thus 85 being in the fifth row and second column is given by

$$h_{3,\,2} = 1^6+2^6+1^4.\,2^2+1^2.\,2^4 = 85.$$

Now since

$$h_{k,\,s} = h_{k,\,s-1}+s^2h_{k-1,\,s},$$

we see that the number in the r-th row and c-th column is equal to the number in the $(r-1)$-th row and $(c-1)$-th column added to c^2 times the number in the $(r-1)$-th row and c-th column.

Thus for $r = 5,\ c = 3,$

$$147 = 21+3^2.\,14.$$

The numbers are therefore readily calculated.

The corresponding algebraic identities are

$$\sum_1^\infty \frac{m^3q^m}{1-q^m} = \sum_1^\infty \frac{q^m}{(1-q^m)^2}+3!\sum_1^\infty \frac{q^{2m}}{(1-q^m)^4},$$

$$\sum_1^\infty \frac{m^5q^m}{1-q^m} = \sum_1^\infty \frac{q^m}{(1-q^m)^2}+5.3!\sum_1^\infty \frac{q^{2m}}{(1-q^m)^4}+5!\sum_1^\infty \frac{q^{3m}}{(1-q^m)^6},\ \text{&c.},$$

for the latter set, and

$$3!\sum_1^\infty \frac{q^{2m}}{(1-q^m)^4} = \sum_1^\infty \frac{m^3q^m}{1-q^m}-\sum_1^\infty \frac{mq^m}{1-q^m},$$

$$5!\sum_1^\infty \frac{q^{3m}}{(1-q^m)^6} = \Sigma \frac{m^5q^m}{1-q^m}-5\Sigma \frac{m^3q^m}{1-q^m}+4\Sigma \frac{mq^m}{1-q^m},\ \text{&c.},$$

for the former.

These are of interest in relation to the well known identity

$$\sum_1^\infty \frac{mq^m}{1-q^m} = \sum_1^\infty \frac{q^m}{(1-q^m)^2}.$$

18. The function B_k is connected with $\xi(n)$ in Glaisher's notation, $\xi(n)$ denoting the sum of the uneven divisors of n minus the sum of the even divisors. Putting

$$\frac{q^m}{(1+q^m)^2} = \beta_m,$$

and considering symmetric functions of the elements β,

$$(1)_b = \Sigma\xi_1(n)q^n,$$

$$-3!(2)_b = \Sigma\{\xi_3(n)-\xi_1(n)\}q^n,$$

$$+5!(3)_b = \Sigma\{\xi_5(n)-5\xi_3(n)+4\xi_1(n)\}q^n,$$

$$-7!(4)_b = \Sigma\{\xi_7(n)-14\xi_5(n)+49\xi_3(n)-36\xi_1(n)\}q^n, \text{ \&c.},$$

$$\Sigma\xi_1(n)q^n = (1)_b,$$

$$\Sigma\xi_3(n)q^n = (1)_b-3!(2)_b,$$

$$\Sigma\xi_5(n)q^n = (1)_b-5.3!(2)_b+5!(3)_b,$$

$$\Sigma\xi_7(n)q^n = (1)_b-21.3!(2)_b+14.5!(3)_b-7!(4)_b, \text{ \&c.}$$

Now since a known result is

$$4\{\Sigma\xi_1(n)q^n\}^2 = 4(1)_b^2 = \Sigma\{-\xi_3(n)+(2n-1)\xi_1(n)\}q^n,$$

we find that

$$B_2 = (1^2)_b = +\frac{1}{4!}\Sigma\{-\xi_3(n)+(6n-5)\xi_1(n)\}q^n.$$

I also find that

$$B_3 = (1^3)_b = \frac{1}{2.4!\,5!}\Sigma\{\xi_5(n)-10(3n-7)\xi_3(n) +(120n^2-420n+259)\xi_1(n)\}q^n,$$

$$(21)_b = \frac{1}{4.5!}\Sigma\{\xi_5(n)-10(n-1)\xi_3(n)+(40n-41)\xi_1(n)\}q^n,$$

and for the cube of the series $\Sigma\xi_1(n)q^n$,

$$(1)_b^3 = \tfrac{1}{64}\Sigma\{\xi_5(n)-6(n-1)\xi_3(n)+(8n^2-12n+3)\xi_1(n)\}q^n.$$

In the course of the work the following result was of service, viz.:

$$\sum_{r=1}^{r=n-1}\xi_1(r)\xi_3(n-r) = \tfrac{1}{16}\{-\xi_5(n)+2(n-1)\xi_3(n)+\xi_1(n)\}.$$

13. Passing to the functions C_k and D_k, we have

$$C_1 = \sum_1^\infty \frac{q^{2m-1}}{(1-q^{2m-1})^2} = \Sigma \Delta_1'(n) q^n,$$

where $\Delta_1'(n)$ is the sum of those divisors of n which have uneven conjugates.

Also $$D_1 = \sum_1^\infty \frac{q^{2m-1}}{(1+q^{2m-1})^2} = \Sigma (-)^{n+1} \Delta_1'(n) q^n,$$

so that it is only necessary to consider the function C_k.

The symmetric functions denoted by brackets

$$(\quad)_c$$

refer to the elements of type

$$a_m = \frac{q^{2m-1}}{(1-q^{2m-1})^2}.$$

We have

$$(k)_c = \sum_1^\infty \frac{q^{k(2m-1)}}{(1-q^{2m-1})^{2k}} = \Sigma\Sigma \binom{2k+s-1}{2k-1} q^{(k+s)(2m-1)},$$

so that if $h+s=d$ be a divisor of n which has an uneven conjugate, the coefficients of q^n in $(k)_c$ is

$$\Sigma \binom{d+k-1}{2k-1},$$

the sum being taken for all divisors d.

This expression being

$$\frac{1}{(2k-1)!} d(d^2-1^2)(d^2-2^2) \ldots \{d^2-(k-1)^2\},$$

the functions $(k)_c$ are expressed in the same manner as $(k)_a$, but now the divisors are such as have uneven conjugates. Thus

$$(2)_c = \frac{1}{3!} \Sigma \{\Delta_3'(n) - \Delta_1'(n)\} q^n,$$

and Glaisher gives $$(1)_c^2 = \tfrac{1}{4} \Sigma \{\Delta_3'(n) - n\Delta_1'(n)\} q^n,$$

whence $$C_2 = (1^2)_c = \frac{1}{4!} \Sigma \{\Delta_3'(n) - 3n\Delta_1'(n) + 2\Delta_1'(n)\} q^n.$$

For the order 3 I find the results

$$(1)_c^3 = \tfrac{1}{64}\Sigma\{\Delta_5'(n) - 3n\Delta_3' + 2n^2\Delta_1'(n)\}\, q^n,$$

$$(2)_c(1)_c = \tfrac{1}{96}\Sigma\{\Delta_5'(n) - (n+4)\,\Delta_3'(n) + 4n\Delta_1'(n)\}\, q^n,$$

$$(3)_c = \frac{1}{5!}\Sigma\{\Delta_5'(n) - 5\Delta_3'(n) + 4\Delta_1'(n)\}\, q^n\},$$

$$(21)_c = \frac{1}{4.5!}\Sigma\{\Delta_5'(n) - 5n\Delta_3'(n) + 4(5n-4)\,\Delta_1'(n)\}\, q^n,$$

$$C_3 = (1^3)_c = \tfrac{1}{5760}\Sigma\{\Delta_5'(n) - 5(3n-8)\,\Delta_3'(n) + 2(15n^2 - 60n + 32)\Delta_1'(n)\}\, q^n.$$

20. The function

$$N_1 = \sum_1^\infty \frac{q^m}{1-q^m} = \Sigma\nu_1(n)\, q^n,$$

which enumerates the divisors of n does not lend itself readily to treatment by means of elliptic functions.

In Glaisher's notation

$$\nu_1(n) \equiv \sigma_0(n),$$

consider the symmetric functions of the terms of the series, viz.:

$$\frac{q}{1-q},\ \frac{q^2}{1-q^2},\ \frac{q^3}{1-q^3},\ \ldots,$$

and denote them by brackets $(\ \ )_0$.

Then $$(k)_0 = \Sigma\binom{d-1}{k-1} q^n,$$

where d is a divisor of n. Thence, if

$$(x-1)(x-2)\ldots\{x-(k-1)\} = x^{k-1} - g_{1,\,k-1}\,x^{k-2} + g_{2,\,k-1}\,x^{k-3} - \ldots,$$

$$(k)_0 = \Sigma\{\sigma_{k-1}(n) - g_{1,\,k-1}\,\sigma_{k-2}(n) + g_{2,\,k-1}\,\sigma_{k-3}(n) - \ldots\}\, q^n,$$

and we have the relations

$$(1)_0 = \Sigma\sigma_0(n)\, q^n,$$

$$(2)_0 = \Sigma\{\sigma_1(n) - \sigma_0(n)\}\, q^n,$$

$$(3)_0 = \Sigma\{\sigma_2(n) - 3\sigma_1(n) + 2\sigma_0(n)\}\, q^n,$$

$$(4)_0 = \Sigma\{\sigma_3(n) - 6\sigma_2(n) + 11\sigma_1(n) - 6\sigma_0(n)\}\, q^n,\ \&c.,$$

and inversely $\Sigma\sigma_0(n)q^n = (1)_0,$

$$\Sigma\sigma_1(n)q^n = (1)_0+(2)_0,$$

$$\Sigma\sigma_2(n)q^n = (1)_0+3(2)_0+(3)_0,$$

$$\Sigma\sigma_3(n)q^n = (1)_0+7(2)_0+6(3)_0+(4)_0, \text{ \&c.},$$

$$\Sigma\sigma_s(n)q^n = (1)_0+\frac{1}{1!}(2^s-1^s)(2)_0+\frac{1}{2!}(3^s-2.2^s+1^s)(3)_0$$

$$+\frac{1}{3!}(4^s-3.3^s+3.2^s-1^s)(4)_0+\ldots.$$

Comparing these formulæ with the symmetric functions $(\ )_a$ of the elements

$$\frac{q}{(1-q)^2}, \quad \frac{q^2}{(1-q^2)^2}, \quad \frac{q^3}{(1-q^3)^2}, \quad \cdots$$

dealt with in § 17, there is no difficulty in establishing the relations

$$(1)_0+(2)_0 = (1)_a,$$

$$6(2)_0+6(3)_0+(4)_0 = (2)_a,$$

$$60(3)_0+60(4)_0+15(5)_0+(6)_0 = 120(3)_a, \text{ \&c.},$$

and, in general,

$$\frac{1}{(k-1)!}(k)_0+\frac{k}{k!}(k+1)_0+\frac{\binom{k}{2}}{(k+1)!}(k+2)_0+\frac{\binom{k}{3}}{(k+2)!}(k+3)_0+\ldots$$

$$+\frac{1}{(2k-1)!}(2k)_0 = (k)_a.$$

It will be observed that

$$\Sigma\sigma_s(n)\,q^n = \Sigma\frac{m^s q^m}{1-q^m}$$

is expressible as a linear function of the symmetric functions $(k)_0$ of the elements

$$\frac{q}{1-q}, \quad \frac{q^2}{1-q^2}, \quad \frac{q^3}{1-q^3}, \quad \ldots.$$

The square of the series

$$\frac{q}{1-q}+\frac{q^2}{1-q^2}+\frac{q^3}{1-q^3}+\ldots=\Sigma\nu_1(n)q^n$$

is expressible by means of the functions $\sigma_1(n)$, $\nu_1(n)$, $\nu_2(n)$.

From the above formulæ it is at once seen that

$$\{\Sigma\nu_1(n)q^n\}^2=\Sigma\sum_{r=1}^{r=n-1}\nu_1(r)\,\nu_1(n-r)q^n=\Sigma\{\sigma_1(n)-\nu_1(n)+2\nu_2(n)\}\,q^n,$$

where $\nu_k(n)$ is the function considered in § 8.

TABLES.

$a_{n,k}$

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	... n
1	1	3	4	7	6	12	8	15	13	18	12	28	14	24	24	31	
2			1	3	9	15	30	45	67	99	135	175	231	306	354	465	
3						1	3	9	22	42	84	140	231	351	551	783	
4										1	3	9	22	51	97	188	
5															1	3	
⋮																	
k																	

$b_{n,k}$

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	... n
1	1	−1	4	−5	6	−4	8	−13	13	−6	12	−20	14	−8	24	−29	
2			1	−1	1	3	−2	1	−5	23	−25	27	−49	74	−62	85	
3						1	−1	1	−2	10	−11	12	−21	31	−13	23	
4										1	− 1	1	− 2	3	+ 5	− 4	
5															1	− 1	
⋮																	
k																	

$c_{n,k}$

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	... n
1	1	2	4	4	6	8	8	8	13	12	12	16	14	16	24	16	
2				1	2	4	8	14	18	28	40	52	70	88	104	140	
3									1	2	4	8	14	24	40	56	
4																1	
⋮																	
k																	

$\nu_k(n)$

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	... n
1	1	2	2	3	2	4	2	4	3	4	2	6	2	4	4	5	
2			1	2	5	6	11	13	17	22	27	29	37	44	44	55	
3						1	2	5	10	15	25	37	52	67	97	117	
4										1	2	5	10	20	30	53	
5															1	2	
⋮																	
k																	

$\mu_k(n)$

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	... n
1	1	0	2	−1	2	0	2	−2	3	0	2	−2	2	0	4	−3	
2			1	0	1	2	1	1	1	6	−1	5	−3	10	0	9	
3						1	0	1	0	5	−1	5	−2	9	3	9	
4										1	0	1	0	2	4	2	
5															1	0	
⋮																	
k																	

For convenience of reference I collect the symbols that have been employed.

$$A_k = \sum\sum\ldots\sum_{m_1<m_2\ldots<m} \frac{q^{m_1+m_2+\ldots+m_k}}{(1-q^{m_1})^2(1-q^{m_2})^2\ldots(1-q^{m_k})^2} = \Sigma a_{n,k}\,q^n,$$

$$B_k = \sum\sum\ldots\sum_{m_1<m_2\ldots<m_k} \frac{q^{m_1+m_2+\ldots+m_k}}{(1+q^{m_1})^2(1+q^{m_2})^2\ldots(1+q^{m_k})^2} = \Sigma b_{n,k}\,q^n,$$

$$P_t = \prod_1^\infty (1-q^{tm}), \qquad Q_t = \prod_1^\infty (1+q^{tm}),$$

$$C_k = \sum\sum\ldots\sum_{m_1<m_2\ldots<m_k} \frac{q^{2m_1+2m_2+\ldots+2m_k-k}}{(1-q^{2m_1-1})^2(1-q^{2m_2-1})^2\ldots(1-q^{2m_k-1})^2} = \Sigma c_{n,k}\,q^n,$$

$$D_k = \sum\sum\ldots\sum_{m_1<m_2\ldots<m_k} \frac{q^{2m_1+2m_2+\ldots+2m_k-k}}{(1+q^{2m_1-1})^2(1+q^{2m_2-1})^2\ldots(1+q^{2m_k-1})^2} = \Sigma d_{n,k}\,q^n,$$

$E_k \equiv A_k$ with $m_1, m_2, \ldots, m_k$ restricted to the forms $5m \pm 1$,

$F_k \equiv B_k$,, ,, ,,

$G_k \equiv A_k$,, ,, $5m \pm 2$,

$H_k \equiv B_k$,, ,, ,,

$$R_t = (1-q^t)(1-q^{4t})(1-q^{6t}) \dots \{1-q^{(5m\pm1)t}\} \dots,$$

$$S_t = (1+q^t)(1+q^{4t})(1+q^{6t}) \dots \{1+q^{(5m\pm1)t}\} \dots,$$

$$T_t = (1-q^{2t})(1-q^{3t})(1-q^{7t}) \dots \{1-q^{(5m\pm2)t}\} \dots,$$

$$U_t = (1+q^{2t})(1+q^{3t})(1+q^{7t}) \dots \{1+q^{(5m\pm2)t}\} \dots,$$

$$N_k = \sum_{m_1<m_2\dots<m_k} \sum \dots \sum \frac{q^{m_1+m_2+\dots+m_k}}{(1-q^{m_1})(1-q^{m_2}) \dots (1-q^{m_k})} = \Sigma \nu_k(n)\, q^n,$$

$$M_k = \sum_{m_1<m_2\dots<m_k} \sum \dots \sum \frac{q^{m_1+m_2+\dots+m_k}}{(1+q^{m_1})(1+q^{m_2}) \dots (1+q^{m_k})} = \Sigma \mu_k(n) q^n,$$

$$J_{2m+1} = 1-3^{2m+1}q+5^{2m+1}q^3-7^{2m+1}q^6+\dots,$$

$$J_{2m} = 1+3^{2m}q+5^{2m}q^3+7^{2m}q^6+\dots,$$

$$Kc_{2m} = 2^{2m}q-4^{2m}q^4+6^{2m}q^9-\dots, \quad Kd_{2m} = 2^{2m}q+4^{2m}q^4+6^{2m}q^9+\dots,$$

$$\gamma_{2m} = 4^{2m}q^5+8^{2m}q^{20}+12^{2m}q^{45}+\dots,$$

$$\delta_{2m} = 2^{2m}+6^{2m}q^{10}+10^{2m}q^{30}+\dots .$$

NOTE ON A PROPERTY OF DIRICHLET'S SERIES

By K. Ananda-Rau.

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1. Let the abscissa of convergence of the Dirichlet's series

$$f(s) = \Sigma a_n e^{-\lambda_n s}$$

be σ_0, and let $f(s)$ be regular for $\sigma > \gamma$, where $\gamma \leqslant \sigma_0$. Let $\beta > \gamma$. The behaviour of $f(\beta+it)$ as $|t| \to \infty$ is extremely complicated, and the results that have been obtained concerning it are far from final. Any little information on this head is therefore of some interest; and it is the object of this note to prove a negative result, namely that $f(\beta+it)$ *cannot tend to a definite finite limit as* $|t| \to \infty$, *except, of course, in the trivial case when* $f(s)$ *is an absolute constant.* This is done in § 3. In § 2 I prove a less precise result, namely that the order* of $f(s)$ for $\sigma = \beta$ cannot be negative, that is to say, *there does not exist any* $\delta > 0$, *such that*

$$f(\beta+it) = O(|t|^{-\delta}), \tag{1}$$

as $|t| \to \infty$.

2. To prove this last result it is clearly sufficient to show that, if (1) is true, then $a_n = 0$ for every n. Let $c > \sigma_0$, $c > \beta > a$, $\lambda_n \leqslant x < \lambda_{n+1}$. Then by a formula due to Perron,†

$$\lim_{T\to\infty} \int_{c-iT}^{c+iT} f(s) \frac{e^{xs}}{s-a}\, as = 2\pi i \sum_{\nu=1}^{n}{}' a_\nu e^{(x-\lambda_\nu)a}, \tag{2}$$

where the accent on the sign of summation means that, in case $x = \lambda_n$, the term $a_n e^{(x-\lambda_n)a}$ in the sum should be replaced by $\frac{1}{2}a_n$.

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* See G. H. Hardy and M. Riesz, "The General Theory of Dirichlet's Series," *Camb Math. Tracts*, No. 18, p. 14.

† *Ibid.*, Theorem 13.

Now, by a known theorem,*

$$f(c+it) = o(|t|).$$

Also $$f(\beta+it) = O(|t|^{-\delta}) = o(|t|),$$

by hypothesis. Hence, by Lindelöf's theorem,†

$$f(\sigma+it) = o(|t|), \tag{3}$$

uniformly for $\beta \leqslant \sigma \leqslant c$.

Let us now apply Cauchy's theorem to the integral

$$\int f(s)\, \frac{e^{xs}}{s-a}\, ds,$$

taken round the rectangle whose corners are $\beta-iT$, $\beta+iT$, $c+iT$, $c-iT$. We easily see that the contributions from the shorter ends tend to zero as $T \to \infty$, in virtue of (3). We thus get

$$\lim_{T\to\infty} \int_{\beta-iT}^{\beta+iT} f(s)\, \frac{e^{xs}}{s-a}\, ds = 2\pi i \sum_{\nu=1}^{n}{}' a_\nu e^{(x-\lambda_\nu)a}. \tag{4}$$

Now, since $$f(\beta+it) = O(|t|^{-\delta}),$$

it follows that the integral in (4) is uniformly convergent with respect to x in any finite interval $\lambda_1 \leqslant x \leqslant H$. Hence, by a classical theorem, it represents a function of x which is continuous for $x \geqslant \lambda_1$. Therefore

$$\sum_{\nu=1}^{n}{}' a_\nu e^{(x-\lambda_\nu)a}$$

is continuous for $x = \lambda_1$, $x = \lambda_2$, and so on. This can only be the case if $a_n = 0$ for every n.

3. I prove in this section that, if $f(\beta+it) \to 0$, then $a_n = 0$ for every n; from which it easily follows that $f(\beta+it)$ cannot tend to a limit. By a formula due to Hadamard‡

$$\lim_{\omega\to\infty} \frac{1}{2\omega} \int_{-\omega}^{+\omega} e^{\lambda_n(\eta+it)} f(\eta+it)\, dt = a^n,$$

where $\eta > \sigma_0$. By the argument used in the last section, it can be shown that in the above integral η may be replaced by β. Since $f(\beta+it) \to 0$,

* *Ibid.*, Theorem 12.

† *Ibid.*, Theorem 14.

‡ Landau, *Handbuch der Lehre von der Verteilung der Primzahlen*, p. 788.

there exists, corresponding to every $\epsilon > 0$, an ω_0 such that for a fixed n and for $|t| \geqslant \omega_0$,

$$|e^{\lambda_n(\beta+it)} f(\beta+it)| < \epsilon.$$

Hence* $$|a_n| \leqslant \overline{\lim_{\omega\to\infty}} \frac{1}{2\omega}\left[\left|\int_{-\omega}^{-\omega_0}\right| + \left|\int_{-\omega_0}^{+\omega_0}\right| + \left|\int_{\omega_0}^{\omega}\right|\right] < 2\epsilon.$$

As a_n is clearly independent of ϵ, it follows that $a_n = 0$ for every n.

4. The result proved in § 2 enables us to extend the properties of $\mu(\sigma)$, proved in Theorem 15 of the *Cambridge Tract* for Dirichlet's series having regions of absolute convergence, to the most general Dirichlet's series. We can, in fact, prove that $\mu(\sigma)$, in its region of definition, is a positive, convex, continuous and decreasing function. When the series has a half plane of absolute convergence $\mu(\sigma)$ is constantly zero after a certain value of σ; but this is not necessarily true when there is no such half-plane.

5. The determination of $\mu(\sigma)$ for a given Dirichlet's series is, in general, a difficult problem. But in the particular case when the series converges throughout the plane,† $\mu(\sigma)$ has an important property, namely that it is a constant. This easily follows from the fact that $\mu(\sigma) \leqslant 1$,‡ and the property of convex functions contained in the following theorem:

Suppose that $\tau(x)$ is an increasing convex function of x defined for $x \geqslant x_0$, and that it is bounded as $x \to \infty$. Then $\tau(x)$ is a constant.

The truth of this is intuitively obvious from geometrical considerations. The arithmetical proof does not present any difficulty.

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* The function $e^{\lambda_n(\beta+it)} f(\beta+it)$ must be understood as the integrand.

† This is also true if only we suppose that the series is summable (λ, κ) all over the plane, for a fixed κ.

‡ Hardy and Riesz, *loc. cit.*, Theorem 12.

ON THE TRIANGULATION METHOD OF DEFINING THE AREA OF A SURFACE

By W. H. YOUNG, Sc.D., F.R.S.

[Read May 15th, 1919.]

1. In a communication to the Royal Society, I have explained how it is possible, by the introduction of the intermediate notion of *the area of a skew curve*, to construct a theory of the area of curved surfaces, which, unlike previous theories, appears to secure all the desiderata, and, in particular, generalises in a satisfactory manner the known theory of *the length of a curve*. Moreover the new theory can itself be extended so as to be applicable to the volumes and hyper-volumes of curved manifolds of any number of dimensions, situated in a space of any greater number of dimensions.

It would seem to lie in the very nature of things that no other equally satisfactory theory can exist. On the other hand, the considerations which led earlier workers to base their researches on a different mode of generalising the concept of the length of a curve, are likely to have some weight, alike with those who have accustomed themselves to reason in the older language, and with those who have a preference for definitions expressible in terms of the simplest possible processes that can be practically carried out.

I propose therefore in the present paper to give a definition of the area of a surface by means of triangulation, which, though it differs in a material point from any hitherto constructed, is such as to satisfy both the classes of thinkers above referred to. Its range of application, however, is of necessity considerably less than that of the definition depending on the concept of the area of a skew curve. As regards this latter point, it is in fact to be remarked that the mere existence of the double integral expressing the area of the surface, to which we are led, should, if possible, correspond to the existence of an area for the surface, and that the limitations on the functions imposed by any method of triangulation are such as to constitute a far more serious falling short of the aim thus set before

us, than in the alternative theory. An entirely new restriction has to be introduced, which appears to involve something of the nature of continuity of the derivates of the functions.

The results obtained in the present paper seem however of interest, not merely for the light they throw on the whole theory, but for themselves. They not only form the basis of a simple theory of the area of curved surfaces, which can be regarded as exhaustive by the working mathematician, or physicist, who is usually prepared to concede any convenient conditions with regard to continuity, but also they appear to be in generality and completeness far in advance of any hitherto obtained.

The success achieved is due in part to the nature of the methods employed, which seem to be entirely new in this connection and involve moreover the utilisation of the new method I have recently given, based on the idea of the area of a skew curve.

The essential novelty in the definition is *the introduction of the idea of order* (*double order*). A surface is to be regarded as having a double order, just as a curve possesses a single order. The expression of the coordinates of a point (x, y, z) on the surface as functions of two independent variables (u, v) is accordingly to be regarded as not merely a convenience, but as corresponding to the essential requirements of the problem of areas. The triangles that we construct will then naturally be ordered, as regards u and v, in a manner analogous to the process adopted in the construction of chords of a curve in order, to give an approximation to the length. This, it may be noted, is precisely the opposite to the course adopted, with only partial success, by Lebesgue.* In fact, he modifies the definition of the length of a curve, by considering it as the limit of the lengths of polygons *not* inscribed in the curve, and it is this modification which he attempts to generalise, when dealing with the areas of surfaces.

One other departure from the usage adopted by the writer just referred to, is worthy of mention. I have endeavoured to avoid a definition expressed in terms of a plurality of limits, and have shewn that, under the conditions given, *a unique repeated limit* for the areas of the inscribed polyhedral surfaces, constructed in accordance with my definition, leads to the desired formula for the area of the surface. We can indeed in two most interesting cases (§§ 26, 27), one relating to a type of surface that appears never to have been discussed, obtain the integral in question as the *unique double limit* of the area of our inscribed polyhedra.

The definition is as follows:—The surface being defined by the

* "Intégrale, longueur, aire."

equations

$$x = x(u, v), \quad y = y(u, v), \quad z = z(u, v), \quad (a \leqslant u \leqslant c,\ b \leqslant v \leqslant d)$$

we first divide up the fundamental (u, v)-rectangle $(a, c\,;\ b, d)$ into sub-rectangles of norms $\bar{h}$ and $\bar{k}$ [that is, we draw successive parallels to the axis of u at distances $\leqslant \bar{k}$, and successive parallels to the axis of v at distances $\leqslant \bar{h}$], and these sub-rectangles we halve by drawing the diagonals, sloping down, say, from left to right. We then take the points on the surface, which are the images of the corners of these semi-rectangles, and join them in a manner similar to that in which these corners are joined so as to form the semi-rectangles in the (u, v)-plane. We thus obtain a set of triangles, forming a polyhedron inscribed in the surface, *related to*, and determined by, our division of the fundamental (u, v)-rectangle into semi-rectangles. This set of triangles has accordingly the same double order as the set of semi-rectangles, this double order being determined by the parameters, or independent variables (u, v), by means of which the surface was defined.

It is the area of this inscribed polyhedral surface which forms an approximation to the area of the portion of our surface corresponding to the fundamental (u, v)-rectangle, when we make, say, first $\bar{k}$ tend to zero, and afterwards $\bar{h}$ decrease indefinitely.

Taking this definition, we then have the following theorem on the existence of an area and its expression as a double integral:—

If the functions $x(u, v)$, $y(u, v)$, and $z(u, v)$, defining the coordinates of a point on a surface, for values of (u, v) in the fundamental (u, v)-rectangle $(a, c\,;\ b, d)$, satisfy the following conditions, the portion of the surface corresponding to this rectangle possesses an area, and this area is given by the double integral

$$\int_a^c \int_b^d \left\{ \left(\frac{\partial(x, y)}{\partial(u, v)}\right)^2 + \left(\frac{\partial(y, z)}{\partial(u, v)}\right)^2 + \left(\frac{\partial(z, x)}{\partial(u, v)}\right)^2 \right\}^{\frac{1}{2}} du\, dv.$$

The conditions fall into two categories. Firstly, those I have already proved sufficient for the application of the former definition of area; these are (i), (ii), and (iii), and are, in particular, all satisfied if the partial derivates of x, y and z are bounded. Secondly an extra condition, introduced *ad hoc*; an extra condition of this kind seems unavoidable.

(i) *Each of the functions $x(u, v)$, $y(u, v)$, and $z(u, v)$ is an absolutely convergent integral with respect to u, and an absolutely convergent integral with respect to v.*

(ii) *The partial derivates of $x(u, v)$, $y(u, v)$, and $z(u, v)$ with respect to v are all numerically less than a summable function $\mu(v)$ of v alone.*

(iii) *The total variations of $x(u, v)$, $y(u, v)$, and $z(u, v)$ with respect to u, for constant v, are bounded functions of v; or, more generally, possess absolutely convergent integrals with respect to $\int \mu(v)\,dv$.**

(iv) *Each pair of the functions $x(u, v)$, $y(u, v)$, and $z(u, v)$ is such that the sum of the two functions, each multiplied by whatever constant we please, constitutes a function whose total variation with respect to v is a continuous function of u.*

In two important cases, in which all these conditions are satisfied, viz.

(A) *When all the partial differential coefficients of $x(u, v)$, $y(u, v)$, and $z(u, v)$ exist everywhere and are continuous functions of (u, v);*

(B) *When $x(u, v)$, $y(u, v)$, and $z(u, v)$ are absolutely convergent double integrals;*

we can, as already remarked, go further, and assert that the areas of the approximating polyhedral surfaces have a unique double limit, and that this unique double limit is

$$\int_a^c \int_b^d \left\{ \left(\frac{\partial(x, y)}{\partial(u, v)}\right)^2 + \left(\frac{\partial(y, z)}{\partial(u, v)}\right)^2 + \left(\frac{\partial(z, x)}{\partial(u, v)}\right)^2 \right\}^{\frac{1}{2}} du\,dv.$$

It is worthy of note that in case (A) a slightly more general result is obtained by more elementary methods. We can indeed, as is shewn in § 33, prove that the concept is now independent of the order on the surface. In fact *any* mode of triangulation leads to the required result, provided only each of the triangles in the (u, v)-plane is such that at least one of its angles lies between γ and $\pi - \gamma$, where γ is any angle, as small as we please, fixed in advance and kept constant during the whole process at all its stages. Thus, in particular, degenerate or evanescent triangles are not admissible.

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* In other words, $\int \frac{\partial x}{\partial u}\,\mu(v)\,dv$ and the integrals got by changing x into y and z, all exist, and are summable functions of u.

2. In order to deal with the expression which occurs in our present problem, we require to define a new kind of integral, as follows.

DEFINITION.—*If $x(U)$, $y(U)$, $z(U)$, ... be any number of positive monotone increasing functions of U, and $\theta(U)$, $\phi(U)$, $\psi(U)$, the same number of continuous functions of U, then, the interval $a \leqslant U \leqslant b$ being supposed divided up into n parts each $< \delta$, at the points $a, u_1, u_2, \ldots, u_{n-1}, b$, the summation*

$$\sum_{r=1}^{n} |\, \theta(u)\{x(u_r)-x(u_{r-1})\} + \phi(u)\{y(u_r)-y(u_{r-1})\} + \ldots \,|,$$

where u is any point whatever in $u_{r-1} \leqslant u \leqslant u_r$, has, as $\delta \to 0$, and therefore also $n \to \infty$, a unique limit. We denote this limit by

$$\int_a^b |\, \theta(U)\, dx(U) + \phi(U)\, dy(U) + \ldots \,|.$$

To prove the uniqueness of the limit, we proceed as follows:—

We shall write out the proof for three terms in the integrand. Suppose first the functions $\theta(U)$, $\phi(U)$, and $\psi(U)$ are always $\geqslant 0$, or always $\leqslant 0$, and let λ_1, λ_2, and λ_3 have the same signs as θ, ϕ, and ψ respectively, where

$$\lambda_1^2 = \lambda_2^2 = \lambda_3^2 = 1.$$

Next write $$\int_a^u \lambda_1 \theta(U)\, dx(U) = X(u), \quad (a \leqslant u \leqslant b),$$

so that $X(u)$ is always positive and monotone increasing. Then

$$\lambda_1 \theta(u)\{x(u_r) - x(u_{r-1})\} = \int_{u_{r-1}}^{u_r} \lambda_1 \theta(u)\, dx(U),$$

and $$X(u_r) - X(u_{r-1}) = \int_{u_{r-1}}^{u_r} \lambda_1 \theta(U)\, dx(U).$$

If therefore δ be chosen so small that the oscillations of the continuous functions $\theta(U)$, $\phi(U)$, and $\psi(U)$ are all $< \epsilon$,

$$|\, \lambda_1 \theta(u)\{x(u_r)-x(u_{r-1})\} - \{X(u_r)-X(u_{r-1})\} \,| < \int_{u_{r-1}}^{u_r} \epsilon\, dx(U),$$

or, which is the same thing,

$$|\, \theta(u)\{x(u_r)-x(u_{r-1})\} - \lambda_1\{X(u_r)-X(u_{r-1})\} \,| < \epsilon \int_{u_{r-1}}^{u_r} dx(U).$$

Changing in turn x into y and z, and simultaneously X into Y and Z,

and adding, we obtain, since the modulus of a sum does not exceed the sum of the moduli,

$$|\theta(u)\{x(u_r)-x(u_{r-1})\}+\phi(u)\{y(u_r)-y(u_{r-1})\}+\psi(u)\{z(u_r)-z(u_{r-1})\}$$
$$-\lambda_1\{X(u_r)-X(u_{r-1})\}-\lambda_2\{Y(u_r)-Y(u_{r-1})\}-\lambda_3\{Z(u_r)-Z(u_{r-1})\}|$$
$$<\epsilon\int_{u_{r-1}}^{u_r}\{dx(U)+dy(U)+dz(U)\}.$$

But the modulus of a difference is not less than the difference (taken positively) of the moduli. Hence the typical term of our summation differs from

$$|\lambda_1\{X(u_r)-X(u_{r-1})\}+\lambda_2\{Y(u_r)-Y(u_{r-1})\}+\lambda_3\{Z(u_r)-Z(u_{r-1})\}|$$

by less than $\epsilon\int_{u_{r-1}}^{u_r}\{dx(U)+dy(U)+dz(U)\}$,

and therefore our summation differs by less than

$$\epsilon\int_a^b\{dx(U)+dy(U)+dz(U)\},$$

from $\Sigma|P(u_r)-P(u_{r-1})|$,

where $P(u)=\lambda_1X(u)+\lambda_2Y(u)+\lambda_3Z(u)$

$$=\int_a^u\{\theta(U)\,dx(U)+\phi(U)\,dy(U)+\psi(U)\,dz(U)\}.$$

But $\Sigma|P(u_r)-P(u_{r-1})$

is known to have a unique limit, viz. the total variation of $P(u)$, and ϵ is as small as we please. This therefore proves the theorem in this case. But, in the general case, we may write

$$\theta(u)=\theta_1(u)-\theta_2(u),$$

where $\theta_1(u)$ and $\theta_2(u)$ are both positive. This therefore merely adds another term between the modulus marks and does not therefore affect our result

Thus the uniqueness of the limit is proved.

3. If $x(u)$, $y(u)$, and $z(u)$ are the integrals of positive functions, the

expression for $P(u)$ becomes

$$\int_a^u \left\{ \theta(U)\frac{\partial x}{\partial U} + \phi(U)\frac{\partial y}{\partial U} + \psi(U)\frac{\partial z}{\partial U} \dots \right\} dU,$$

and therefore our limit, being the total variation of $P(u)$ is

$$\int_a^b \left| \theta(U)\frac{\partial x}{\partial U} + \phi(U)\frac{\partial y}{\partial U} + \psi(U)\frac{\partial z}{\partial U} \dots \right| dU.$$

Hence we can evidently say that, if $x(u)$, $y(u)$, and $z(u)$ are *any* integrals with respect to u, and $\theta(u)$, $\phi(u)$, and $\psi(u)$ any continuous functions, the summation

$$\sum_{r=1}^{r} \left| \theta(u) \{x(u_r) - x(u_{r-1})\} + \dots \right|,$$

has a unique limit, and this is

$$\int_a^b \left| \theta(U)\frac{\partial x(U)}{\partial U} + \dots \right| dU.$$

In fact we only have to express $x(u)$, $y(u)$, $z(u)$, ... as the difference of the integrals of two positive functions and apply the preceding result.

4. Let $\quad x = x(u, v), \quad y = y(u, v) \quad (a \leqslant u \leqslant c,\ b \leqslant v \leqslant d)$

define a correspondence between the (x, y) and the (u, v)-planes, such that* the area of any and every contour in the (x, y)-plane which is the image of the perimeter of any and every rectangle, with sides parallel to the axes of u and v, is expressed by the integral of the function

$$B(u, v) = \frac{\partial(x, y)}{\partial(u, v)} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}$$

over the area of that rectangle.

If now the fundamental rectangle $(a, c;\ b, d)$ be divided up in any manner whatever, by means of parallels to the axes of u and v, into M sub-rectangles R_i of span less than a norm δ, then, if C_i denote the area of the contour which is the image of R_i, we have

$$C_i = \iint_{R_i} B(u, v)\, du\, dv, \tag{1}$$

* See my recent communication "On a Formula for an Area," *Proc. London Math. Soc.*, Ser. 2, Vol. 18 (1920), pp. 339–374.

and therefore

$$\left|\int_a^c\int_b^d B(u,v)\,du\,dv\right| \leqslant \sum_{i=1}^{M} |C_i| \leqslant \int_a^c\int_b^d |B(u,v)|\,du\,dv. \qquad (2)$$

We have now the following theorem :—

THEOREM 1.—*If the norm δ tend to zero in any manner, the sum $\sum_{i=1}^{M} |C_i|$ of the areas, taken positively, of the contours which are the images of the sub-rectangles of span $< \delta$, into which the fundamental (u, v)-rectangle has been divided, tends to a unique limit, independent of the mode in which $\delta \to 0$, and of the particular form of the sub-rectangles ; moreover, this limit is*

$$\int_a^c\int_b^d |B(u,v)|\,du\,dv.$$

The proof of this theorem need not be given here at length, as the argument is closely analogous to that employed in a communication of mine to the Royal Society, shortly to be published "On the Area of Surfaces."

We first prove that a unique limit exists.

It is then evident from (2) that this limit cannot exceed the double integral in question.

To prove that the limit cannot be less than the double integral, we define a function $\phi(u, v)$ as given by this limit, the fundamental rectangle being replaced by the rectangle $(a, c\,;\ u, v)$; $\phi(u, v)$ is then monotonely monotone, and is therefore not less than the double integral of its mixed differential coefficient $\dfrac{\partial^2\phi}{\partial u\,\partial v}$, which exists except at a plane set of points of content zero. Now the double increment of $\phi(u, v)$,

$$\phi(u+h, v+k)-\phi(u, v+k)-\phi(u+h, v)+\phi(u, v)$$

is, by the definition of ϕ, our limit itself, with the fundamental rectangle replaced by the small rectangle $(u, v\,;\ u+h, v+h)$.

Hence, by (2),

$$\left|\frac{1}{hk}\int_u^{u+h}\int_v^{v+k} B(u,v)\,du\,dv\right|$$

$$\leqslant \{\phi(u+h, v+k)-\phi(u, v+k)-\phi(u+h, v)-\phi(u, v)\}/hk,$$

whence, letting $h \to 0$, $k \to 0$,

$$\frac{\partial^2 \phi}{\partial u \, \partial v} \geqslant |B(u, v)|,$$

and therefore

$$\phi(u, v) \geqslant \int_a^u \int_b^v \frac{\partial^2 \phi}{\partial u \, \partial v} \, du \, dv \geqslant \int_a^u \int_b^v |B(u, v)| \, du \, dv.$$

This completes the proof of the theorem.

5. In place of the contours C_i, we shall now consider a system of triangles. The *vertices* of these triangles will be taken to correspond to the *vertices* of such a system of sub-rectangles as we have supposed in the (u, v)-plane. The sides of the triangles will, of course, not be the images of the lines joining the corresponding points, but will be determined by these, and will accordingly be said, like the triangles, to be "related."

In the (u, v)-plane the triangles, that is the semi-rectangles, will for definiteness be supposed constructed by adding to the parallels already drawn, the diagonals of the sub-rectangles sloping down, say, from left to right, *i.e.* making positive intercepts on the axes of u and v.

We then have, without *any* restriction on the generality of the correspondence, the following theorem :—

THEOREM 2.—*If we have any (first) mode of division of the fundamental (u, v)-rectangle into M sub-rectangles, whose images are the contours $C_1, C_2, \ldots, C_M$, and η be any chosen positive quantity, we can find ϵ' so small that, when the norms $\bar{h}$ and $\bar{k}$ are both $\leqslant \epsilon'$, and we divide the fundamental (u, v)-rectangle in any (second) mode into sub-rectangles of norms $\bar{h}$ and $\bar{k}$, and these into semi-rectangles by means of the diagonals, sloping down from left to right, then the sum $\sum_{n=1}^{m} |D_n|$ of the areas of the "related" triangles is connected with the sum $\sum_{r=1}^{M} |C_r|$ of those of the contours, the areas in each case taken positively, by the relation*

$$\sum_{r=1}^{M} |C_r| \leqslant \sum_{n=1}^{m} |D_n| + \eta,$$

provided only the dividing lines of the first mode are included in those of the second mode.

To prove this theorem, let us group the semi-rectangles, of norms $\leqslant \epsilon'$, according to the rectangles R_j of the first mode of division which

they exactly fill up. In this way the perimeter of each rectangle R_j is divided up at certain marked points, to which correspond the vertices in order of a polygonal figure F_j, inscribed in the contour C_j.

By definition, the letters denoting areas,

$$C_j = \operatorname*{Lt}_{\epsilon' \to 0} F_j;$$

therefore, if ϵ' is conveniently small,

$$\sum_{j=1}^{M} C_j \leqslant \sum_{j=1}^{M} F_j + \eta. \tag{3}$$

On the other hand, the area F_j of the polygonal figure is defined as half the resultant couple of a system of forces, represented by the sides of the figure. This couple is the same as the resultant couple of the system of forces represented by the sides in order of the triangles D_n, "related" to the semi-rectangles, grouped together to form the rectangle R_j. This is therefore none other than the resultant of the couples yielded by each triangle D_n separately.

Hence it follows that the magnitude of our resultant couple is not greater than the sum of the areas, taken positively, of the triangles $|D_n|$.

Thus
$$\sum_{j=1}^{M} |F_j| \leqslant \sum_{n=1}^{m} |D_n|. \tag{4}$$

From (3) and (4) the theorem at once follows.

6. It is evident from Theorems 1 and 2 that, *if the correspondence is such as was contemplated in* § 4, however we divide up the fundamental (u, v)-rectangle into m_i semi-rectangles of norms $\bar{h}_i$, $\bar{k}_i$, in successive stages, so that $\bar{h}_i$ and $\bar{k}_i$ tend towards zero as $i \to \infty$, we shall have

$$\int_a^c \int_b^d |B(u, v)|\, du\, dv \leqslant \text{lower limit}_{i \to \infty} \sum_{n=1}^{m_i} |D_n|,$$

provided the dividing parallels at each stage are retained at all subsequent stages.

We may accordingly say that, *for such modes of construction by successive interpolation of parallels in the* (u, v)*-plane,* our double integral is less than, or equal to, every possible double limit of the sum $\sum_{n=1}^{m} |D_n|$ of the areas of the triangles "related" to the semi-rectangles, when the norms $\bar{h}$ and $\bar{k}$ tend *in any manner* to zero.

7. We shall now shew that, if the correspondence is such that the conditions of § 4 are satisfied, in virtue of certain restrictions already determined in my former paper,* and if we add an additional condition, we can connect the sum $\sum_{n=1}^{m} |D_n|$ directly with the double integral as limit. At present, however, the limit, though determinate, is not *a unique double limit*, but *a unique repeated limit*, the norms tending successively to zero, in a definite order, prescribed by the extra condition. In this result, unlike that of the preceding article, only the law of the norms is essential, the particular mode in which the sub-rectangles are constructed in no way affecting the result.

8. Denoting by (u, v) that vertex of our semi-rectangle where we have a right angle, and by $(u+h, v)$ and $(u, v+k)$ the other vertices, we have, appertaining to each (u, v), two pairs (h, k), one in which both members are positive, and one in which both are negative; except in the case when (u, v) lies on the periphery of the fundamental rectangle, where there is only one such pair (h, k).

In this way we obtain

$$|D_n| = | \tfrac{1}{2}\{x(u+h, v)-x(u, v)\}\{y(u, v+k)-y(u, v)\} - \tfrac{1}{2}\{y(u+h, v)-y(u, v)\}\{x(u, v+k)-x(u, v)\} | ,$$

and we have to consider the double summation of this expression all over the fundamental rectangle, as (u, v) moves over all the points of intersection of the dividing parallels at the stage considered.

In order to treat this summation, we require the result in generalised integration, given in §§ 2 and 3.

Supposing $x(u, v)$ and $y(u, v)$ to be any integrals with respect to v, and treating u and h as parameters, our result enables us to state that

$$\sum_{b}^{d} | \{x(u+h, v)-x(u, v)\}\{y(u, v+k)-y(u, v)\} - \{y(u+h, v)-y(u, v)\}\{x(u, v+k)-x(u, v)\} |$$

tends, as the norm $\bar{k}$ approaches zero in *any* manner, to a unique limit, and this is

$$\int_b^d dV \left| \frac{\partial y(u, V)}{\partial V}\{x(u+h, V)-x(u, V)\} - \frac{\partial x(u, V)}{\partial V}\{y(u+h, V)-y(u, V)\} \right| .$$

* See below, § 17.

Summing this expression for each value of u determined by our divisions, and for the positive and negative values of h appertaining to that value of u, we have

$$\operatorname*{Lt}_{k\to 0} 2 \sum_a^c \sum_b^d |D_n|$$

$$= \sum_a^c \int_b^d dV \left| \frac{\partial y(u.\,V)}{\partial V} \{x(u+h,\,V)-x(u,\,V)\} - \frac{\partial x(u,\,V)}{\partial V} \{y(u+h,\,V)-y(u,\,V)\} \right|$$

$$= \int_b^d dV \sum_a^c \left| \frac{\partial y(u,\,V)}{\partial V} \int_u^{u+h} \frac{\partial x(U,\,V)}{\partial U}\, dU - \frac{\partial x(u,\,V)}{\partial V} \int_u^{u+h} \frac{\partial y(U,\,V)}{\partial U}\, \partial U \right|$$

$$= \int_b^d dV \sum_a^c \left| \int_u^{u+h} \left\{ \frac{\partial x(U,\,V)}{\partial U} \frac{\partial y(u,\,V)}{\partial V} - \frac{\partial x(u,\,V)}{\partial V} \frac{\partial y(U,\,V)}{\partial U} \right\} dU \right|$$

$$\leqslant \int_b^d dV \sum_a^c \int_u^{u+h} \left| \frac{\partial x(U,\,V)}{\partial U} \frac{\partial y(u,\,V)}{\partial V} - \frac{\partial x(u,\,V)}{\partial V} \frac{\partial y(U,\,V)}{\partial U} \right| dU$$

$$\leqslant \int_b^d dV \int_a^c \left| \frac{\partial x(U,\,V)}{\partial U} \frac{\partial \phi_r(U,\,V)}{\partial V} - \frac{\partial y(U,\,V)}{\partial U} \frac{\partial \psi_r(U,\,V)}{\partial V} \right| dU$$

$$\leqslant \int_a^c dU \int_b^d \left| \frac{\partial x}{\partial U} \frac{\partial \phi_r}{\partial V} - \frac{\partial y}{\partial U} \frac{\partial \psi_r}{\partial V} \right| dV, \qquad (\alpha)$$

where $$\phi_r(U,\,V) = y(u,\,V), \qquad \psi_r(U,\,V) = x(u,\,V), \qquad (\beta)$$

when $$u \leqslant U < u+h,$$

as in § 25 of my paper on "A Formula for an Area," so that

$$\left.\begin{aligned} \frac{\partial \phi_r(U,\,V)}{\partial V} &= \frac{\partial y(u,\,V)}{\partial V} \\ \frac{\partial \psi_r(U,\,V)}{\partial V} &= \frac{\partial x(u,\,V)}{\partial V} \end{aligned}\right\}. \qquad (\gamma)$$

9. We now make the extra hypothesis that, *for any and every pair of constants P and Q, positive, negative, or zero,*

$$Py(U,\,V) - Qx(U,\,V),$$

*has with respect to V a total variation which is a continuous function of U.**

Since this total variation is expressed by

$$\int \left| P\frac{\partial y(U,V)}{\partial V} - Q\frac{\partial x(U,V)}{\partial V} \right| dV,$$

and, by the definition of $\phi_r(U, V)$ and $\psi_r(U, V)$, we have, for

$$u \leqslant U \leqslant u+h.$$

$$\int \left| P\frac{\partial y(u,V)}{\partial V} - Q\frac{\partial x(u,V)}{\partial V} \right| dV = \int \left| P\frac{\partial \phi_r(U,V)}{\partial V} - Q\frac{\partial \psi_r(U,V)}{\partial V} \right| dV,$$

our hypothesis as to the continuity gives us, when $V \to \infty$, and therefore the norm $\bar{h}_r \to 0$, and accordingly $u \to U$,

$$\operatorname*{Lt}_{r\to\infty} \int \left| P\frac{\partial \phi_r}{\partial V} - Q\frac{\partial \psi_r}{\partial V} \right| dV = \int \left| P\frac{\partial y}{\partial V} - Q\frac{\partial x}{\partial V} \right| dV\,(\delta).$$

10. We shall now require the following theorem on the integration of successions :—

THEOREM.—*If $f_n(x)$ and $\phi_n(x)$ describe successions of functions of $x = (x_1, x_2, \ldots, x_n)$ such that, for all values of the constants P and Q,*

$$\operatorname*{Lt}_{n\to\infty} \int |\, Pf_n(x) - Q\phi_n(x) \,|\, dx = \int |\, Pq(x) - Q\phi(x) \,|\, dx, \tag{5}$$

and†

$$\operatorname*{Lt}_{\substack{E\to 0\\ n\to\infty}} \int_E f_n(x)\, dx = 0, \qquad \operatorname*{Lt}_{\substack{E\to 0\\ n\to\infty}} \int_E \phi_n(x)\, dx = 0, \tag{6}$$

then

$$\operatorname*{Lt}_{n\to\infty} \int |\, f_n(x)\, g(x) - \phi_n(x)\, \gamma(x) \,|\, dx = \int |\, q(x)\, g(x) - \phi(x)\, \gamma(x) \,|\, dx, \tag{7}$$

* As regards this hypothesis, it will be noticed that it is, as it should be, invariant for projective transformation of the (x, y)-plane into itself. Such a transformation, in fact, multiplies the area of each triangle by the determinant of transformation, while at the same time it multiplies the double integral by the same constant. The new condition, on the other hand, transforms into itself. Moreover it is evident that we could not advantageously replace this condition by a simple one relating to x and to y separately, unless it should appear that the former is a consequence of the latter : projective transformation would in fact turn such a formally simpler condition into one of the form adopted in the present article.

† As usual E denotes here on the one hand the set of points over which we integrate, and on the other its content which tends to zero. The limit has to be a unique double limit.

provided the integrals involved exist and are absolutely convergent, and*

$$\operatorname*{Lt}_{\binom{E\to 0}{n\to\infty}} \int_E f_n(x)\, g(x)\, dx = 0, \qquad \operatorname*{Lt}_{\binom{E\to 0}{n\to\infty}} \int_E \phi_n(x)\, \gamma(x)\, dx = 0. \tag{8}$$

Since (6) holds, we have

$$\operatorname*{Lt}_{\binom{E\to 0}{n\to\infty}} \int_E |\, Pf_n(x) - Q\phi_n(x)\,|\, dx = 0,$$

whence it follows, by a known theorem,† *that* (5) *holds for integration over any set of points*, although in the enunciation it is only postulated that it holds over every "interval."

Let then, in the first instance, $g(x)$ *and* $\gamma(x)$ *be bounded functions.* Then we can find functions $g_r(x)$ and $\gamma_r(x)$ which differ from $g(x)$ and $\gamma(x)$ respectively by less than e at every point, and assume only a finite number of values. Let $S_1, S_2, \ldots, S_m$ be a finite number of sets of points over which *both* $g_r(x)$ and $\gamma_r(x)$ are constant, say on S_s,

$$g_r(x) = P_s, \qquad \gamma_r(x) = Q_s.$$

Then, by what was remarked above, we have for each index s,

$$\operatorname*{Lt}_{n\to\infty} \int_{S_s} |\, P_s f_n(x) - Q\ \phi_n(x)\,|\, dx = \int_{S_s} |\, P_s q(x) - Q_s \psi(x)\,|\, dx,$$

and therefore, adding these relations, since the sets $S_1, S_2, \ldots, S_m$ are non-overlapping and fill up our whole interval,

$$\operatorname*{Lt}_{n\to\infty} \int |\, g_r(x) f_n(x) - \gamma_r(x)\ \phi_n(x)\,|\, dx = \int |\, g_r(x)\, q(x) - \gamma_r(x)\, \psi(x)\,|\, dx. \tag{7a}$$

We now apply the inequality‡

$$\left| \int |\lambda(x)|\, dx - \int |\mu(x)|\, dx \right| \leqslant \int |\lambda(x) - \mu(x)|\, dx. \tag{a}$$

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* It will be seen that that on the right exists if those on the left do so.

† W. H. Young, "On Successions of Integrals and Fourier Series" (1911), *Proc. London Math. Soc.*, Ser. 2, Vol. 11, p. 61.

‡ The proof of this inequality is elementary. The left-hand side is

$$\left| \int \{|\lambda(x)| - |\mu(x)|\}\, dx \right| \leqslant \int \Big| |\lambda(x)| - |\mu(x)| \Big|\, dx,$$

where, since the modulus of a sum $\leqslant$ the sum of the moduli,

$$|\lambda(x)| - |\mu(x)| \leqslant |\lambda(x) - \mu(x)| \geqslant |\mu(x)| - |\lambda(x)|.$$

This gives, on the one hand,

$$\left| \int |g(x) f_n(x) - \gamma(x)\, \phi_n(x)|\, dx - \int |g_r(x) f_n(x) - \gamma_r(x)\, \phi_n(x)|\, dx \right| \leqslant e \int \{|f_n(x)| + |\phi_n(x)|\}\, dx,$$

and, on the other, a similar inequality, with $f_n(x)$ replaced by $q(x)$ and $\phi_n(x)$ by $\psi(x)$. Hence, using (5), we obtain, by (7a),

$$\operatorname*{Lt}_{n\to\infty} \left| \int |g(x) f_n(x) - \gamma(x)\, \phi_n(x)|\, dx - \int |g(x)\, q(x) - \gamma(x)\, \psi(x)|\, dx \right| \leqslant 2e \int \{|q(x)| + |\psi(x)|\}\, dx.$$

Since e is as small as we please, this implies (7) *which is accordingly proved, when $g(x)$ and $\gamma(x)$ are bounded functions.*

Next let $g(x)$ and $\gamma(x)$ be unbounded, and let us define auxiliary bounded functions

$$g_r(x) = g(x), \qquad \gamma_r(x) = \gamma(x),$$

wherever $|g(x)| \leqslant r$ and $|\gamma(x)| \leqslant r$, while at the remaining points, forming a set E, say,

$$g_r(x) = 0, \qquad \gamma_r(x) = 0.$$

Then, by our preceding result, (7a) holds. Also the content E tends to zero as $r \to \infty$, since at each point of E either

$$|g(x)| > r \quad \text{or} \quad |\gamma(x)| > r.$$

Thus, by the inequality (a),

$$\left| \int |g(x) f_n(x) - \gamma(x)\, \phi_n(x)|\, dx - \int |g_r(x) f_n(x) - \gamma_r(x)\, \phi_n(x)|\, dx \right| \leqslant \int_E |g(x) f_n(x) - \gamma(x)\, \phi_n(x)|\, dx,$$

where, by (8), the right-hand side has the unique double limit zero, when $n \to \infty$, $r \to \infty$.

Also,* by (a),

$$\left| \int |g(x)\,q(x) - \gamma(x)\,\psi(x)|\,dx - \int |g_r(x)\,q(x) - \gamma_r(x)\,\psi(x)|\,dx \right| \leqslant \int_E |g(x)\,q(x) - \gamma(x)\,\psi(x)|\,dx.$$

Thus by the last two inequalities, together with $(7a)$, we have, r and n being supposed large enough,

$$-\epsilon \leqslant \int |g(x)\,f_n(x) - \gamma(x)\,\phi_n(x)|\,dx - \int |g(x)\,q(x) - \gamma(x)\,\psi(x)|\,dx \leqslant \epsilon.$$

Letting $n \to \infty$, we get, since ϵ is as small as we please, the required result, which is accordingly proved in all its generality,

$$\operatorname*{Lt}_{n\to\infty} \int |g(x)\,f_n(x) - \gamma(x)\,\phi_n(x)|\,dx = \int |g(x)\,q(x) - \gamma(x)\,\psi(x)|\,dx. \quad (7)$$

11. Applying this theorem to our present theory, we replace $f_n(x)$ and $\phi_n(x)$, $g(x)$ and $\psi(x)$, by $\frac{\partial \phi_r}{\partial V}$ and $\frac{\partial \psi_r}{\partial V}$, $\frac{\partial y}{\partial V}$ and $\frac{\partial x}{\partial V}$, respectively. The relation (δ) then corresponds to (5) of the preceding article, and we have to make such further hypotheses that the relations (6) are fulfilled. The functions $\frac{\partial x}{\partial U}$ and $\frac{\partial y}{\partial U}$ replace now the $g(x)$ and $\gamma(x)$ of the last article,

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* We have here assumed, not only that for all values of n,

$$g(x)\,f_n(x) - \gamma(x)\,\phi_n(x)$$

is summable, but also that $g(x)\,q(x) - \gamma(x)\,\psi(x)$

is summable. We can however prove that the latter assumption is unnecessary, the summability of this function following from our other conditions. Indeed we can, without this assumption, replace the last inequality by

$$-\epsilon \leqslant \int |g(x)\,f_n(x) - \gamma(x)\,\phi_n(x)|\,dx - \int |g_r(x)\,q(x) - \gamma_r(x)\,\psi(x)|\,dx \leqslant \epsilon.$$

It at once follows that the limits when $n \to \infty$ of the former integral are the same as the limits when $r \to \infty$ of the latter integral. Hence also, as r and n are independent of one another, these limits are unique and finite and equal to one another.

Now considering the latter integral, we notice that the integrand is positive, and converges, as $r \to \infty$, except at a set of content zero, while the integral itself converges, and therefore nowhere diverges. Therefore, by a known theorem, given in my paper "On Semi-Integrals," in Vol. 9 of these *Proceedings*, p. 300, the limiting function of the integrands $g(x)\,q(x) - \gamma(x)\,\psi(x)$ is summable, as was asserted.

and we have to lay such additional restrictions on the variables that on the one hand the relations (8) are verified, and on the other hand the functions

$$\frac{\partial x}{\partial U}\frac{\partial y}{\partial V}-\frac{\partial y}{\partial U}\frac{\partial u}{\partial V}$$

and

$$\frac{\partial x}{\partial U}\frac{\partial \phi_r}{\partial V}-\frac{\partial y}{\partial U}\frac{\partial \psi_r}{\partial V}$$

for all values of r, are summable.

These are precisely the same requirements as we had to make in the former paper on "A Formula for an Area," and are accordingly satisfied by any one of the sets of conditions there given.* Supposing then such a set of conditions verified $\left(e.g.\ \frac{\partial x}{\partial U},\ \frac{\partial y}{\partial V},\ \frac{\partial x}{\partial V},\ \frac{\partial y}{\partial U}\text{ all bounded}\right)$ and also the additional condition of § 9, we have the result

$$\operatorname*{Lt}_{r\to\infty}\int\left|\frac{\partial x}{\partial U}\frac{\partial \phi_r}{\partial V}-\frac{\partial y}{\partial U}\frac{\partial \psi_r}{\partial V}\right|dV=\int\left|\frac{\partial x}{\partial U}\frac{\partial y}{\partial V}-\frac{\partial y}{\partial U}\frac{\partial x}{\partial V}\right|dV.$$

But any one of the hypotheses made was such that each of the integrals in the equation just written down is less than, or equal to, a summable function of U. Therefore we may integrate this equation term-by-term with respect to U. Thus we finally prove that all the limits as $\bar{h}\to 0$, of

$$\operatorname*{Lt}_{\bar{h}\to\infty}\ 2\sum_{a}^{e}\sum_{b}^{d}|D_n|\leqslant\operatorname*{Lt}_{r\to\infty}\int_a^e dU\int_b^d\left|\frac{\partial x}{\partial U}\frac{\partial \phi_r}{\partial V}-\frac{\partial y}{\partial U}\frac{\partial \psi_r}{\partial V}\right|dV,$$

are

$$\leqslant\int_a^e dU\int_b^d\left|\frac{\partial(x,\,y)}{\partial(U,\,V)}\right|dV=\int_a^e\int_b^d|B(u,\,v)|\,du\,dv. \qquad (9)$$

We have then proved that the sum of the absolute values of the areas of the triangles tends, when $\bar{h}\to 0$ to a unique limit (depending on the h's), which, if then $\bar{h}\to 0$, has one or more repeated limits, each of these being less than or equal to the double integral of the modulus of the Jacobian. The conditions under which this has been proved are (1) *any of the sets of conditions under which the algebraic sum of the areas of the triangles had been shewn to tend to the double integral of the Jacobian as unique double limit, together with* (2) *the extra condition that, for any and every pair of constants P and Q,*

$$Py\,(U,\,V)-Qx(U,\,V),$$

* See below, § 17.

has with respect to V a total variation which is a continuous function of U.

12. We notice that apparently here in order to arrive at a unique (repeated) limit, we have, not merely to choose a particular sequence of $\bar{h}$'s, but also a particular form of divisions, that is a particular succession of systems of h's. It will be shewn however immediately that all such choices lead to a unique repeated limit, given by the double integral of the modulus of the Jacobian.

In the meanwhile it is evident from § 6, that we can so choose the form of the sub-rectangles (namely so that the dividing parallels at each stage are retained at all subsequent stages), that the repeated limit obtained is not less than and is therefore actually

$$= \int_a^c \int_b^d |B(u, v)| \, du\, dv.$$

This shews that, under the conditions specified, the *upper* repeated limit is the integral of the modulus of the Jacobian.

We proceed however to prove the more general result which includes this as a special case.

13. The statement made at the end of § 11 in the language of limits is clearly equivalent to the following :—

Under the conditions we have assumed, whatever value h may have, provided*

$$\bar{h} \leqslant \delta \leqslant \epsilon', \tag{10}$$

where δ depends on ϵ, and ϵ' is a fixed quantity independent of ϵ and as small as we please, we can find $\bar{k} \leqslant \epsilon'$, depending on $\bar{h}$, so that, provided only the h's and k's are not greater than the norms $\bar{h}$ and $\bar{k}$ respectively,

$$\sum_{n=1}^{m} |D_n| \leqslant \int_a^c \int_b^d |B(u, v)| \, du\, dv + \epsilon, \tag{11}$$

this result being perfectly independent of the mode in which the actual division of the fundamental (u, v)-rectangle into sub-rectangles is effected.

We will, for brevity, denote the configuration of sub-rectangles in the (u, v)-plane so obtained by G. It is a *variable configuration*, since $\bar{h}$ may be chosen at will, provided (10) is satisfied.

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* See below, § 17.

The statement made at the beginning of the present article is equally true if we replace the fundamental (u, v)-rectangle by any other rectangle inside it; the double integral and the summation will then apply to that rectangle, and the quantities δ and $\bar{k}$ will be correspondingly determined.

14. Consider now a standard mode of division (fixed), by which the fundamental (u, v)-rectangle is divided up into a configuration, which we will denote by F, consisting of M sub-rectangles $R_1, R_2, \ldots, R_M$, the images of which are the contours $C_1, C_2, \ldots, C_M$. Then, by Theorem 1, we may choose F so that

$$\int_a^c \int_b^d |B(u, v)|\, du\, dv - \eta \leqslant \sum_{r=1}^{M} |C_r|, \tag{12}$$

where η is as small as we please.

Let us now define a new variable configuration Γ, with the same norms $\bar{h}, \bar{k}$, as the variable configuration G, the dividing parallels of Γ being those of G, with those of F added. Then, since $\bar{h}$ and $\bar{k}$ are each $\leqslant \epsilon'$, we can, by Theorem 2, choose ϵ' so small that

$$\sum_{r=1}^{M} |C_r| \leqslant \sum_{n=1}^{\mu} |\Delta_n| + \eta,$$

where $\sum_{n=1}^{\mu} |\Delta_n|$ denotes the usual summation taken relative to Γ, and $\sum_{n=1}^{m} |D_n|$ that relative to G.

Thus, by (12),

$$\int_a^c \int_b^d |B(u, v)|\, du\, dv - 2\eta \leqslant \sum_{n=1}^{\mu} |\Delta_n|. \tag{13}$$

15. The sub-rectangles of Γ then fall into two classes :—

$$(\Gamma)_1 = (G)_1,$$

consisting of those sub-rectangles of G, each of which is internal to a single rectangle of the standard F; and

$$(\Gamma)_2,$$

consisting of those sub-rectangles of Γ which are cut out of the remaining rectangles $(G)_2$ of G by the dividing parallels of F.

These latter sub-rectangles $(\Gamma)_2$ evidently form bands on each side of the parallels of the standard F, the width of the vertical bands being $\leqslant \bar{h}$, and the height of the horizontal bands $\leqslant \bar{k}$. Since $\bar{h}$ and $\bar{k}$ are both $\leqslant \epsilon'$, these bands will be internal to strips E of fixed width ϵ' on each side of the parallels of the standard F.

Now these fixed strips divide themselves up into a fixed finite number N of rectangles, each of which, as already remarked in § 13, may be taken to replace the fundamental rectangle in the discussion there undertaken: moreover the quantity ϵ there appearing may be replaced by η/N. Since there is only a finite number of rectangles—namely $N+1$, if we include the fundamental rectangle—and therefore only $N+1$ quantities δ and $\bar{k}$, we may agree to denote the *least* of these by δ and $\bar{k}$ respectively.

As the sub-rectangles $(\Gamma)_2$ only fill up our variable bands *inside* the fixed strip E, the sum of the triangles $|\Delta_n|$, which are related to the semi-rectangles of $(\Gamma)_2$ internal to any one of the fixed N rectangles of our strips, will not exceed a similar sum relating to semi-rectangles including these and filling up that fixed rectangle. In other words, having chosen our dependent norm $\bar{k}$ in the manner indicated, we not only have for Γ, as before for G,

$$\sum_{n=1}^{\mu} |\Delta_n| \leqslant \int_a^c \int_b^d |B(u, v)|\, du\, dv + \epsilon,$$

but also the sum of all those of the triangles $|\Delta_n|$ which are related to semi-rectangles of $(\Gamma)_2$ inside any one of the fixed N rectangles of our strips E, is

$$\leqslant \iint |B(u, v)|\, du\, dv + \eta/N,$$

the double integral being taken over this particular rectangle.

Hence the sum of those triangles $|\Delta_n|$ related to the semi-rectangles $(\Gamma)_2$ will be

$$\leqslant \iint_E |B(u, v)|\, du\, dv + \eta,$$

the double integral being taken over the fixed strips E of breadth ϵ'.

But, η being chosen small enough, ϵ' is as small as we please, and therefore this double integral is as small as we please, and may be taken to be $\leqslant \eta'$, where $\eta' \to 0$ when $\eta \to 0$.

Since the sub-rectangles $(\Gamma)_1$ are included among those of the scheme G, it is now evident that

$$\sum_{n=1}^{\mu} |\Delta_n| \leqslant \sum_{n=1}^{m} |D_n| + \eta' + \eta. \tag{14}$$

Hence, by (13),

$$\int_a^c \int_b^d |B(u, v)|\, du\, dv - 2\eta \leqslant \sum_{n=1}^{m} |D_n| + \eta' + \eta, \tag{15}$$

and therefore, by (11),

$$\int_a^c \int_b^d |B(u, v)|\, du\, dv - 3\eta - \eta' \leqslant \sum_{n=1}^{m} |D_n| \leqslant \int_a^c \int_b^d |B(u, v)|\, du\, dv + \epsilon, \tag{16}$$

where ϵ and η are independent, and $\eta' \to 0$ when $\eta \to 0$, and this relation (16) holds, provided the norms $\bar{h}$ and $\bar{k}$ are such that

$$\bar{h} \leqslant \delta \leqslant \epsilon',$$

where δ depends on ϵ, and ϵ' on η, and $\bar{k}$ depends on $\bar{h}$ for smallness.

16. Letting first $\bar{k} \to 0$, the construction of the actual sub-rectangles being otherwise any we please, the central member of (16) has, as we saw, a unique limit, depending on $\bar{h}$. Then, letting $\bar{h} \to 0$, in any and every manner, we get a certain set of limits, whose lower bound we denote by L, and the upper bound by U. We then have, by (16),

$$\int_a^c \int_b^d |B(u, v)|\, du\, dv - 3\eta - \eta' \leqslant L \leqslant U \leqslant \int_a^c \int_b^d |B(u, v)|\, du\, dv + \epsilon.$$

Letting $\epsilon \to 0$ and $\eta \to 0$, and therefore $\eta' \to 0$, we have

$$L = U = \int_a^c \int_b^d |B(u, v)|\, du\, dv = \int_a^c \int_b^d \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du\, dv. \tag{17}$$

This proves the statement made in § 7, that, *under the conditions hypothecated, the sum* $\sum_{n=1}^{m} |D_n|$ *of the areas, taken positively, of the triangles in the* (x, y)*-plane related to the semi-rectangles in the* (u, v)*-plane, tends to a unique repeated limit, when first the norm* $\bar{k} \to 0$, *and then* $\bar{h} \to 0$, *and this limit is*

$$\int_a^c \int_b^d \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du\, dv = \int_a^c \int_b^d |B(u, v)|\, du\, dv.$$

17. The conditions hypothecated may be here recapitulated :—

(I) *Conditions obtained in the earlier paper, in virtue of which*

$\sum_{n=1}^{m} D_n$, *the sum of our triangles, with proper signs, tends as* $\bar{h} \to 0$, $\bar{k} \to 0$ *to a unique double limit* $\int_a^c \int_b^d B(u, v)\, du\, dv$;

(i) $x(u, v)$ *and* $y(u, v)$ *are absolutely convergent integrals with respect to* u, *and with respect to* v, *separately* ;

(ii) $$\left|\frac{\partial y}{\partial v}\right| \leqslant \mu(v), \qquad \left|\frac{\partial x}{\partial v}\right| \leqslant M(v),$$

where $\mu(v)$ *and* $M(v)$ *are summable functions of* v *alone* ;

(iii) *the total variations of* $x(u, v)$ *and* $y(u, v)$ *with respect to* u, *for constant* v, *are bounded functions of* v ; *or, more generally, possess absolutely convergent integrals with respect to* $\int \mu(v)\, dv$ *and* $\int M(v)\, dv$ *respectively* ;

(II) *Condition characteristic for the present problem* ;

For any and every pair of constants P *and* Q,

$$Py(u, v) - Qx(u, v),$$

has with respect to v *a total variation which is a continuous function of* u.

It will be of course noticed that, whereas in the conditions of the former paper u and v might be interchanged, since the limit involved was a unique double limit, we can only make this change in our present conditions (I) and (II), if we proceed to the limit in the reverse order, making first $\bar{h} \to 0$ and then $\bar{k} \to 0$.

We shall return to this question in § 24.

18. We have now to apply our results to the theory of surfaces. Let our surface be defined by

$$x = x(u, v), \qquad y = y(u, v), \qquad z = z(u, v), \tag{17}$$

where $x(u, v)$, $y(u, v)$ and $z(u, v)$ are such functions of (u, v) that our conditions are satisfied for each pair of these functions, (u, v) lying in the fundamental rectangle $(a, c\,;\ b, d)$.

In my former paper on areas, I divided up the fundamental rectangle by means of parallels to its sides into M sub-rectangles $R_1, R_2, \ldots, R_M$, of span less than a norm δ. Corresponding to the perimeters of these

sub-rectangles, we had on the surface skew curves, or contours $C_1, C_2, \ldots, C_M$ [whose projections on the plane of (x, y) are of course the curves whose *directed* areas in § 4 were denoted by $C_1, C_2, \ldots, C_M$].

It was then shewn that, however the division into sub-rectangles was performed, the sum of the areas of the skew curves tended, as $\delta \to 0$, to a unique limit, and this we took, by definition, to be the area I of the portion of the surface which was the image of the fundamental (u, v)-rectangle.

It was then shewn that, provided our conditions (I) of § 16 be satisfied for each pair of dependent variables, viz. (x, y), (y, z), and (z, x),

$$I = \int_a^c \int_b^d \left\{ \left(\frac{\partial(x, y)}{\partial(u, v)}\right)^2 + \left(\frac{\partial(y, z)}{\partial(u, v)}\right)^2 + \left(\frac{\partial(z, x)}{\partial(u, v)}\right)^2 \right\}^{\frac{1}{2}} du\, dv$$

$$= \int_a^c \int_b^d |J(u, v)|\, du dv. \tag{18}$$

19. We propose now to shew that, supposing in addition the condition (II) of § 16 to be satisfied for each pair of dependent variables, we can find a system of triangulation which leads to a unique repeated limit which is again the area I: in fact, we can inscribe in our portion of the surface a polyhedron with triangular faces, whose surface tends to I as its unique repeated limit, when first $\bar{k} \to 0$, then $\bar{h} \to 0$. This system of triangulation obeys the law of order implicitly involved in the definition of the surface by the equations (17).

As in the earlier sections of this communication, we divide up the fundamental (u, v)-rectangle into semi-rectangles with norms $\bar{h}$ and $\bar{k}$. Related to these we have our system of triangles inscribed in the surface. The vertices of the triangles, being the images of the vertices of the sub-rectangles, are thus arranged in double order on the surface, just as the corresponding points are arranged in double order on the plane, by means of the values of their coordinates u and v.

As $\bar{h}$ and $\bar{k}$ tend to zero, the vertices of our triangles at all stages (whether these stages are countably infinite or non-countably infinite) form an everywhere dense set of points in double order on the surface.

20. Now the area of a skew curve is a vector, which is defined as the unique limit of the area of a polygonal figure inscribed in it, with vertices in the progressive order of the curve, when the lengths of the sides of the polygons tend to zero. On the other hand, the area of the polygon is

defined as the resultant of a system of forces, represented in magnitude, line of action and sense by the sides of the polygon.

We have then again the argumentation of § 5 *supra*, where, it will be observed, *the fact that the polygon and contour were plane in no way entered.* Theorem 2 therefore holds as it stands, interpreting the images in our present sense on the surface, instead of in the plane.

The theorem, which we restate here, is as follows: in the corollary which comes after, the double integral which occurs is of course that denoted in § 17 by I.

THEOREM 2′.—*If we have any (first) mode of division of the fundamental (u, v)-rectangle into M sub-rectangles whose images are the contours $C_1, C_2, \ldots, C_M$, and η be any chosen positive quantity, we can find ϵ' so small that, when the norms $\bar{h}$ and $\bar{k}$ are both $\leqslant \epsilon'$, and we divide the fundamental (u, v)-rectangle in any (second) mode into sub-rectangles of norms $\bar{h}$ and $\bar{k}$, and these into semi-rectangles by means of the diagonals, sloping down from left to right, then the sum $\sum_{n=1}^{m} |D_n|$ of the moduli of the areas of the related triangles, is connected with the sum $\sum_{r=1}^{M} |C_r|$ of those of the contours, of the relation*

$$\sum_{r=1}^{M} |C_r| \leqslant \sum_{n=1}^{m} |D_n| + \eta,$$

provided only the dividing lines of the first mode are included in those of the second mode.

COR.—*If the correspondence fulfils the conditions demanded in* § 17,

$$\int_a^c \int_b^d |J(u, v)| \, du \, dv \leqslant \text{lower limit}_{i \to \infty} \sum_{n=1}^{m_i} |D_n|,$$

provided the dividing parallels at each stage be retained at all subsequent stages, as the norms $\bar{h}_i$ and $\bar{k}_i$ tend to zero.

21. Now let the projection of each of our inscribed triangles D_n on the axial planes be respectively $D_{x,n}$, $D_{y,n}$, and $D_{z,n}$. Then, by a simple inequality,

$$\sum_{n=1}^{m} |D_n| = \sum_{n=1}^{m} \{(D_{x,n})^2 + (D_{y,n})^2 + (D_{z,n})^2\}^{\frac{1}{2}}$$

$$\leqslant \left[\left\{ \sum_{n=1}^{m} |D_{x,n}| \right\}^2 + \left\{ \sum_{n=1}^{m} |D_{y,n}| \right\}^2 + \left\{ \sum_{n=1}^{m} |D_{z,n}| \right\}^2 \right]^{\frac{1}{2}},$$

and therefore, since each of the three terms squared on the right has been shewn to have a unique limit when $\bar{k} \to 0$, which has again a unique limit when $h \to 0$, and this repeated limit is the integral of the modulus of the corresponding Jacobian, $\sum_{n=1}^{m} |D_n|$ *approaches, under these circumstances, one or more limits* ($\bar{k} \to 0$, $\bar{h} \to 0$), *all of which are*

$$\leqslant \Big[\Big\{ \int_a^c \int_b^d \left| \frac{\partial (y, z)}{\partial (u, v)} \right| du\, dv \Big\}^2 + \Big\{ \int_a^c \int_b^d \left| \frac{\partial (z, x)}{\partial (u, v)} \right| du\, dv \Big\}^2 + \Big\{ \int_a^c \int_b^d \left| \frac{\partial (x, y)}{\partial (u, v)} \right| du\, dv \Big\}^2 \Big]^{\frac{1}{2}},$$

and therefore

$$\leqslant \int_a^c \int_b^d \sqrt{\left[\left(\frac{\partial (x, y)}{\partial (u, v)} \right)^2 + \left(\frac{\partial (y, z)}{\partial (u, v)} \right)^2 + \left(\frac{\partial (z, x)}{\partial (u, v)} \right)^2 \right]} du\, dv = I, \qquad (19)$$

by a theorem proved in my communication to the Royal Society. This result for our surface corresponds exactly to (9) for the plane.

22. This result, stated in the language of limits, may then be restated in the form adopted in § 13, the double integral in (11) being replaced by our present double integral I.

The argument used in that article then applies word for word in the present case, our surface taking the place of the plane. We thus arrive at the result, which takes the place of (15),

$$I - 2\eta \leqslant \sum_{n=1}^{m} |D_n| + \eta' + \eta, \qquad (20)$$

and therefore, by (16),

$$I - 3\eta - \eta' \leqslant \sum_{n=1}^{m} |D_n| \leqslant I + \epsilon, \qquad (21)$$

where ϵ and η are independent arbitrary small quantities, and $\eta' \to 0$, when $\eta \to 0$, this last relation (21) holding, provided the norms $\bar{h}$ and $\bar{k}$ are such that

$$\bar{h} \leqslant \delta \leqslant \epsilon',$$

where δ depends on ϵ, and ϵ' on η, and $\bar{k}$ depends on $\bar{h}$, for smallness.

23. Hence, as in § 16, letting first $\bar{k} \to 0$, the construction of the actual sub-rectangles being otherwise arbitrary, the central member of

(21) tends to a unique limit, depending on $\bar{h}$. Then letting $\bar{h} \to 0$ in any manner, we get a certain set of limits, whose lower bound L and upper bound U are such that

$$I - 3\eta - \eta' \leqslant L \leqslant U \leqslant I + \epsilon,$$

so that letting $\epsilon \to 0$, $\eta \to 0$, we see that

$$L = U = I.$$

We have thus proved that, provided the three pairs of the functions $x(u, v)$, $y(u, v)$, *and* $z(u, v)$, *defining our surface, satisfy the conditions given in* § 17, *the area* I *of the part of the surface corresponding to the fundamental* (u, v)*-rectangle may be defined as the limit of the area of a polyhedral surface, with triangular faces, inscribed in the surface, under the following restrictions :—*

(i) *The vertices of the polyhedron form a doubly ordered set, the order being determined by the parameters* u *and* v *in such a way that these vertices are the images of the corners of sub-rectangles, got by dividing the fundamental* (u, v)*-rectangle by parallels to its sides, while the triangles are formed by joining these vertices on the surface in the same way as their images in the* (u, v)*-plane, when the sub-rectangles are halved by the diagonals sloping down from left to right, i.e. making positive intercepts on the axes of* u *and* v.

(ii) *The dimensions of the sub-rectangles in the* (u, v)*-plane parallel to* $u = \text{const.}$ *and* $v = \text{const.}$ *being less than, or equal to, the norms* $\bar{h}$ *and* $\bar{k}$ *respectively, the limit in question is the repeated limit, when first* $\bar{k} \to 0$, *and then* $\bar{h} \to 0$, *this limit is unique and equal to the double integral*

$$I = \int_a^c \int_b^d \left\{ \left(\frac{\partial(x, y)}{\partial(u, v)}\right)^2 + \left(\frac{\partial(y, z)}{\partial(u, v)}\right)^2 + \left(\frac{\partial(z, x)}{\partial(u, v)}\right)^2 \right\}^{\frac{1}{2}} du\, dv.$$

We recall that the area I, so defined, is the same as I had previously defined it by means of the contours on the surface itself, which are the images of the sub-rectangles in the (u, v)-plane. But in that case the limit was *a unique double limit,* $\bar{h}$ *and* $\bar{k}$ *tending independently, or dependently, to zero in any way.*

Moreover the former definition was more general in its application, since the present process has only been justified under the same conditions as the former, together with the extra condition that, *for all values of the*

constants A, B, and C, the total variation of each of the functions

$$Ax(u, v)+By(u, v), \quad By(u, v)+Cz(u, v), \quad Cz(u, v)+Ax(u, v),$$

with respect to v should be continuous functions of u.

24. The order (first $\bar{k} \to 0$, then $\bar{h} \to 0$), in which the limiting process is to be pursued, is prescribed by the form of the characteristic conditions last referred to. If it be desired that the other possible process (first $\bar{h} \to 0$, then $\bar{k} \to 0$) should lead to the same limit, we must of course interchange u and v in all our conditions.

If the two processes are both to lead to the double integral I as unique repeated limit, the condition of § 17, not only for (x, y), but for all three variables (x, y, z), will evidently undergo the following changes: condition (iii) will disappear, while conditions (ii) and (iv) will be doubled, namely by adding to them the analogous conditions with u and v interchanged.

25. The interesting question then naturally presents itself, as to whether there are not convenient though more restricted conditions under which, not only the repeated limits, but *all* the limits are equal, so that there is a unique double limit.

It is known that the answer to this question is in the affirmative in a class of cases which, though extremely restricted as regards theoretical generality, are of some importance in ordinary working analysis. It will perhaps be a convenience to the reader if I quote this known result* in full, more especially as it will enable him to judge rapidly as to the nature of the extension about to be explained in the following articles:—

"If $x(u, v)$, $y(u, v)$, and $z(u, v)$ are analytic functions of (u, v), which are regular inside, and on the boundary of the fundamental rectangle, or region, in the (u, v)-plane, and are such that the Jacobians of the pairs of these functions are not all simultaneously zero, then, however we divide up the (u, v)-plane into triangles satisfying the following conditions:—

(1) the sides of all the triangles are less than δ;

(2) none of the angles of any of the triangles exceeds ω, where δ and ω are arbitrary fixed positive quantities, of which $\frac{1}{3}\pi < \omega < \pi$;

* *Encyk. d. Math. Wiss.*, Vol. 3 (3), 1, p. 64.

then the sum $\sum_{n=1}^{m} |D_n|$ of the moduli of the areas of the 'related' triangles, inscribed in the surface $x = x(u, v)$, $y = y(u, v)$, $z = z(u, v)$ has a unique limit, when $\delta \to 0$, and this limit is

$$\iint \left\{ \left(\frac{\partial(x, y)}{\partial(u, v)}\right)^2 + \left(\frac{\partial(y, z)}{\partial(u, v)}\right)^2 + \left(\frac{\partial(z, x)}{\partial(u, v)}\right)^2 \right\}^{\frac{1}{2}} du\, dv$$

over the fundamental region."

26. It will be noticed that the mode of triangulation, adopted in the theorem just quoted, is somewhat *more* general than that we have employed, owing to the fact that the type of triangle has no reference to the u and v lines, and somewhat *less* general in that it lays a restriction on all three angles of the triangles in the (u, v)-plane. Thus none of the conditions we have obtained so far constitute a complete generalisation of those quoted in the preceding paragraph. There is however no difficulty in generalising them as they stand. We can, in fact, by elementary methods prove the following theorem:—

THEOREM.—*If the first partial differential coefficients* $\frac{\partial x}{\partial u}, \frac{\partial x}{\partial v}, \frac{\partial y}{\partial u}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial u}, \frac{\partial z}{\partial v}$ *all exist everywhere and are continuous functions of* (u, v), *and the fundamental rectangle, or other region of the* (u, v)*-plane of which the surface is the image, be divided up into triangles satisfying the single condition that none of the triangles is degenerate, having all three vertices on a straight line, and that one at least of the three angles lies between an assigned fixed small angle and its supplement, and the limiting process be carried out in such a manner that throughout it this limitation with respect to one of the angles of each triangle is maintained, then the sum of the absolute values of the areas of the triangles related to these, namely those inscribed in the surface whose vertices are the images of their vertices, has a unique double limit given by the usual double integral*

$$\iint \sqrt{\left[\left(\frac{\partial(x, y)}{\partial(u, v)}\right)^2 + \left(\frac{\partial(y, z)}{\partial(u, v)}\right)^2 + \left(\frac{\partial(z, x)}{\partial(u, v)}\right)^2\right]} du\, dv$$

over the fundamental region.

The proof of this theorem will be given in full* elsewhere.

* See below, § 33.

27. The following theorem is in some respects a still more interesting result than that just given, the interest lying partly in the far greater difficulty of the ideas involved in the proof of it, and partly because it constitutes what from one point of view might seem the most natural generalisation of the known theory of the length of a curve.

There are in fact two modes of generalising the concepts of *function of bounded variation* and of *integral* in passing from one to two variables. In the one mode we retain a single dependent variable, while increasing the independent variables to two; in the other mode both the independent and the dependent variables are increased to two: and we may say that the latter generalisation corresponds to that theory of the area of a surface, exposed in my previous paper and used in the present one, which is based on the subsidiary concept of the area of a skew curve. The former mode of generalisation suggests the *a priori* probability of the theorem we now proceed to give.

THEOREM.—*If* $x(u, v)$, $y(u, v)$, *and* $z(u, v)$ *are double integrals, and the method of triangulation explained in this paper be adopted with this extension that, in proceeding to the limit,* $\bar{h}$ *and* $\bar{k}$ *are to diminish in any manner whatever, a unique double limit exists for the sum* $\sum_{n=1}^{m} |D_n|$, *and this is given by the usual double integral* I.

It will be noticed that this theorem neither includes, nor is included in, the theorem last given; even if the mode of triangulation in that theorem be modified, so as to satisfy the restriction of order imposed by us on the triangles. This suggests that the proper generalisation in dealing with the theory of surfaces of the concept of a function of bounded variation is that virtually employed in the earlier part of the present paper and in its predecessor.

28. We remark first that, as in the earlier part of the paper, it is only necessary to carry out the discussion for the triangles in the (x, y)-plane. For in §§ 21–23 the mode of proceeding to the limit was only conditioned by the mode necessary for the triangles in each of the three axial planes and did not otherwise enter into the argument.

We remark secondly that, under the hypotheses made in the enunciation, our conditions of § 17 are fulfilled. Indeed that the extra condition (iv) is satisfied is evident, as well as the condition (i); also the condition (ii) is fulfilled, as well as the corresponding condition with U and V interchanged [in virtue of which (iii) is satisfied], as is at once

seen by writing

$$y(u, v) = \int_a^u \int_b^v \{g_1(U, V) - g_2(U, V)\}\, dU\, dV,$$

where $g_1(U, V) \geqslant 0$, $g_2(U, V) \geqslant 0$. Therefore

$$\left| \frac{\partial y(u, v)}{\partial v} \right| = \int_a^u \{g_1(U, v) - g_2(U, v)\}\, dU \leqslant \int_a^b \{g_1(U, v) - g_2(U, v)\}\, dU$$

$$\leqslant \mu(v),$$

where $\mu(v)$ is a summable function of v, since $g_1(U, V)$ and $g_2(U, V)$ have by hypotheses integrals in $(a, c; b, d)$.

Hence, by our former results, the two repeated limits of $\sum_{n=1}^{m} |D_n|$ are unique and equal to our double integral. It remains therefore only to convince ourselves of the uniqueness of the double limit. We shall however not assume our previous results, but start again *ab initio*.

29. Considering then any double succession of norms

$$\bar{h}_1 > \bar{h}_2, \ \ldots \to 0,$$

$$\bar{k}_1 > \bar{k}_2, \ \ldots \to 0,$$

and taking our triangles $\sum_{n=1}^{m} |D_n|$ at the stage characterised by the pair of norms $(\bar{h}_r, \bar{k}_s)$, we have, since x and y are integrals, transforming the expression for D_n, given in § 8,

$$D_n = \tfrac{1}{2} \int_u^{u+h} \frac{\partial x(U, v)}{\partial U} dU \int_v^{v+k} \frac{\partial y(u, V)}{\partial V} dV - \int_u^{u+h} \frac{\partial y(U, v)}{\partial U} dU \int_v^{v+k} \frac{\partial x(u, V)}{\partial V} dV$$

$$= \tfrac{1}{2} \int_u^{u+h} \int_v^{v+k} e_{r,s}(U, V)\, dU\, dV, \qquad (22)$$

where, in the sub-rectangle in question,

$$e_{r,s}(U, V) = \frac{\partial x(U, v)}{\partial U} \frac{\partial \phi_r(U, V)}{\partial V} - \frac{\partial y(U, v)}{\partial U} \frac{\partial \psi_r(U, V)}{\partial V}, \qquad (23)$$

$\phi_r(U, V)$ and $\psi_r(U, V)$ being used in the same sense as before (§ 8), so

that, for

$$u \leqslant U \leqslant u+h,$$

$$\phi_r(U, V) = y(u, V), \qquad \psi_r(U, V) = x(u, V). \tag{24}$$

Let us write, for points in the same sub-rectangle,

$$f_s(U, V) = \frac{\partial x(U, v)}{\partial U}\,\frac{\partial y(U, V)}{\partial V} - \frac{\partial y(U, v)}{\partial U}\,\frac{\partial x(U, V)}{\partial V}, \tag{25}$$

where, in accordance with what was already pointed out, we have

$$\left|\frac{\partial x(U, v)}{\partial U}\right| \leqslant \lambda(U), \qquad \left|\frac{\partial y(U, v)}{\partial U}\right| \leqslant \Lambda(U). \tag{26}$$

We thus have, by well known inequalities,

$$\left|\left|\int_u^{u+h}\int_v^{v+k} e_{r,s}(U, V)\,dU\,dV\right| - \left|\int_u^{u+h}\int_v^{v+k} f_s(U, V)\,dU\,dV\right|\right|$$

$$\leqslant \left|\int_u^{u+h}\int_v^{v+k} \{e_{r,s}(U, V) - f_s(U, V)\}\,dU\,dV\right|$$

$$\leqslant \int_u^{u+h}\int_v^{v+k} |e_{r,s}(U, V) - f_s(U, V)|\,dU\,dV$$

$$\leqslant \int_u^{u+h} dU \int_v^{v+k} \lambda(U)\left|\frac{\partial}{\partial V}\{y(U, V) - \phi_r(U, V)\}\right| dV$$

$$+ \int_u^{u+h} dU \int_v^{v+k} \Lambda(U)\left|\frac{\partial}{\partial V}\{x(U, V) - \psi_r(U, V)\}\right| dV.$$

Summing all over the rectangle $(a, c\,;\ b, d)$, and denoting the two summations of double integrals, taken positively, which occur on the left by $E_{r,s}$ and $F_{r,s}$, we have

$$|E_{r,s} - F_{r,s}| \leqslant \int_a^c dU \int_b^d \lambda(U)\left|\frac{\partial}{\partial V}\{y(U, V) - \phi_r(U, V)\}\right| dV$$

$$+ \int_a^c dU \int_b^d \Lambda(U)\left|\frac{\partial}{\partial V}\{x(U, V) - \psi_r(U, V)\}\right| dV. \tag{27}$$

30. Now using the expression for $y(u, v)$ as an integral (§ 28), we have

$$y(U, V) - \phi_r(U, V) = \{y_1(U, V) - \phi_{r,1}(U, V)\} - \{y_2(U, V) - \phi_{r,2}(U, V)\},$$

where

$$y_1(U, V) = \int_b^V\int_a^U g_1(U, V)\,dU\,dV, \qquad y_2(U, V) = \int_b^V\int_a^U g_2(U, V)\,dU\,dV,$$

and $\phi_{r,1}(U, V)$ and $\phi_{r,2}(U, V)$ are connected with $y_1(U, V)$, $y_2(U, V)$ in the same way as $\phi_r(U, V)$ with $y(U, V)$, (24).

We have therefore

$$\left|\frac{\partial}{\partial V}\{y(U, V)-\phi_r(U, V)\}\right|$$

$$\leqslant \frac{\partial}{\partial V}\{y_1(U, V)-\phi_{r,1}(U, V)\}+\frac{\partial}{\partial V}\{y_2(U, V)-\phi_{r,2}(U, V)\},$$

where each of the last two differential coefficients is $\geqslant 0$, since they are equal to $\int_u^U g_1(U, V)\,dU$ and $\int_u^U g_2(U, V)\,dU$ respectively.

Hence, taking, for example, the first of the two integrals on the right in (27), we see that this is

$$\leqslant \int_a^c dU\,\lambda(U)\int_b^d\Big[\frac{\partial}{\partial V}\{y_1(U, V)-\phi_{r,1}(U, V)\}$$

$$+\frac{\partial}{\partial V}\{y_2(U, V)-\phi_{r,2}(U, V)\}\Big]dV.$$

Performing the integration with respect to V, we get four integrals from this first integral, and four from the second integral, all of the same type; the first, for instance, being

$$\int_a^c \lambda(U)\{y_1(U, d)-\phi_{r,1}(U, d)\}\,dU.$$

Since $\phi_{r,1}(U, d)$ tends boundedly to $y_1(U, d)$ as limit, when $r\to\infty$, and the whole integral is an expression independent of the k's, this integral, and therefore also each of our eight integrals, has a unique double limit zero, when $r\to\infty$, $s\to\infty$. Thus, by (27),

$$E_{r,s}-F_{r,s}\equiv\sum_{n=1}^{m}|D_n|-\sum_a^c\sum_b^d\int_u^{u+h}\int_v^{v+k}f_r(U, V)\,dU\,dV$$

tends as $r\to 0$, $s\to 0$, *to the unique double limit zero.*

31. By this argumentation we may, in the consideration of the limits, replace in the integrand $e_{r,s}(U, V)$ by $f_r(U, V)$, that is u by U. Similarly we may replace v by V, that is we may replace $\sum_{n=1}^{m}|D_n|$, that is $E_{r,s}$, by

$$\sum_a^c\sum_b^d\left|\int_u^{u+h}\int_v^{v+k}\left\{\frac{\partial x(U, V)}{\partial U}\,\frac{\partial y(U, V)}{\partial V}-\frac{\partial y(U, V)}{\partial U}\,\frac{\partial x(U, V)}{\partial V}\right\}dU\,dV\right|,$$

which is the total increment of

$$\int_a^U \int_b^V \frac{\partial(x, y)}{\partial(U, V)}\, dU dV,$$

and has therefore as $\bar{h}_r \to 0$, $\bar{k}_s \to 0$, a unique double limit, namely the total variation of this last integral, that is

$$\int_a^c \int_b^d \left| \frac{\partial(x, y)}{\partial(U, V)} \right| dU dV.$$

Thus, finally, our theorem (§ 27) is proved, since we always get the same unique limit however our sequences of norms and the actual construction of the sub-rectangles be chosen, viz.

$$\operatorname*{Lt}_{\substack{(\bar{h}\to 0)\\(\bar{k}\to 0)}} \sum_{n=1}^{m} |D_n| = \int_a^c \int_b^d \left| \frac{\partial(x, y)}{\partial(U, V)} \right| dU dV.$$

32. We may remark that the special case of the theorem of § 26, in which the construction is carried out in the manner of the present paper may very simply be proved by the same method as the preceding theorem. In fact, if instead of assuming x and y to be double integrals, *we suppose* $\frac{\partial x}{\partial U}$, $\frac{\partial x}{\partial V}$, $\frac{\partial y}{\partial U}$, and $\frac{\partial y}{\partial V}$ *to be continuous functions of* U, V, we can assume the sub-rectangles so small that the oscillation of these four functions in each sub-rectangle is numerically less than ϵ. If then B be the upper bound of these four functions,

$$|E_{r,s} - F_{r,s}| \leqslant B \sum_a^c \sum_b^d \int_u^{u+h} \int_v^{v+k} \epsilon\, dU dV \leqslant B\epsilon(c-a)(b-d),$$

and is therefore as small as we please. Thus, as before, we may replace $E_{r,s}$ by $F_{r,s}$, and similarly $F_{r,s}$ by

$$\sum_a^c \sum_b^d \left| \int_a^{u+h} \int_v^{v+k} \frac{\partial(x, y)}{\partial(U, V)}\, dU dV \right|.$$

Hence, as in § 31, we get for $E_{r,s}$ the unique double limit

$$\int_a^c \int_b^d \left| \frac{\partial(x, y)}{\partial(U, V)} \right| dU dV,$$

which proves the theorem.

33. If we wish to prove that, when the partial differential coefficients of x, y, z are continuous functions of (u, v), the double limit, obtained by the triangulation of the surface, is our double integral I, *under the sole condition that one of the angles of every one of the triangles in the* (u, v)*-plane lies between* γ *and* $\pi-\gamma$, the method of this paper will not immediately apply, and a more elementary treatment is preferable. For this we must refer elsewhere. The reader will however be able to formulate the proof for himself, if he bears in mind that the triangle Δ_n whose vertices are

$$(u, v), \quad (u+h, v+k), \quad (u+h', v+k'),$$

has related to it in the (x, y)-plane a triangle $D_{z,n}$ whose area is

$$\tfrac{1}{2}\begin{vmatrix} x(u+h, v+k)-x(u, v), & y(u+h, v+k)-y(u, v) \\ x(u+h', v+k')-x(u, v), & y(u+h', v+k')-y(u, v) \end{vmatrix},$$

which differs from

$$\tfrac{1}{2}\begin{vmatrix} h\frac{\partial x}{\partial u}+k\frac{\partial x}{\partial v}, & h\frac{\partial y}{\partial u}+k\frac{\partial y}{\partial v} \\ h'\frac{\partial x}{\partial u}+k'\frac{\partial x}{\partial v}, & h'\frac{\partial y}{\partial u}+k'\frac{\partial y}{\partial v} \end{vmatrix}, \tag{A}$$

by quantities of the type

$$\begin{vmatrix} h\epsilon+k\epsilon', & hA+kB \\ h'\epsilon_1+k'\epsilon_1', & h'A'+k'B' \end{vmatrix},$$

where ϵ and ϵ', ϵ_1 and ϵ_1' are all less than a fixed small quantity η, and A, B, A', B' all less than a fixed quantity M. The sum of these latter quantities is

$$\leqslant 6\sqrt{(h^2+k^2)}\sqrt{(h'^2+k'^2)}\,\eta M \leqslant |hk'-h'k|\,\frac{\alpha\beta}{\alpha\beta\sin C}\,6\eta M,$$

where α, β are sides of the related (u, v)-triangle, and C is the angle included by them. Thus, if

$$\operatorname{cosec} C < \operatorname{cosec}\gamma,$$

where γ is fixed, the ratio of our triangle $D_{z,n}$ in the (x, y)-plane to its related triangle A_r in the (u, v)-plane is $\leqslant B\eta\operatorname{cosec}\gamma$, where B is a fixed quantity.

On the other hand, our determinant (A) may be written

$$\begin{vmatrix} h, & k \\ h', & k' \end{vmatrix} \cdot \begin{vmatrix} \frac{\partial x}{\partial u}, & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u}, & \frac{\partial y}{\partial v} \end{vmatrix}.$$

Thus the ratio of our two related triangles $|D_{z,n}|/|\Delta_n|$ may be written

$$\frac{\partial(x, y)}{\partial(u, v)} + \eta_1.$$

Similar expressions hold for $D_{x,n}$ in the (y, z)-plane, and $D_{y,n}$ in the (z, x)-plane. Thus, D_n denoting the triangle inscribed in the surface, so that

$$|D_n|/|\Delta_r| = \{(D_{x,n})^2 + (D_{y,n})^2 + (D_{z,n})^2\}^{\frac{1}{2}}/|\Delta_r|,$$

we get, by a lemma easily proved,

$$|D_n| - |\Delta_n| \left\{ \left(\frac{\partial(y, z)}{\partial(u, v)}\right)^2 + \left(\frac{\partial(z, x)}{\partial(u, v)}\right)^2 + \left(\frac{\partial(x, y)}{\partial(u, v)}\right)^2 \right\}^{\frac{1}{2}} \leqslant \{\eta_1^2 + \eta_2^2 + \eta_3^2\}^{\frac{1}{2}}.$$

Hence, by the theory of Cauchy integration, the required result follows.

34. It remains only to shew that our mode of triangulation (§ 33) includes that described in the theorem quoted in § 25 from the *Encyklopädie*

Let A, B, and C be the angles of our triangle Δ_n, and suppose

$$A < \omega, \quad B < \omega, \quad C < \omega.$$

Then

$$B + C > \pi - \omega,$$

and therefore either B or C is $> \frac{1}{2}(\pi - \omega)$: suppose, for instance, this angle is B, then

$$\tfrac{1}{2}(\pi - \omega) < B < \omega.$$

If therefore we put $\frac{1}{2}(\pi - \omega) = \gamma$, we have

$$\gamma < B \leqslant \pi - 2\gamma,$$

a fortiori

$$\gamma < B \leqslant \pi - \gamma,$$

which is our condition (§ 33).

Hence any triangulation which satisfies the requirements of the theorem quoted in § 25 satisfies also our requirements in § 33.

Moreover in our case γ is any angle we please, whereas in the theorem

quoted $\omega > \frac{1}{3}\pi$, and therefore $\gamma < \frac{1}{3}\pi$. Further we prove similarly that one at least of the remaining two angles A and C is in that theorem restricted in the same way as B.

ERRATA TO PAPER APPEARING IN VOL. 15.

P. 366, last line, *for* $f_+(x)$ *read* $f_-(x)$.

P. 367, relation 4, *for* $x_1 - x$ *read* $x_1 + h_1 - x$.

P. 371, line 4, *add* "By Theorem 1, removing a sub-set of content zero, we ensure that $f^-(x)$ and $f_-(x)$ are also finite."

,, line 5, *after* $f^+(x)$ *add* "$f^-(x)$, $f_-(x)$."

,, line 8, *after* "is," *add* "when we replace the sub-set of content zero."

,, line 16, *add* $0 < f_-(x) \leqslant f^-(x) < A$.

,, line 25, *insert first* $f^-(x) < A$,

and *insert last* $f_-(x) > 0$.

,, lines 4 and 5 from bottom, *change* + *into* $\pm$ (twice).

P. 372, *delete* lines 10 and 11 from bottom up to the full stop.

P. 373, line 9, *change* "the right-hand" *into* "one."

[I have to thank Prof. Hobson for the detection of this slip in the proof and for the correction.—Grace Chisholm Young.]

THE COMPLEX MULTIPLICATION OF WEIERSTRASSIAN ELLIPTIC FUNCTIONS

By W. E. H. BERWICK.

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IT has recently been shown by Dr. G. B. Mathews* that the complex multiplication of the lemniscate function, defined by $j = 1728$, is intimately connected with the function

$$\psi_{a+bi}(u) = \sigma(a+bi)u/\sigma(u)^{N(a+bi)}.$$

The present paper is mainly concerned with a discussion of the analogous functions $\psi_{a+bi\sqrt{m}}(u)$ which exist among any system of elliptic functions admitting complex multiplication. Just as in real multiplication

$$\wp(u) - \wp(\mu u) = \psi_{\mu+1}\psi_{\mu-1} \div \psi_\mu^2,$$

and there exists the same recurrence-formula

$$\psi_{\mu+\nu}\psi_{\mu-\nu}\psi_{\pi+\rho}\psi_{\pi-\rho} + \psi_{\mu+\pi}\psi_{\mu-\pi}\psi_{\rho+\nu}\psi_{\rho-\nu} + \psi_{\mu+\rho}\psi_{\mu-\rho}\psi_{\nu+\pi}\psi_{\nu-\pi} = 0.$$

After the determination of the first complex ψ, which can frequently be obtained most conveniently from a transformation formula, the value of every other complex ψ_μ can be obtained by rational operations alone, *i.e.* without the labour of evaluating those roots of the transformation equation of order $N(\mu)$ which are rational in the field $[j, i\sqrt{m}]$.

The functions arising under the discriminant $\Delta = 20$ are discussed in considerable detail. In $[i\sqrt{5}]$ there are two classes of ideals, so that $\Delta = 20$ defines two systems of elliptic functions admitting multiplication by every member of the modulus $(1, i\sqrt{5})$. The two systems are connected by a quadratic transformation, and their invariants can be taken to be conjugate in the field $[i\sqrt{5}]$. Due to a non-principal prime ideal

$$\mathfrak{p} \equiv (p, c+i\sqrt{5})$$

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* "A direct Method in the Multiplication Theory of the Lemniscate Functions and other Elliptic Functions," *Proc. London Math. Soc.*, Ser. 2, Vol. 14 (1915), pp. 467–475.

there are $p-1$ solutions,

$$\pm u_1, \ \pm u_2, \ \ldots, \ \pm u_{\frac{1}{2}(p-1)},$$

distinct from zero, of the congruences

$$pu \equiv (c+i\sqrt{5})\, u \equiv 0 \pmod{2\omega_1, 2\omega_2},$$

and the irreducible polynomial

$$\prod_{1, \frac{1}{2}(p-1)} [\wp(u)-\wp(u_r)]$$

divides ψ_μ whenever $\mu \equiv 0 \pmod{\mathfrak{p}}$. It is shown how this polynomial can be isolated by the quadratic relations connecting the two systems of functions.

More generally, every ideal $\mathfrak{a}$ in an imaginary quadratic field can be associated with a unique polynomial $\psi_\mathfrak{a}$, which divides ψ_μ when μ is a member of $\mathfrak{a}$. In certain cases $\psi_\mathfrak{a}$ can be obtained from transformation properties without the labour involved in carrying out the "greatest common measure" process.

1. *The Periods and Multipliers.*

The most general system of elliptic functions admitting complex multiplication has the primitive pair of periods

$$2\omega_1 = a\Omega, \quad 2\omega_2 = (b+ci\sqrt{m})\,\Omega, \tag{1}$$

wherein b, c, a, m are rational integers, the last three of them positive, and m divisible by no square factor. In order that μ may be a multiplier $\wp(\mu u)$ must have the periods $2\omega_1$, $2\omega_2$ for which it is necessary and sufficient that

$$\left.\begin{aligned} \mu a &= x'a + y'(b+ci\sqrt{m}), \\ \mu(b+ci\sqrt{m}) &= x''a + y''(b+ci\sqrt{m}), \end{aligned}\right\} \tag{2}$$

with rational integral values of x', y', x'', y''. When μ_1, μ_2 are multipliers $\mu_1+\mu_2$ is another, hence the whole aggregate of multipliers forms a compound modulus

$$\mathfrak{m} \equiv (\mu_1, \mu_2, \mu_3, \ \ldots, \ x_1\mu_1+x_2\mu_2+x_3\mu_3+\ldots)$$

of numbers in the quadratic field $[i\sqrt{m}]$. Such a modulus being always reducible to a two-member basis (α_1, α_2), with α_1 rational, it follows that every multiplier has the form

$$y_1\alpha_1+y_2\alpha_2$$

for rational integral values of y_1, y_2. Rational integers are the only rational numbers which satisfy (2): accordingly $a_1 = 1$ and

$$\mathfrak{m} = (1, a).$$

The simplest cases of multiplication by complex members of the field $[i\sqrt{m}]$ occur when, in (1),

$$a, \quad b+ci\sqrt{m}$$

form a two-member basis of an ideal in the field. Every complex integer of the field is then a multiplier and, conversely, every multiplier is a complex integer. The aggregate of multipliers thus coincides with $\mathfrak{o}$, the integral basis of $[i\sqrt{m}]$, which has the two-member basis

$$(1, i\sqrt{m}) \equiv (1, a) \quad \text{when} \quad m \equiv 1, 2 \pmod 4,$$

and $$[1, \tfrac{1}{2}(1+i\sqrt{m})] \equiv (1, a) \quad \text{when} \quad m \equiv 3 \pmod 4.$$

If, in this case,
$$\omega = \frac{b+ci\sqrt{m}}{a},$$

so that
$$a^2\omega^2 - 2ab\omega + b^2 + mc^2 = 0,$$

$$k = dv\,(a^2, 2ab, b^2+mc^2),$$

$$a^2 = ka_0, \quad 2ab = kb_0,$$

$$b^2+mc^2 = kc_0,$$

$$a_0\omega^2 - b_0\omega + c_0 = 0, \tag{3}$$

then
$$4a_0c_0 - b_0^2 = \Delta_0,$$

$$\left.\begin{aligned} \Delta_0 &= 4m \text{ for } m \equiv 1, 2 \pmod 4, \\ \Delta_0 &= m \quad \text{for } m \equiv 3 \pmod 4. \end{aligned}\right\} \tag{4}$$

Taking the set of non-equivalent reduced ideals in the field $[i\sqrt{m}]$, whose class number is h, to be

$$(a_1, b_1+a) \equiv (1, a), \quad (a_2, b_2+a), \quad \ldots, \quad (a_h, b_h+a), \quad a_r > 0,$$

there are h fundamental allied sets of elliptic functions admitting complex multiplication by every member of $(1, a)$, and their respective primitive periods are

$$(a_1, b_1+a)\,\Omega^{(1)}, \quad (a_2, b_2+a)\,\Omega^{(2)}, \quad \ldots, \quad (a_h, b_h+a)\,\Omega^{(h)}. \tag{5}$$

Kronecker has proved that the absolute invariants

$$j\left(\frac{b_1+a}{a_1}\right), \quad j\left(\frac{b_2+a}{a_2}\right), \quad \ldots, \quad j\left(\frac{b_h+a}{a_h}\right)$$

of the *allied* moduli

$$(b_1+a)/a_1 \equiv a, \quad (b_2+a)/a_2, \quad \ldots, \quad (b_h+a)/a_h,$$

of which the first, a, is called the principal modulus, are the roots of an algebraic equation (Klassengleichung)

$$j^h+C_1 j^{h-1}+C_2 j^{h-2}+\ldots+C_h = 0,$$

whose coefficients are rational integers. This equation is irreducible in the field of rational numbers and Abelian in the quadratic field $[i\sqrt{m}]$.

In the more general case where

$$a, \quad b+ci\sqrt{m}$$

do not form the basis of an ideal in $[i\sqrt{m}]$ the discriminant of the equation (3)

$$a_0\omega^2-b_0\omega+c_0 = 0,$$

is

$$\Delta = 4a_0c_0-b_0^2 = n^2\Delta_0 \quad (n \geqslant 2),$$

and the number of non-equivalent moduli $(b+ci\sqrt{m})/a$ belonging to the discriminant $n^2\Delta_0$ is lh. A set of elliptic functions with the primitive periods

$$a\Omega, \quad (b+ci\sqrt{m})\Omega$$

is connected by a transformation of the n-th degree with one (and only one) of the h fundamental sets (5) with primitive periods

$$a_r\Omega^{(r)}, \quad (b_r+a)\,\Omega^{(r)},$$

and the absolute invariant $j[(b+ci\sqrt{m})/a]$ is a root of the irreducible equation

$$j^l+\Gamma_1^{(r)} j^{l-1}+\Gamma_2^{(r)} j^{l-2}+\ldots+\Gamma_l^{(r)} = 0,$$

whose coefficients are complex integers in the field $[j\{(b_r+a)/a_r\}]$. The modulus of multipliers for every such system is

$$\mathfrak{m} \equiv (1, na).$$

If $p_1, p_2, \ldots, p_c$ are the distinct prime factors of n, a general system of elliptic functions possesses

$$T(n) \equiv n\left(1+\frac{1}{p_1}\right)\left(1+\frac{1}{p_2}\right)\ldots\left(1+\frac{1}{p_c}\right)$$

distinct transformations of the n-th order. When the system admits com-

plex multiplication, the $T(n)$ transformed moduli may include among them moduli which are equivalent to each other or to the original modulus or to one of its allied moduli or are connected with one of these two latter by a transformation of degree less than n. In any such case

$$l < T(n),$$

and each of the allied moduli gives rise to the same number l of *proper* transformations.

As a numerical example take

$$m = 5, \quad \Delta_0 = 20, \quad h = 2,$$

with two reduced ideals $(1, i\sqrt{5})$, $(2, 1+i\sqrt{5})$ and the corresponding allied moduli $i\sqrt{5}$, $\frac{1}{2}(1+i\sqrt{5})$. Under cubic transformation $i\sqrt{5}$ becomes

$$3i\sqrt{5}, \quad \tfrac{1}{3}i\sqrt{5}, \quad \tfrac{1}{3}(1+i\sqrt{5}), \quad \tfrac{1}{3}(2+i\sqrt{5}),$$

and only the first two of these are proper transformed moduli, the other two being equivalent to the allied modulus $\frac{1}{2}(1+i\sqrt{5})$. Similarly on cubic transformation of $\frac{1}{2}(1+i\sqrt{5})$ we obtain the principal modulus $i\sqrt{5}$ twice and two (l) new moduli

$$\tfrac{3}{2}(1+i\sqrt{5}), \quad \tfrac{1}{7}(3+6i\sqrt{5}).$$

2. *Periodic Properties of* $\psi_\mu(u)$.

Taking the primitive periods

$$2\omega_1 = a\Omega, \quad 2\omega_2 = (b+ci\sqrt{m})\,\Omega,$$

with

$$\mathfrak{m} = (1, f+gi\sqrt{m}),$$

$$\mu = x+y\,(f+gi\sqrt{m})$$

(x, y rational integers),

$$\sigma(u) = u - \frac{g_2u^5}{240} - \frac{g_3u^7}{840} - \ldots,$$

I put

$$\psi_\mu(u) = e^{Auu}\sigma(\mu u)/\sigma(u)^M. \tag{6}$$

It is then found that

$$\psi_\mu(u+2\omega_1) = (-)^{P_1}\, e^{Q_1}\psi_\mu(u),$$

$$\psi_\mu(u+2\omega_2) = (-)^{P_2} e^{Q_2}\psi_\mu(u),$$

where

$$P_1 = x + \frac{(fc-bg+ag)y}{c} + \frac{agy}{c}\left\{x + \frac{(fc-bg)y}{c}\right\} - M,$$

$$P_2 = \frac{g(ab-b^2-mc^2)y}{ac} + \left\{1 - \frac{(b^2+mc^2)gy}{ac}\right\}\left(x+fy+\frac{bgy}{c}\right) - M,$$

both rational integers,

$$Q_1 = u\left[4A\omega_1 + \frac{2\mu agy\eta_2}{c} + 2\left\{\mu\left(x+\frac{fc-bg}{c}y\right) - M\right\}\eta_1\right]$$

$$+4A\omega_1^2 + \frac{2\mu agy\eta_2\omega_1}{c} + 2\left\{\mu\left(x+\frac{fc-bg}{c}y\right) - M\right\}\eta_1\omega_1,$$

$$Q_2 = (u+\omega_2)\left[4A\omega_2 - \frac{2\mu g(b^2+mc^2)y\eta_1}{ac} + 2\left\{\mu\left(x+fy+\frac{bgy}{c}\right) - M\right\}\eta_2\right].$$

Taking $M = \mu\mu' = \{x+y(f+gi\sqrt{m})\}\{x+y(f-gi\sqrt{m})\}$,

$$Q_1 = 2(u+\omega_1)[2A\omega_1 - \{(b-ci\sqrt{m})\eta_1 - a\eta_2\}\mu gy/c],$$

$$Q_2 = 2(\omega_2/\omega_1)(u+\omega_1)[2A\omega_1 - \{(b-ci\sqrt{m})\eta_1 - a\eta_2\}\mu gy/c].$$

Hence, if

$$A = \frac{\mu gy}{2c\omega_1}\{(b-ci\sqrt{m})\eta_1 - a\eta_2\} = C\mu y = Cy\{x+y(f+gi\sqrt{m})\}, \quad (7)$$

$$\frac{\psi_\mu(u+2\omega_1)}{\psi_\mu(u)} = \pm 1, \qquad \frac{\psi_\mu(u+2\omega_2)}{\psi_\mu(u)} = \pm 1.$$

The zeros of $\psi_\mu(u)$ occur when

$$\mu u \equiv 0 \pmod{2\omega_1, 2\omega_2},$$

and are all simple. Due to the zeros u_0, $-u_0$ (which involve each other), when distinct, $\psi_\mu(u)$ has the linear factor $\wp(u)-\wp(u_0)$. When $u = \omega_1$, a half period, the corresponding factor is $\sqrt{\{\wp(u)-\wp(\omega_1)\}}$. A ψ_μ which vanishes at two half-periods vanishes at all three, contains the factor $\wp'$, and is doubly periodic. When ψ_μ vanishes at ω_1 only it is skew-periodic in $2\omega_1$. Hence putting

$$\wp(\omega_1) = e_1, \quad \wp(\omega_2) = e_2, \quad \wp(\omega_3) = e_3,$$

(i) when $$\psi_\mu(u+2\omega_1) = \psi_\mu(u+2\omega_2) = \psi_\mu(u),$$

$$\psi_\mu = I_{\frac{1}{2}(M-1)}(\wp) \quad \text{or} \quad I_{\frac{1}{2}(M-4)}(\wp)\,\wp',$$

according as M is odd or even ;

(ii) when $\psi_\mu(u+2\omega_1+2\omega_2) = \psi_\mu(u+2\omega_2) = -\psi_\mu(u+2\omega_1) = -\psi_\mu(u)$,

$$\psi_\mu = I_{\frac{1}{2}(M-2)}(\wp)\sqrt{(\wp-e_1)};$$

(iii) when $\psi_\mu(u+2\omega_1+2\omega_2) = -\psi_\mu(u+2\omega_2) = \psi_\mu(u+2\omega_1) = -\psi_\mu(u)$,

$$\psi_\mu = I_{\frac{1}{2}(M-2)}(\wp)\sqrt{(\wp-e_2)};$$

(iv) when $\psi_\mu(u+2\omega_1+2\omega_2) = -\psi_\mu(u+2\omega_2) = -\psi_\mu(u+2\omega_1) = \psi_\mu(u)$,

$$\psi_\mu = I_{\frac{1}{2}(M-2)}(\wp)\sqrt{(\wp-e_3)}.$$

In each case $I_s(\wp)$ is a rational integral function of $\wp(u)$ of degree s.

The zeros of $\psi_\mu(u)$ are included among those of $\psi_{\mu\nu}(u)$, ν being any complex integer such that $\mu\nu$ belongs to $\mathfrak{m}$. Hence ψ_μ is a factor of $\psi_{\mu\nu}$ and in particular of $\psi_{\mu\mu'} \equiv \psi_M$.

If $\mu \equiv x+y(f+gi\sqrt{m})$, ν are any two complex numbers and

$$N(\mu) = (x+fy)^2+g^2my^2,$$

$$P(\mu) = y\{x+y(f+gi\sqrt{m})\},$$

then $$N(\mu+\nu)+N(\mu-\nu) = 2N(\mu)+2N(\nu),$$

and $$P(\mu+\nu)+P(\mu-\nu) = 2P(\mu)+2P(\nu)$$

identically. It follows that

$$\wp(\nu u)-\wp(\mu u) = \frac{\sigma(\mu-\nu)\,u\sigma(\mu+\nu)u}{\sigma^2(\mu u)\,\sigma^2(\nu u)} = \frac{\psi_{\mu-\nu}\psi_{\mu+\nu}}{\psi_\mu^2\psi_\nu^2}, \tag{8}$$

and, in particular, $$\wp(\mu u) = \wp(u)-\frac{\psi_{\mu-1}\psi_{\mu+1}}{\psi_\mu^2},$$

exactly as in real multiplication. The identity

$$[\wp(\nu u)-\wp(\mu u)][\wp(\rho u)-\wp(\pi u)]+[\wp(\pi u)-\wp(\mu u)][\wp(\nu u)-\wp(\rho u)]$$
$$+[\wp(\rho u)-\wp(\mu u)][\wp(\pi u-\wp(\nu u)] = 0$$

gives immediately, after (8), the recurrence formula

$$\psi_{\mu+\nu}\psi_{\mu-\nu}\psi_{\pi+\rho}\psi_{\pi-\rho}+\psi_{\mu+\pi}\psi_{\mu-\pi}\psi_{\rho+\nu}\psi_{\rho-\nu}+\psi_{\mu+\rho}\psi_{\mu-\rho}\psi_{\nu+\pi}\psi_{\nu-\pi} = 0,$$

from which all real ψ's can be calculated from ψ_1, ψ_2, ψ_3, ψ_4, and all real and complex ψ's from those for

$$\mu = 1, 2, 3, gi\sqrt{m}, 1+gi\sqrt{m}, 2+gi\sqrt{m} \text{ when } f = 0,$$

or $$\mu = 1, 2, 3, f+gi\sqrt{m}, 1+f+gi\sqrt{m} \text{ when } f = \tfrac{1}{2}.$$

The invariants g_2, g_3 can always be chosen to belong to the field $[j]$: in fact the general solution of

$$\frac{1728g_2^3}{g_2^3-27g_3^2}=j \quad \text{or} \quad \frac{g_2^3}{g_3^2}=\frac{27j}{j-1728}$$

is $$g_3=\lambda g_2=27j\lambda^3/(j-1728).$$

When λ is so chosen that $\frac{1}{4}g_2$, g_3 are real or complex integers the coefficients of the real ψ's are integral, and consequently ψ_μ, a factor of $\psi_{\mu\mu'}$, has integral coefficients too.

In dealing with a numerical case it is first necessary to calculate the constant C, of (7), which is homogeneous and of the same dimensions as $\sqrt{g_2}$ or g_3/g_2. The same constant defines all the ψ's and in particular ψ_μ, ψ_ρ where (μ), (ρ) are prime ideals of the first degree with different norms p_1, p_2. It follows that C is a modular function belonging to each of the stages p_1, p_2 and therefore to the first stage, so, when g_2, g_3 are chosen to belong to $[j]$, C lies in the field $[j, i\sqrt{m}]$.

3. *The first System of Elliptic Functions under* $\Delta=20$.

For this system the primitive periods are

$$2\Omega, \quad 2i\sqrt{5}\,\Omega$$

and $$\mathfrak{m}=\mathfrak{o}=(1, i\sqrt{5}).$$

Also $$j(i\sqrt{5})=320(1975+884\sqrt{5})=2^6.5\sqrt{5}(2+\sqrt{5})^4(4-\sqrt{5})^3,$$

$$j(i\sqrt{5})-1728=2^3(2+\sqrt{5})(2\sqrt{5}+1)^2(4+\sqrt{5})^2.$$

I take provisionally $$g_2=12(10+3\sqrt{5})\lambda^2,$$

$$g_3=16(14+9\sqrt{5})\lambda^3,$$

$$g_2^3-27g_3^2=2^6.3^6(-2+\sqrt{5})\lambda^6,$$

so that $$\wp'^2=4\wp^3-12(10+3\sqrt{5})\lambda^2\wp-16(14+9\sqrt{5})\lambda^3$$

$$=4(\wp+4\lambda)[\wp^2-4\lambda\wp-(14+9\sqrt{5})\lambda^2],$$

and $$\wp(1+i\sqrt{5})\,\Omega=-4\lambda.$$

Putting $$\mu=x+yi\sqrt{5}, \quad M=x^2+5y^2,$$

$$\psi_\mu(u)=e^{Cy(x+yi\sqrt{5})uu}\,\sigma(x+yi\sqrt{5})\,u/\sigma(u)^M,$$

$$\psi_\mu(u+2\Omega)=\psi_\mu(u+2i\sqrt{5}\,\Omega)=(-)^{xy}\psi_\mu(u+2\Omega+2i\sqrt{5}\,\Omega)=(-)^{xy}\psi_\mu(u).$$

Hence $\psi_\mu(u) = I_{\frac{1}{2}(M-1)}(\wp)$ when M is odd,

$\psi_\mu = I_{\frac{1}{2}(M-4)}(\wp)\wp'$ when x, y are both even,

$\psi_\mu = I_{\frac{1}{2}(M-2)}(\wp)\sqrt{(\wp+4\lambda)}$ when x, y are both odd.

Equating to zero the coefficients of u^2, u^4 in the identity

$$\psi_{i\sqrt5} = e^{Ci\sqrt5 uu}\frac{\sigma(i\sqrt5\,u)}{\sigma(u)^5} = i(\sqrt5\,\wp^2+B_1\lambda\wp+B_2\lambda^2),$$

after eliminating B_1, B_2 from the coefficients of u^{-2}, u^0, and obtaining their highest common factor as polynomials in C, I find

$$C = -i(3+\sqrt5)\,\lambda,$$

$$\psi_{i\sqrt5} = i\{\sqrt5\,\wp^2+(15+5\sqrt5)\,\lambda\wp+(42+13\sqrt5)\lambda^2\}.$$

An alternative method of calculating the division equation

$$\psi_{i\sqrt5} \equiv i(\sqrt5\,\wp^2+B_1\lambda\wp+B_2\lambda^2) = 0$$

is applicable here and to certain other values of m. The modulus $i\sqrt{(5q)}$ being connected with its allied modulus $i\sqrt{(q/5)}$ by quintic transformation, the sextic resolvent satisfied by any symmetric function of $\wp(\frac{2}{5}\varpi)$, $\wp(\frac{4}{5}\varpi)$ has a root rational in $[j\{i\sqrt{(5q)}\}]$, when q is prime to 5, and the rational root belongs to the division values

$$\wp\{\tfrac{2}{5}i\sqrt{(5q)}\,\Omega\}, \qquad \wp\{\tfrac{4}{5}i\sqrt{(5q)}\,\Omega\}.$$

In the present case $q = 1$ and Klein's principal parameter τ, given by

$$\frac{(\tau^2-10\tau+5)^3}{-\tau} = j(i\sqrt5) = 320(1975+884\sqrt5),$$

has the rational value $-5\sqrt5$.* Hence, after Halphen,†

$$t = [(B_1/\sqrt5)^2-4B_2/\sqrt5]\,\lambda^2,$$

$$t^3 = -(g_2^3-27g_3^2)/\tau = 18^3(5-\sqrt5)^3\,\lambda^6/125,$$

$$t = 18(5-\sqrt5)\,\lambda^2/5,$$

$$B_1 = -\frac{\sqrt5}{\lambda}\,\frac{6g_3t^2}{t^3-2g_2t^2+g_2^3-27g_3^2} = 15+5\sqrt5,$$

$$B_2 = \tfrac{1}{4}\sqrt5(\tfrac{1}{5}B_1^2-t\lambda^{-2}) = 42+13\sqrt5,$$

* G. B. Mathews, *Proc. London Math. Soc.*, Ser. 1, Vol. 21 (1890), p. 240.

† *Fonctions Elliptiques*, Vol. 3, p. 5.

and $$\psi_{i\sqrt5} = i\{\sqrt5\,\wp^2+(15+5\sqrt5)\lambda\wp+(42+13\sqrt5)\lambda^2\},$$

$$C = -\tfrac{1}{5}iB_1 = -i(3+\sqrt5)\lambda,$$

as before.

Comparison of coefficients in (6) now gives immediately

$$\psi_{1+i\sqrt5} = (1+i\sqrt5)\{\wp^2+(3+3\sqrt5-3i-i\sqrt5)\lambda\wp + (14+3\sqrt5-12i-4i\sqrt5)\lambda^2\}\sqrt{(\wp+4\lambda)},$$

and

$$\psi_{2+i\sqrt5} = (2+i\sqrt5)\{\wp^4+(5+3\sqrt5-6i-2i\sqrt5)\lambda\wp^3 + (27+9\sqrt5-54i-24i\sqrt5)\lambda^2\wp^2+(43+9\sqrt5-144i-96i\sqrt5)\lambda^3\wp + (-16+3\sqrt5-258i-56i\sqrt5)\lambda^4\},$$

while $\psi_{x-yi\sqrt5}$ is obtained from $\psi_{x+yi\sqrt5}$ by the substitution $\begin{pmatrix} i, & \sqrt5 \\ -i, & \sqrt5 \end{pmatrix}$.

Using the known formulæ

$$\psi_2 = -\wp',$$

$$\psi_3 = 3\wp^4-\tfrac{3}{2}g_2\wp^2-3g_3\wp-\tfrac{1}{16}g_2^2$$

$$= 3\{\wp^4-(60+18\sqrt5)\lambda^2\wp^2-(224+144\sqrt5)\lambda^3\wp-(435+180\sqrt5)\lambda^4\},$$

the recurrence relation gives

$$\psi_{3+i\sqrt5} = \{(3+i\sqrt5)\wp^6+(24+18\sqrt5-12i-6i\sqrt5)\lambda\wp^5 + (354+99\sqrt5-339i-162i\sqrt5)\lambda^2\wp^4 + (804+648\sqrt5-1884i-1216i\sqrt5)\lambda^3\wp^3 + (1941-252\sqrt5-8436i-2265i\sqrt5)\lambda^4\wp^2 - (8880+2142\sqrt5+8628i+5382i\sqrt5)\lambda^5\wp - (10350+7155\sqrt5+7581i+4178i\sqrt5)\lambda^6\}\sqrt{(\wp+4\lambda)}.$$

Due to the prime ideal $(3, 1+i\sqrt5)$ ψ_3 and $\psi_{1+i\sqrt5}$ have a common linear factor which vanishes when $u = \frac{2}{3}(1-i\sqrt5)\Omega$. I denote this factor by $\mathfrak{P}_{(3,\,1+i\sqrt5)}$ when its leading coefficient is reduced to

$$(1+i\sqrt5)/(1+i) \quad \text{or} \quad \tfrac{1}{2}(1+\sqrt5-i+i\sqrt5).$$

Similarly ψ_3, $\psi_{1-i\sqrt5}$, $\psi_{2+i\sqrt5}$ have a common linear factor whose zeros are $\pm\frac{2}{3}(1+i\sqrt5)\Omega$.

By the greatest common measure process I easily find

$$\psi_{1+i\sqrt5} = (1+i\sqrt5)\{\wp+\tfrac12(1-3i)(1+i\sqrt5)\lambda\}$$
$$\times\{\wp+\tfrac12(5+3\sqrt5-3i-3i\sqrt5)\lambda\}\sqrt{(\wp+4\lambda)},$$

$$\psi_3 = 3\{\wp^2+(1+3\sqrt5)\lambda\wp+15\lambda^2\}\{\wp^3-(1+3\sqrt5)\lambda\wp-(29+12\sqrt5)\lambda^2\}$$
$$= 3\{\wp+\tfrac12(1-3i)(1+i\sqrt5)\lambda\}\{\wp+\tfrac12(1+3i)(1-i\sqrt5)\lambda\}$$
$$\times\{\wp^2-(1+3\sqrt5)\lambda\wp-(29+12\sqrt5)\lambda^2\},$$

$$\psi_{2+i\sqrt5} = (2+i\sqrt5)\{\wp+\tfrac12(1+3i)(1-i\sqrt5)\lambda\}$$
$$\times\{\wp^3+\tfrac12(9+3\sqrt5-15i-3i\sqrt5)\lambda\wp^2+(6+3\sqrt5-42i-12i\sqrt5)\lambda^2\wp$$
$$+\tfrac12(-31+3\sqrt5-69i-57i\sqrt5)\lambda^3\}.$$

It follows immediately that

$$\wp\tfrac23(1-i\sqrt5)\Omega = -\tfrac12(1-3i)(1+i\sqrt5)\lambda,$$

$$\wp\tfrac23(1+i\sqrt5)\Omega = -\tfrac12(1+3i)(1-i\sqrt5)\lambda,$$

$$\wp[\tfrac23(1-i\sqrt5)\Omega+(1-i\sqrt5)\Omega] = \wp\tfrac13(1-i\sqrt5)\Omega$$
$$= -\tfrac12(5+3\sqrt5-3i-3i\sqrt5)\lambda,$$

$$\wp\tfrac13(1+i\sqrt5)\Omega = -\tfrac12(5+3\sqrt5+3i+3i\sqrt5)\lambda,$$

that $\wp(\frac23\Omega)$ and $\wp(\frac23 i\sqrt5\,\Omega)$ are the roots of the irreducible quadratic

$$\wp^2-(1+3\sqrt5)\lambda\wp-(29+12\sqrt5)\lambda^2 = 0,$$

and $\wp\frac29(2-i\sqrt5)\Omega$, $\wp\frac49(2-i\sqrt5)\Omega$, $\wp\frac89(2-i\sqrt5)\Omega$,

the zeros of the irreducible cubic factor of $\psi_{2+i\sqrt5}$.

Accordingly

$$\psi_{(3,\,1+i\sqrt5)} = \frac{1+i\sqrt5}{1+i}\left\{\wp+\frac{(1-3i)(1+i\sqrt5)}{2}\lambda\right\},$$

$$\psi_{(3,\,1-i\sqrt5)} = \tfrac12(1+\sqrt5+i-i\sqrt5)\{\wp+\tfrac12(1+3i)(1-i\sqrt5)\lambda\},$$

and I also write $\psi_{(2,\,1+i\sqrt5)} = (1+i)\sqrt{(\wp+4\lambda)}$.

The polynomials ψ_7, $\psi_{3+i\sqrt5}/\sqrt{(\wp+4\lambda)}$, $\psi_{4-i\sqrt5}$, $\psi_{-1+2i\sqrt5}$ have a common irreducible cubic factor $\psi_{(7,\,3+i\sqrt5)}$. To calculate this factor by the greatest common measure process is not feasible owing to the magnitude of the coefficients involved, $\psi_{4-i\sqrt5}$ and $\psi_{-1+2i\sqrt5}$ being each of the tenth degree.

4. *The Second System of Elliptic Functions under* $\Delta = 20$.

The primitive periods are

$$4\Omega', \quad 2(1+i\sqrt{5})\Omega',$$

and $$j[\tfrac{1}{2}(1+i\sqrt{5})] = 320(1975-884\sqrt{5}),$$

$$g_2' = 12(10-3\sqrt{5})\theta^2,$$

$$g_3' = 16(14-9\sqrt{5})\theta^3,$$

$$\wp(2\Omega') = -4\theta,$$

$$\mathfrak{m} = \mathfrak{o} = (1, i\sqrt{5}).$$

For this system I write χ instead of ψ. Then, with $\mu = x+i\sqrt{5}\,y$,

$$\chi_\mu(u) = e^{C'y(x+i\sqrt{5}y)uu}\frac{\sigma(x+i\sqrt{5}\,y)\,u}{\sigma(u)^{x^2+5y^2}} = \chi_\mu(u+4\Omega')$$

$$= (-)^{xy}\chi_\mu(u \pm 2\Omega' + 2i\sqrt{5}\,\Omega');$$

$\chi_\mu = I_{\frac{1}{2}(M-1)}(\wp)$ when $M \equiv x^2+5y^2$ is odd,

$\chi_\mu = I_{\frac{1}{2}(M-4)}(\wp)\wp'$ when x, y are both even,

$\chi_\mu = I_{\frac{1}{2}(M-2)}(\wp)\sqrt{(\wp+4\theta)}$ when x, y are both odd.

Since g_2', g_3' are obtained from g_2, g_3 by the substitution $(\sqrt{5}, -\sqrt{5})$, and since the coefficients of $\sigma(u)$ are rational integral functions of g_2, g_3, it is evident that the equation

$$e^{C'i(-\sqrt{5})u^2}\frac{\sigma(-i\sqrt{5}\,u)}{\sigma(u)^5} = i\{-\sqrt{5}\,\wp^2+B_1'\theta\wp+B_2'\theta^2\}$$

will be satisfied identically when B_1', B_2' are the conjugates of B_1, B_2 in $[\sqrt{5}]$. Hence

$$C' = -i(3-\sqrt{5})\,\theta,$$

$$B_1' = 15-5\sqrt{5},$$

$$B_2' = 42-13\sqrt{5},$$

i.e. $$\chi_{-i\sqrt{5}} = i\{-\sqrt{5}\,\wp^2+(15-5\sqrt{5})\theta\wp+(42-13\sqrt{5})\,\theta^2\}$$

and $$\chi_{i\sqrt{5}} = i\{\sqrt{5}\,\wp^2+(-15+5\sqrt{5})\theta\wp+(-42+13\sqrt{5})\theta^2\}.$$

In general the substitution $\begin{pmatrix} i, & \sqrt{5}, & \lambda \\ -i, & -\sqrt{5}, & \theta \end{pmatrix}$ changes $\psi_{x+yi\sqrt{5}}$ into $\chi_{x+yi\sqrt{5}}$.

Thus $\chi_{1+i\sqrt{5}} = (1+i\sqrt{5})\{\wp^2+(3-3\sqrt{5}+3i-i\sqrt{5})\theta\wp$

$$+(14-3\sqrt{5}+12i-4i\sqrt{5})\theta^2\}\sqrt{(\wp+4\theta)},$$

$$\chi_{(3,\,1+i\sqrt{5})} = \frac{1+i\sqrt{5}}{1-i}\left\{\wp+\frac{(1+3i)(1+i\sqrt{5})}{2}\theta\right\},$$

with zeros at $\pm\frac{2}{3}(-1+i\sqrt{5})\Omega'$, and so on.

5. *Quadratic Relations between the two Systems under* $\Delta = 20$.

If
$$X = \wp(u;\ 2\Omega_1,\ 2\Omega_2;\ g_2,\ g_3),$$
$$Y = \wp(u;\ 2\Omega_1,\ \Omega_2;\ \gamma_2,\ \gamma_3),$$
$$e = \wp(\Omega_2;\ 2\Omega_1,\ 2\Omega_2;\ g_2,\ g_3),$$

it is easily proved that

$$\left.\begin{aligned}\gamma_2 &= 60e^2-4g_2,\\ \gamma_3 &= 88e^3-8eg_2,\\ Y &= \frac{X^2-eX+3e^2-\frac{1}{4}g_2}{X-e}\end{aligned}\right\} \qquad (9)$$

So taking X to be a function of the first system under $\Delta = 20$,

$$\left.\begin{aligned}2\Omega_1 &= 2\Omega,\\ 2\Omega_2 &= 2(1+i\sqrt{5})\,\Omega,\\ g_2 &= 12(10+3\sqrt{5})\lambda^2,\\ g_3 &= 16(14+9\sqrt{5})\lambda^3,\\ e &= -4\lambda,\\ \gamma_2 &= 4.12(10-3\sqrt{5})\lambda^2,\\ \gamma_3 &= -8.16(14-9\sqrt{5})\lambda^3,\end{aligned}\right\} \qquad (10)$$

so that $\gamma_2 = g_2'$, $\gamma_3 = g_3'$ when $\lambda = -\frac{1}{2}\theta$.

A prime number p which $\equiv 3$ or $7 \pmod{20}$ breaks up into two non-principal prime ideals in $[i\sqrt{5}]$, while $2p$ is the product of two complex integral factors $(x+yi\sqrt{5})(x-yi\sqrt{5})$ each divisible by one prime ideal factor of p. Due to the prime ideal $(p,\ x+yi\sqrt{5}) \equiv \mathfrak{p}$ there is an irre-

ducible polynomial

$$\psi_{\mathfrak{p}} \equiv \psi_{(p,\, x+yi\sqrt{5})}$$

of degree $\frac{1}{2}(p-1)$, whose zeros are

$$\pm 2\Omega s\,\{x+5y+i\sqrt{5}(x-y)\}/p \quad [s = 1, 2, \ldots, \tfrac{1}{2}(p-1)],$$

but, as already mentioned, it is practically impossible to calculate $\psi_{\mathfrak{p}}$ by the greatest common measure process even when $p = 7$.

When $p = 7$, $\qquad x+yi\sqrt{5} = 3+i\sqrt{5}$,

and the roots of $\qquad \psi_{3+i\sqrt{5}}/\sqrt{(\wp+4\lambda)} = 0$

are
$$\wp_1 = \wp\left[\frac{8+2i\sqrt{5}}{7}\,\Omega\right], \quad \wp_6 = \wp\left[\frac{8+2i\sqrt{5}}{7}\,\Omega+(1+i\sqrt{5})\Omega\right],$$

$$\wp_2 = \wp\left[\frac{16+4i\sqrt{5}}{7}\,\Omega\right], \quad \wp_5 = \wp\left[\frac{16+4i\sqrt{5}}{7}\,\Omega+(1+i\sqrt{5})\Omega\right],$$

$$\wp_3 = \wp\left[\frac{10+6i\sqrt{5}}{7}\,\Omega\right], \quad \wp_4 = \wp\left[\frac{10+6i\sqrt{5}}{7}\,\Omega+(1+i\sqrt{5})\Omega\right].$$

Three of the zeros, $\frac{1}{7}(8+2i\sqrt{5})\Omega$, $\frac{1}{7}(16+4i\sqrt{5})\Omega$, $\frac{1}{7}(10+6i\sqrt{5})\Omega$, which are also zeros of $\psi_{4-i\sqrt{5}}$, lie in the parallelogram with vertices at

$$0, \quad 2\Omega, \quad (1+i\sqrt{5})\Omega, \quad (3+i\sqrt{5})\Omega,$$

and the other three, which are not zeros of $\psi_{4-i\sqrt{5}}$, lie in the contiguous parallelogram whose vertices are

$$(1+i\sqrt{5})\Omega, \quad (3+i\sqrt{5})\Omega, \quad (2+2i\sqrt{5})\Omega, \quad (4+2i\sqrt{5})\Omega.$$

From (9) above the roots of

$$X^2-eX+3e^2-\tfrac{1}{4}g_2 = Y(X-e)$$

are
$$X_1 = \wp(u;\ 2\Omega_1, 2\Omega_2;\ g_2, g_3),$$

$$X_2 = \wp(u+\Omega_2;\ 2\Omega_1, 2\Omega_2;\ g_2, g_3).$$

Hence it must be possible to reduce the equation

$$\psi_{x+yi\sqrt{5}}/\sqrt{(\wp+4\lambda)} = 0$$

to an equation of degree $\frac{1}{2}(p-1)$ by arranging it in terms of

$$\wp^2+4\lambda\wp+(18-9\sqrt{5})\lambda^2, \quad \wp+4\lambda.$$

On calculation, with $p = 7$,

$$\begin{aligned}
&\psi_{3+i\sqrt5}/\sqrt{(\wp+4\lambda)}\\
=\ &(3+i\sqrt5)(\wp+4\lambda)^6+(-48+18\sqrt5-12i-30i\sqrt5)\,\lambda(\wp+4\lambda)^5\\
&+(594-261\sqrt5-99i+198i\sqrt5)\,\lambda^2(\wp+4\lambda)^4\\
&+(-4860+1944\sqrt5+1620i-864i\sqrt5)\,\lambda^3(\wp+4\lambda)^3\\
&+9(2-\sqrt5)(594-261\sqrt5-99i+198i\sqrt5)\,\lambda^4(\wp+4\lambda)^2\\
&+81(2-\sqrt5)^2(-48+18\sqrt5-12i-30i\sqrt5)\lambda^5(\wp+4\lambda)\\
&+729(2-\sqrt5)^3(3+i\sqrt5)\lambda^6
\end{aligned}$$

(in reciprocal form)

$$\begin{aligned}
=\ &(3+i\sqrt5)\{\wp^2+4\lambda\wp+(18-9\sqrt5)\lambda^2\}^3\\
&+2\lambda(-6+9\sqrt5-6i-9i\sqrt5)(\wp+4\lambda)\{\wp^2+4\lambda\wp+(18-9\sqrt5)\lambda^2\}^2\\
&+4\lambda^2(48-9\sqrt5-15i-12i\sqrt5)(\wp+4\lambda)^2\{\wp^2+4\lambda\wp+(18-9\sqrt5)\lambda^2\}\\
&+8\lambda^3(-45-87i+20i\sqrt5)(\wp+4\lambda)^3.
\end{aligned}$$

Hence, after (9), the cubic

$$\begin{aligned}
&(3+i\sqrt5)\,\wp^3+(6-9\sqrt5+6i+9i\sqrt5)\,\theta\wp^2\\
&\qquad+(48-9\sqrt5-15i-12i\sqrt5)\,\theta^2\wp+(45+87i-20i\sqrt5)\,\theta^3
\end{aligned}$$

should be a factor of $\chi_{3+i\sqrt5}$, and therefore

$$\begin{aligned}
\psi\equiv\ &(3+i\sqrt5)\wp^3+(6+9\sqrt5-6i+9i\sqrt5)\lambda\wp^2\\
&+(48+9\sqrt5+15i-12i\sqrt5)\lambda^2\wp+(45-87i-20i\sqrt5)\lambda^3,
\end{aligned}$$

a factor of $\psi_{3+i\sqrt5}$. On division it is found that

$$\begin{aligned}
\psi_{3+i\sqrt5}=\psi\sqrt{(\wp+4\lambda)}\,\{&\wp^3+\tfrac12(-3+3\sqrt5-9i-9i\sqrt5)\lambda\wp^2\\
&+(-6+3\sqrt5-72i-18i\sqrt5)\lambda^2\wp+\tfrac12(101+3\sqrt5-135i-135i\sqrt5)\lambda^3\}.
\end{aligned}$$

Thus

$$\begin{aligned}
\psi_{(7,\,3+i\sqrt5)}=\psi\div(1+i)=\ &\tfrac12(3+i\sqrt5)(1-i)\wp^3+(9\sqrt5-6i)\lambda\wp^2\\
&+\tfrac12(63-3\sqrt5-33i-21i\sqrt5)\lambda^2\wp\\
&-(21+10\sqrt5+66i+10i\sqrt5)\lambda^3,
\end{aligned}$$

with zeros at $\pm\frac17(8+2i\sqrt5)\Omega$, $\pm\frac17(16+4i\sqrt5)\Omega$, $\pm\frac17(10+6i\sqrt5)\Omega$, while

the zeros of the other cubic factor of $\psi_{3+i\sqrt{5}}$ are

$$\pm\tfrac{1}{7}(15+9i\sqrt{5})\Omega, \quad \pm\tfrac{1}{7}(9+11i\sqrt{5})\Omega, \quad \pm\tfrac{1}{7}(3+13i\sqrt{5})\Omega.$$

By the same quadratic transformation one can determine the irreducible factors of $\psi_{x+yi\sqrt{5}}$ when

$$x^2+5y^2 = 2p, \quad p = 3 \text{ or } 7 \pmod{20}:$$

one of them, $\psi_{(p,\, x+yi\sqrt{5})}$, divides ψ_μ whenever μ is a member of the prime ideal $(p, x+yi\sqrt{5})$. Each non-principal prime ideal of $[i\sqrt{5}]$ is therefore associated with a unique polynomial $\psi_{\mathfrak{p}}$.

More generally if $\mathfrak{a} \equiv (\mu, \mu', \ldots)$ is any ideal, whether prime or composite, in $[i\sqrt{m}]$, and $\psi_{\mathfrak{a}}$ is defined as the highest common factor of *all* the polynomials ψ_μ, $\psi_{\mu'}$, &c., there exists a similar correspondence between $\mathfrak{a}$ and $\psi_{\mathfrak{a}}$. It should be noted that $\psi_{\mathfrak{a}}\psi_{\mathfrak{b}}$ is not identical with $\psi_{\mathfrak{ab}}$, though the latter polynomial is divisible by $\psi_{\mathfrak{a}}\psi_{\mathfrak{b}}$ when $\mathfrak{a}$, $\mathfrak{b}$ are co-prime.

The quadratic transformation affords a sufficient means of isolating the polynomial associated with a prime ideal $\mathfrak{p}$ whenever $2N(\mathfrak{p})$ is the norm of an algebraic integer in $[i\sqrt{m}]$. A discussion of the application of higher order transformations to this problem, in cases where the quadratic transformation is insufficient, is reserved to another occasion.

ERRATUM.—P. 157, line 17, *for* $\frac{1}{7}(3+6i\sqrt{5})$, *read* $\frac{1}{7}(2+3i\sqrt{5})$.

ON THE MATHEMATICAL EXPRESSION OF THE PRINCIPLE OF HUYGENS—II.

By Sir JOSEPH LARMOR.

[Read November 13th, 1919.]

THE following considerations are in amplification and further development of a previous paper on this subject, *Proc. Lond. Math. Soc.*, Ser. 2, Vol. 1 (1903), p. 1.

In any medium which transmits physical action, the disturbance sent out across any surface S, which surrounds the true sources, is wholly determined by knowledge of the stress which is exerted, on the medium beyond, across each element of S. It would also be determined by knowledge of the strain which is imparted to the medium outside at each element of S. These determinations must agree: and the reason is that the two specifications over the surface S, one in terms of stress the other in terms of strain, are not independent, one determines the other.

Let us fix attention on the case of sound waves in a gas, completely determined by a single propagated velocity potential ϕ. The stress condition is that the change of pressure $-\rho\partial\phi/\partial t$, and therefore the value of ϕ as regards sound of given frequency, is given over each element δS. The strain (in this case velocity) condition is that the value of $\partial\phi/\partial n$, the gradient of ϕ along the normal, is everywhere given over S. If we had a formula for the effect of a single type of source, namely of an impressed pressure given over an element and merely absent over the remainder of the surface, or else of the normal velocity given there and maintained null over the other elements of the surface, the total effect would be expressible as an integral taken over the surface.

Symmetry indicates how this is to be effected when the surface S is an infinite plane, say that of xy. When the velocity of propagation is infinite, ϕ is the ordinary static potential $f(t)/r$, and the respective solutions are $\frac{\phi_0}{2\pi}\delta\Omega$ and $-\frac{1}{2\pi}\left(\frac{\partial\phi}{\partial n}\right)_0\delta S/r$; where $\delta\Omega$ is the element of solid angle sub-

tended, equal to $\delta S \frac{\partial}{\partial z}\left(\frac{1}{r}\right)$ or $\cos(zr)\,\delta S/r^2$, and the subscript zero indicates local value at δS. The possibility of integration of such formulæ rests on the principle that a gradient differentiation of any function of r can be transferred from one end of r, the source, to the other, at which the disturbance is to be estimated, by mere change of sign. The corresponding forms for a potential propagated with finite speed c are $f\left(t-\frac{r}{c}\right)\Big/r$ and $\frac{\partial}{\partial z}\cdot f\left(t-\frac{r}{c}\right)\Big/r$. Very near the origin these can be expanded by Taylor's series into $f(t)/r-c^{-1}f'(t)+\ldots$ and $\partial/\partial z$ of the same. The source or singular region at the origin must be specified completely in the first terms which alone involve inverse powers of r: thus as regards its determination the value of c does not enter, and the case is the same as the previous one of the static Laplacean potential. The disturbance transmitted across the unlimited plane is thus expressed for the point r, z by either of the formulæ

$$\phi_1 = \int \frac{1}{r} f\left(t-\frac{r}{c}\right) dS, \qquad \phi_2 = -\int \frac{\partial}{\partial z}\left\{\frac{1}{r} F\left(t-\frac{r}{c}\right)\right\} dS,$$

in which the surface data are

$$f(t) = -\frac{1}{2\pi}\left(\frac{\partial \phi}{\partial z}\right)_0, \qquad F(t) = \frac{1}{2\pi}\phi_0.$$

It is to be noted that in the expression for ϕ_2, the function F must be kept inside the bracket: otherwise the element of the integral would not be a propagated potential. In fact this second formula is the same as

$$\phi_2 = -\int \frac{\cos(rz)}{cr} F'\left(t-\frac{r}{c}\right) dS - \int \frac{\cos(rz)}{r^2} F\left(t-\frac{r}{c}\right) dS,$$

of which it is the first term alone that represents radiation sent away from the system, the second term being, owing to the higher inverse power of r, a local alternating disturbance which does not involve steady emission of energy.

For the special case of radiation transmitted across a wave front, or surface of constant phase, of any form, the transmission will be by rays of energy, whose paths are determined by the condition that time of transit does not vary sensibly for change to any adjacent path, and therefore that the ray travels by the quickest possible route at each stage,

though it may not remain the path of shortest time for the whole course, if the latter is too long. In the case of a uniform medium to which alone the formulæ, as here written, apply the rays are straight, and their intensity is expressed in the Fresnel manner in terms of the elements of wave front from which the disturbance comes. There ought to be agreement as regards the expressions ϕ_1 and ϕ_2; and that is so in this special case of plane fronts, even for the separate elements, for as regards any function of $t-z/c$, the operator $\partial/\partial z$ is the same as $-c^{-1}\partial/\partial t$.

Here f and F may change with position so that the disturbance varies from place to place on the plane of resolution; but if this plane is not a wave front the functions do not involve only z and t, and the formulæ for ϕ_1 and ϕ_2 will be altered. The advancing disturbance is, except in the above special case, decomposed into elements at the plane or other surface of resolution in two different ways; these must be equivalent as regards total effect, but we are not as yet entitled to assert that either of them represents a resolution of the source into local equivalent physical parts. And this should not cause surprise: the strain specification expresses assigned local strain combined with constraint everywhere else on the surface S: while the stress specification is local but with no constraining or other condition elsewhere. Thus the latter, being self-contained as regards the element, has better claim to be a true physical resolution of the source of disturbance, as will presently be confirmed.

The second of these two special modes of resolution, that by means of stress conditions, is in keeping with the Kirchhoff formula expressing a quantitative formulation of the principle of Huygens as regards pressural waves; but the analytical possibility of the first one also was calculated to throw doubt on the definiteness of any such resolution of the disturbance into components emerging from elementary surface sources. It is this point that we are now to consider more closely.

In the previous paper the general problem was approached from the point of view of Green's analysis in his fundamental *Essay* of 1828, which formed in fact incidentally the earliest initiation of the theory of continuous analytical functions and their singularities. If the field of activity due to a set of sources inside S and another set outside S is supposed to be mapped out, then it was established at once on these principles that the outside effect of the sources inside can be represented as due to an equivalent distribution of sources over S: and the inside effect of the sources outside is due to the same distribution with sign reversed. It thus appears that there are an infinite number of distributions over S which can represent in the outside region any given set of internal sources: for the external sources of the theorem can be chosen

at will. And this applies whether S is a wave front or not: the resolution of the disturbance actually transmitted into contributions from secondary sources distributed over S is not unique, and the theory of definite rays transmitting the disturbance from the various elements of S seems therefore to be in danger of failure, which ought to be avoidable.

The proposition now to be advanced is that only one of these resolutions is physically correct. Indeed when we consider the matter, ample reason appears for the problem being definite. A state of stress and strain is continually transmitted up to the surface S from the actual sources inside, and we are to find a distribution of secondary sources that will send it on to the outside as it arrives, without sending anything back. For the sending of a disturbance back into the interior would be an alteration of the physical circumstances, would in fact add to and confuse the effect of the assigned true sources inside. Hence the correct secondary sources are the distribution which produces the effect of the true sources on the side beyond them, but produces no backward effect on their own side. The distribution of such sources was determined in the previous paper by an immediate process indicated above; and this now puts in evidence both its possibility and its uniqueness.

Again it will suffice to illustrate from the case of sound waves which depend on only one potential ϕ of scalar type. The value of ϕ outside due to the given true sources inside and no sources outside is as before

$$\phi = \frac{1}{4\pi}\int \frac{1}{r}\left(\frac{\partial\phi}{\partial n}\right)_0 dS - \frac{1}{4\pi}\int \frac{\partial}{\partial n}\left(\frac{\phi_0}{r}\right) dS,$$

in which under the integrals ϕ_0 and $(\partial\phi/\partial n)_0$ denote the values at the surface-element δS at time $t-r/c$: cf. *loc. cit.*, Vol. 1, p. 7. Now

$$\frac{\partial}{\partial n}(\phi_0) = -\frac{\cos(rn)}{c}\left(\frac{\partial\phi}{\partial t}\right)_0$$

and *if the surface S be a wave-front*

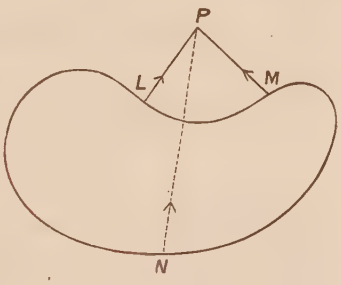
$$\left(\frac{\partial\phi}{\partial n}\right)_0 = -\frac{1}{c}\left(\frac{\partial\phi}{\partial t}\right)_0.$$

Thus, effecting the differentiation,

$$\phi = \frac{1}{4\pi}\int \frac{1+\cos(rn)}{r}\left(\frac{\partial\phi}{\partial n}\right)_0 dS + \frac{1}{4\pi}\int \frac{\cos(rn)}{r^2}\,\phi_0 dS.$$

The latter term involving r^{-2} is merely a local alternation or fluctuation of disturbance; so that it is the former alone that expresses the acoustic radiation transmitted from each secondary source at an element δS. In the forward direction along δn it is $\frac{2}{4\pi r}\left(\frac{\partial \phi}{\partial n}\right)_0 \delta S$, whence it continuously falls off in sideway directions until in the backward direction it has become null. When the surface S is a wave front, an actual ray is thus propagated normally from the group of adjacent local elements δS in the forward direction, but none in the backward direction. For in dealing with radiation as such, it is implied that it is estimated far enough from the source for the local fluctuating field represented by the other integral term in the formula to be ignored. It is to be remembered also that in this mathematical analysis the surface S is merely a geometrical boundary in the continuous medium, and does not at all act as an obstacle; each secondary source on an element δS of it is to be regarded as radiating freely in all directions without any relation to the sources on the other elements.

We still consider a surface S which is a wave front: and it is to be noted that this usually implies a single point source, though with its radiation modified by transit through different media, as in the familiar case of optics. The disturbance that reaches an external point P will travel to it along definite paths which are those of quickest propagation across the region between P and the surface. In a uniform medium they will be straight paths, in fact the normals to the surface, shown as PL, PM, PN in the diagram: it is only the elements of disturbance crossing S near L, M and N that can sensibly affect P. But when such secondary sources on S are made determinate by the condition that they produce no backward radiation, the contribution starting from N in the direction along the normal drawn inward ought to be obliterated, and that is in fact secured by the formula above developed: therefore NP is not a physical ray, for no energy travels from N along it. The rays are restricted to be the outward-directed paths of shortest time, or more strictly of quickest propagation. The difficulty of Fresnel's early theory of diffraction, as to why a ray is not also propagated backward, is thus surmounted.



[*July* 10*th*, 1920.—If, more generally, the surface S over which the

emerging disturbance is dissected is not a wave-front, let ν be the direction of the ray at the element δS and (νn) its inclination to the normal. Then

$$\left(\frac{\partial \phi}{\partial n}\right)_0 = \cos(\nu n)\left(\frac{\partial \phi}{\partial \nu}\right)_0 = -\frac{\cos(\nu n)}{c}\left(\frac{\partial \phi}{\partial t}\right)_0,$$

$$\frac{\partial}{\partial n}(\phi_0) = \frac{\cos(rn)}{c}\left(\frac{\partial \phi}{\partial t}\right)_0,$$

and the formula becomes

$$\phi = \frac{1}{4\pi}\int \frac{\cos(rn)-\cos(\nu n)}{cr}\left(\frac{\partial \phi}{\partial t}\right)_0 dS + \frac{1}{4\pi}\int \frac{\cos(rn)}{r^2}\phi_0 dS.$$

in which the second term is as before purely local. In this form it applies to any coherent system of rays. The case of a simple pulsating point-source of sound is interesting: for it, r_1 being the distance of this source from δS,

$$\phi = \frac{A}{r_1}\cos\left(nt - \frac{r_1}{c}\right),$$

thus the transmitted disturbance can be expressed* as

$$\phi = -\int \frac{An}{4\pi c}\,\frac{\cos(rn)-\cos(r_1 n)}{rr_1}\sin\left(nt - \frac{r_1+r}{c}\right) dS,$$

which puts in evidence the factor of attenuation $(rr_1)^{-1}$ and the phase depending on the path r_1+r_2, and shows that it is only the elements δS that lie near the line joining the source to any point which produce the disturbance at that point.]

But now consider the more general and indeterminate distribution of secondary sources, in which they are derived as representing the actual sources inside S together with any assumed fictitious sources outside. The presence of the latter does not affect the validity of the new system of secondary sources, provided it is taken as a whole, as regards outside space: but that system also sends disturbance into the inside space, rays thus travel and energy is propagated backwards as well as outwards. If

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* Cf. Kirchhoff, *Wied. Ann.*, Vol. 18, 1883, formula (13).

the surface S happened to continue to be a wave front after the fictitious external sources are superposed NP would be a ray; and LP, MP would continue to be rays, but of altered intensities such however that the aggregate disturbance at P would remain the same. Usually S would not remain a wave front, and the rays passing between it and P would start from it not normally: it is only under very special circumstances that disturbances from different sources could consolidate into one system of rays at all. Anyhow this somewhat artificial illustration suffices to enforce the point, that all these mathematically equivalent distributions of secondary sources are fictitious except the one that happens to have been hit upon originally for this case of pressural waves involving a potential, by Kirchhoff. For the disturbance at each place outside would remain the same whatever fictitious external distribution of sources is superposed, but the paths by which the energy travels across the medium to that place would be different: as a ray is the definite path of actual transfer of energy, this illustration in terms of rays is specially graphic, indeed is of the essence of the matter. The corresponding general formulæ for optical and electrical radiation can now be derived on the lines expounded in the previous paper: for example the expressions for optical radiation may be developed from § 5. Enough has been said to illustrate the principles that are involved.

The Law of Oblique Diffraction of Transverse Waves.

The expression for the electric or optical radiation diffracted from an element of wave front δS, which strangely does not seem hitherto to have been determined at all, is perhaps obtained most simply by taking the element at the origin in the plane xy, so that the electric force F is along x and the magnetic force (equal to F/c in electrodynamic units) is along y, the axis of z being thus the normal to the front. We have to determine the disturbance propagated in the direction of the radius vector r making an angle θ with the normal to the front and such that the plane of θ makes an angle ψ with the direction (x) of the electric force in the front. It is easier, and also is a variation on the procedure of the previous paper, to resolve the equivalent electric and magnetic oscillators on δS into linear components in the directions of the polar elements δr, $r\,\delta\theta$, $r\sin\theta\,.\,\delta\psi$. When we attend only to the terms of lowest order in r^{-1}, representing true radiation, the first component produces no effect in the direction of r. The effects of the others involve only the amplitudes of radiation of a

linear magnetic vibrator $f'/4\pi$ and a linear electric vibrator $\phi'/4\pi$ (in the previous notation) in directions transverse to their lengths: the former produces in its equatorial plane electric amplitude $f'/4\pi r$ parallel to the element, and magnetic amplitude $f'/4\pi cr$ at right angles, the latter produces magnetic amplitude $\phi'/4\pi c^2 r$ parallel to its direction and electric amplitude $-\phi'/4\pi cr$ at right angles, in each case the argument of the functions f' and ϕ' being $t-r/c$.

By combining the contributions from all the components of the equivalent sources on δS the result comes out in the comparatively simple form

$$\text{electric amplitude} \quad \begin{matrix}\text{along } r\delta\theta \\ \text{along } r\sin\theta.\delta\psi\end{matrix} \quad \begin{matrix}\cos\psi \\ \sin\psi\end{matrix} (1+\cos\theta)\frac{\delta S}{4\pi r} f'\left(t-\frac{r}{c}\right),$$

$$\text{magnetic amplitude} \quad \begin{matrix}\text{along } r\delta\theta \\ \text{along } r\sin\theta.\delta\psi\end{matrix} \quad \begin{matrix}\sin\psi \\ -\cos\psi\end{matrix} (1+\cos\theta)\frac{\delta S}{4\pi cr} f'\left(t-\frac{r}{c}\right),$$

in which $f(t)$ is the electric amplitude on the original wave front of radiation on δS, which is in the direction of ψ null, and $cf(t)$ is the magnetic amplitude in the direction on the original wave front at right angles.

The resultant electric and magnetic disturbances along any direction r are at right angles, as they ought to be in radiation. The ray diffracted from δS along r is thus plane polarized of electric amplitude $(1+\cos\theta)\frac{\delta S}{4\pi r} f'\left(t-\frac{r}{c}\right)$ with the plane of its electric vibration inclined to the plane through the normal to δS at an angle equal to the azimuth of r around the normal, the sign of this angle being conveniently recovered by considering directions of r nearly normal to δS for which the electric vibration must be nearly parallel to x. From this formula, applied for directions near the normal, the doctrine of rays may be developed on the usual lines.

For ordinary light the result is the same as for equal beams polarized in directions at right angles: in the intensity the azimuth ψ thus disappears, and the obliquity factor $1+\cos\theta$ in the amplitude alone survives squared for the intensity of unpolarized diffracted light.

The resultant ray as integrated from all the elements such as δS is normal to the front as usual: and the factor $1+\cos\theta$ ensures, as in the previous case of waves of pressure, that while energy is propagated forward in the ray-direction of θ null, no energy travels backward in the other formally possible ray-direction specified by θ equal to π.

In the direction of the actual ray the intensity of the electric force propagated from an element of surface is $f'/2\pi r$ parallel to x, and of the magnetic force $f'/2\pi cr$ parallel to y, in which the argument of f' is $t-r/c$. Integrated over the wave front for an alternating train, the result should be f and f/c: as may be verified by the usual procedure of Fresnel.

[*July 10th*, 1920.—It should not be forgotten that the diffraction of elastic waves of general type was treated very clearly and completely by Stokes, with results virtually equivalent to those in the text, including the crucial factor $1+\cos(rn)$, as early as 1849 in Part I, Sec. iii (cf. § 33) of the classical memoir "On the Dynamical Theory of Diffraction," *Camb. Phil. Trans.*, reprinted in *Math. and Phys. Papers*, Vol. 2. As however he was chiefly concerned with the direction of polarization of the diffracted light, the formulæ were not developed. The disturbance was taken as transmitted by traction, not displacement, of the medium, as was natural: cf. *supra*, p. 171. But it has not been sufficiently noted that his analysis completely evades any discussion of the mode of transmission. Thus (§ 32) "the disturbance transmitted during the interval τ ... occupies a film of the medium, of thickness $b\tau$, and consists of a displacement $f(bt')$ and a velocity $bf'(bt')$": and the procedure is to deduce from the general formulæ applicable to an unlimited unrestricted medium the way in which this disturbance in these local elements of volume will spread. There is thus no room for any ambiguity in Stokes' mode of analysis, and an appeal to the mode of propagation of energy is not necessary: though he appears to have overlooked (or rather was not concerned with) the result that his formula allows no backward propagation of disturbance into the region within the surface on which the disturbance is thus dissected.

In § 36 he, however, appears to state that for common light the obliquity factor for intensity would involve $1+\cos^2\theta$ instead of $(1+\cos\theta)^2$: this would violate the canon last mentioned, and indeed is not consonant with modern ideas of the nature of ordinary light.

The law of polarization which has emerged above does not seem to agree with the famous law of Stokes, the subject of so much experiment more or less relevant by him and others, namely that the plane of (magnetic) vibration of the diffracted disturbance is parallel to the vibration of the original beam. It would not be surprising that electrodynamic secondary sources should thus give a different result from secondary sources in a mechanical elastic medium, as is in fact put in evidence in the different laws for reflection of waves in the two cases. The argument of Stokes from symmetry, which is so effective for diffraction from

particles and the polarization of the blue of the sky, thus does not here seem to apply because the magnetic vibration is as important as the electric, as in p. 11 of the earlier paper. But further scrutiny of the discrepancy must be deferred. On this question, and on the ambiguities resulting from modes of resolution that have produced embarrassment, cf. Lord Rayleigh's summary, in "Wave Theory," *Ency. Brit.* or *Scientific Papers*, Vol. 3, p. 165.]

Curved Wave Fronts.

We have considered the oblique diffraction from elements δS of a plane wave-front. Curvature, if its radius is very great compared with the wave-length, may be presumed not to sensibly affect the result. To examine this question, we may try to express analytically the train of coherent radiation advancing from a front of finite curvatures.

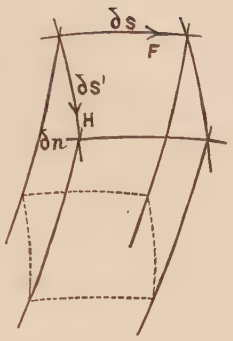
Indeed before any general formulæ for diffraction can be practically applicable, before we can pass for optical systems of rays much beyond the elementary mode of arguments of Huygens and Fresnel confined to directions near the ray, it would be necessary to have an electrodynamic specification of the general optical train on which the formulæ are to operate. In the absence of knowledge of any previous attempt in this direction, the following sketch is offered for scrutiny and expansion.

The criterion of a coherent train is that there must be a system of wave fronts or surfaces of equal phase. The rays, or paths of the energy, are for an isotropic medium the curves intersecting this system of surfaces orthogonally. If the rays are curved the medium must be of varying density: thus the velocity of propagation c' will be a function of position in the medium, unless the rays are in straight lines.

Let us divide one of the wave fronts into elementary curvilinear rectangles by drawing on it the two sets of lines of curvature. The geometrical rays that issue normally to it from each line of curvature will form a sheet or surface. The two sets of surfaces thus obtained, and the set of wave fronts, form a triply orthogonal system: therefore by the theorem of Dupin, all their curves of intersection are lines of curvature on the surfaces on which they lie.

Let us consider a train of waves such that at one of the fronts the electric force F is directed along one of the sets of lines of curvature, of element δs, and the magnetic force H along the other set, of element $\delta s'$.

Then the circuital relations of Ampère and Faraday, when applied around the faces of the curvilinear element of volume $\delta s\,\delta s'\,\delta n$, give (with electromagnetic units) six equations of types



$$\delta n \frac{\partial}{\partial n}(H\delta s') = -\frac{\partial}{\partial t}\left(\frac{F}{c'^2}\right)\delta s'\,\delta n,$$

$$\delta n \frac{\partial}{\partial n}(F\delta s) = -\frac{\partial H}{\partial t}\delta s\,\delta n,$$

together with the relations that $F\delta s$ and $H\delta s'$ remain constant, but only up to the first order and so only for a short range, in passing to the opposite sides of the rectangle $\delta s\,\delta s'$,—together also with the relations that F and H are along the directions of the lines of curvature on the consecutive wave front (in the recognized shorthand infinitesimal sense of that term) indicated by the dotted rectangle in the diagram. If this be allowed, the invariance of $F\delta s$ and of $H\delta s'$ along the respective lines of curvature determines the distribution of F and H as regards intensity, over any one selected wave front, the directions however being fixed. Then on the other wave fronts their directions will also be along the lines of curvature, and their intensities will be determined by the equations first written. These equations take the form, if R, R' are the principal radii of curvature of the front,

$$\frac{\partial H}{\partial n} - \frac{H}{R'} = -\frac{1}{c'^2}\frac{\partial F}{\partial t}, \qquad \frac{\partial F}{\partial n} - \frac{F}{R} = -\frac{\partial H}{\partial t};$$

we derive $$\frac{\partial}{\partial n}FH - \left(\frac{1}{R}+\frac{1}{R'}\right)FH = -\tfrac{1}{2}\frac{\partial}{\partial t}\left(\frac{1}{c'^2}F^2+H^2\right).$$

This affords a verification; for it expresses, on Poynting's principle, that the integrated flux of energy (equal in intensity to $FH/4\pi$) out of the element of volume is the rate of diminution of the energy inside it. It expresses the law of intensity along the ray.

On elimination, F and H are determined separately by the two equations

$$\left(\frac{\partial}{\partial n}-\frac{1}{R'}\right)\left(\frac{\partial}{\partial n}-\frac{1}{R}\right)F = \frac{1}{c'^2}\frac{\partial^2 F}{\partial t^2}, \qquad \left(\frac{\partial}{\partial n}-\frac{1}{R}\right)\left(\frac{\partial}{\partial n}-\frac{1}{R'}\right)H = \frac{1}{c'^2}\frac{\partial^2 H}{\partial t^2},$$

which are different as regards the second order, such as has been

neglected. Radiation of general type may be replaced by a beam polarized along δs, together with an equal one polarized along $\delta s'$. If the wave fronts are plane, this is the usual equation of simple transmission of disturbance. If c' is uniform, as for a curved train in free space, $\partial R/\partial n$ and $\partial R/\partial n'$ are each -1, when δn is measured inward as here.

The practical important case in which an exact specification for curved fronts is available is that of the radiation from a simple bipolar (or multipolar) source: for from that case of spherical fronts all other systems of rays may be constructed by superposition.

A NEW THEORY OF MEASUREMENT: A STUDY IN THE LOGIC OF MATHEMATICS

By NORBERT WIENER.

[Read November 13th, 1919.]

Introduction.

It is a deeply rooted popular idea that mathematics is but another name for measurement. Notwithstanding the fact that the existence of such non-metrical branches of mathematics as projective and descriptive geometry, the theory of groups, the algebra of logic, &c., prove this notion false, it is nevertheless true that the applications of mathematics have, up to the present time, been, almost without exception, applications of measurement. The natural sciences, in so far as they have been regarded as at all amenable to a mathematical treatment, have reduced themselves to the correlation of different ranges of measurement—of space, time, and mass, in the case of physics, of intensity of stimulus and intensity of sensory experience, in the case of psychophysics, and so on. Now, things do not, in general, run around with their measures stamped on them like the capacity of a freight-car: it requires a certain amount of investigation to discover what their measures are. It is, then, a necessary preliminary to the most complete scientific work that we should possess an analysis of the process through which we go in measuring the magnitude of a thing.

Now, a very beautiful theory of measurement has been developed in the third volume of the *Principia Mathematica* of A. N. Whitehead and B. Russell. It depends on the consideration of "vector-families"; that is, of sets of vectors such that it is possible to go from any point of the field ordered by the vectors to one and only one point along any given vector. These vectors correspond to definite increments of the measures of the terms in the common field of the vectors: thus, in the "vector-family" representing distances to the right along a given line, from a given point, an increment of the distance of this given point from a point to the extent of one inch, or of two inches, &c., is a vector. Since it is always possible to leave a point by any vector one pleases, our system of measurement

must contain magnitudes larger by any desired amount than any given magnitude. As a consequence, measurement by "vector-families" breaks down when we have to deal with ranges of quantities that are essentially limited. As the authors of the *Principia* say of their theory, "We exclude magnitudes which have a definite maximum, unless they are circular, like the angles at a point, or the distances on an elliptic straight line."* The authors of the *Principia* justify this omission by the further statement, "But, except when they are circular, such magnitudes are of little importance." This is, in general, true, but there is one exception which is well worth considering. Perhaps the least satisfactory and most discussed portion of the unsatisfactory and much discussed theory of measurement is that which deals with the measurement of the intensities and qualities of sense-data.† Now, the intensities and qualities of sense-data are not susceptible to increments of arbitrary magnitude. There is no degree of loudness as much greater than that of a foghorn at close range as the loudness of this is greater than that of the ticking of a watch at the distance of ten feet. There is no note as much higher than that of the cricket as the latter is higher than the lowing of an ox. There is no object as much more intensely red than a drop of blood in the sunlight as the latter is than a piece of grey flannel. Nevertheless, we often do speak of the measure of the loudness or pitch of a note or the intensity of a colour, and if Weber's law is to have any meaning in any other form than that in which it refers to "just noticeable" differences in intensity or quality,‡ we must be able to establish some intrinsic criterion whereby we can determine the ratio of one difference in sensation quality or intensity to another, which will not presuppose that such an interval can be increased by any numerical factor whatever.

The first steps which are essential to the measurement of sensory qualities and intensities are the determination of the fundamental experience by means of which this measurement is to be performed, and the derivation from this crude, uncouth experience of functions which will have certain comparatively neat properties, and which hence will form a mere convenient starting point for the process of measurement proper. Our measurement of sensation-intensities obviously has its origin in the consideration of intensity-intervals between sensations: that is sufficiently

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* Vol. 3, p. 340, lines 1–4.

† Except in one respect, which we shall indicate later, the theories of qualities and intensities coincide.

‡ Cf. "Studies in Synthetic Logic," *Proc. Camb. Phil. Soc.*, Vol. 18, Part 1, pp. 24–28, §§ 7, 8, by Norbert Wiener.

indicated by the fact that our measurement of a sensation-intensity always reduces itself, sooner or later, to the determination of its ratio to some standard intensity, while such a proposition as "x is twice as intense as y," is simply a paraphrase for some such statement as "The interval of intensity between x and y equals that between y and some sensation of zero intensity." The fundamental experience for which we are searching is not, however, of the form "The interval between x and y equals that between u and v." Not every two intervals which seem equal are equal: two intervals may only be subliminally different. Nevertheless, our only direct method of determining with what intervals an interval is equal lies, as we shall show, in the determination, first, of with what intervals it *seems* equal. But even the seeming equality of two intervals is not quite what we want: two intervals seem equal, as far as we are concerned, when and only when neither seems greater than the other. We shall take, then, as the relation which forms the basis of the measurement of sensation-intensities, "The interval between x and y seems less than that between u and v," where intervals are regarded as possessing signs, ascending intervals being regarded as positive, and "less than" is to be interpreted as "*algebraically* less than." In taking this as our primitive experience, we do not mean to assert—in fact we should categorically deny—that this relation is given as such in our experience, and that no further analysis of it is possible: what we *do* assert is that it represents a much more minute analysis of the basis of our measurements of sensation-intensities than any yet given, and forms a convenient starting point for a theory of sensation-intensities.

Although our initial relation enables us to give a sort of order to all sensation-intervals which seem greater or less than other intervals in some definite way (*i.e.* as loudness-intervals or as brightness-intervals, &c.), we shall, in our subsequent work, limit its range of application to positive supraliminal intervals, or intervals which seem greater than some interval between a thing and itself. We do this, because zero-intervals have certain properties which interfere with our subsequent theory. Later on, we shall use as our criterion of the genuine equality of two intervals the fact that all the intervals which are indistinguishable from (*i.e.* neither noticeably greater than nor noticeably less than) either are indistinguishable from the other. It will be possible, therefore, for us to say that two intervals may be indistinguishable in magnitude, yet that one is greater than the other. Let the interval between x and y and the interval between x and z be a pair of intervals of this sort. Then y must be indistinguishable from z by direct comparison, yet we must be able to say that y and z are not really of the same intensity, and hence that the interval

between y and z, though indistinguishable from the zero-interval, is not of measure zero, or else it will not seem natural to call the interval between x and y of really different size than that between x and z. Now, it appears that an interval seems either to be one of difference or of identity. If it seems to be one of difference, it is indistinguishable from intervals of difference, and from those only, while all intervals of identity seem of the same magnitude. Hence, if our criterion of the genuine equality of two intervals is that all the intervals which seem identical with either seem identical with the other, if an interval be subliminal, it is genuinely identical in magnitude with the zero-interval. We thus obtain the result that the interval between y and z, whose difference from zero we wish to secure, is genuinely identical in magnitude with a zero-interval. To avoid this, we limit our discussion at first to positive supraliminal intervals.

As we said above, we regard two intervals as genuinely equal* when and only when all the intervals which are indistinguishable from (*i.e.* seem neither greater nor less than) either are indistinguishable from the other. Genuine equality is, as may readily be seen, a reflexive, symmetrical and transitive relation. It is possible, therefore, to group all positive supraliminal intervals into naturally exclusive sets, such that no member of any set is genuinely equal to a member of another set, and every member of a set is genuinely equal to every member of the set. The relation between two terms which consists in their being separated by some interval belonging to one of these sets we shall call the *vector* associated with the set. We are able, then, to regard any positive supraliminal interval as an instance of one of these vectors. We can next define the vector corresponding to a subliminal interval as follows: since all intervals representing a vector are equal, we should naturally regard the difference of an interval belonging to one vector from an interval belonging to another vector as independent with respect to its magnitude of the particular intervals chosen from these vectors. Now, the difference of two intervals may be readily defined when their upper ends coincide as the interval from the lower end of the interval from which the subtraction is to be made to the lower end of the subtracted interval. Furthermore, every subliminal interval, if its upper end does not lie in the neighbourhood of the maximum possible magnitude of sensations of the appropriate sort, may be regarded in the above manner as the difference of two supraliminal intervals with coincident upper ends. In the neighbourhood of the maximum possible magnitude of sensations of the appropriate sort, every subliminal interval may be regarded as the difference in an analogous

* Cf. "Studies in Synthetic Logic."

sense of two intervals with coincident *lower* ends. It will be found, then, that the vector corresponding to a subliminal interval may be defined in terms of two supraliminal vectors whose difference it forms, as the relation between two terms when one either first ascends an interval belonging to one, and from that point descends an interval belonging to the other, or first descends an interval belonging to the second, and then ascends an interval belonging to the first. Now, we wish to confine our discussion to vectors made up of positive intervals. To this end, it is necessary that the subtracted vector should be smaller than that from which it is to be subtracted. We need next, therefore, a criterion of the relative magnitudes of two vectors.

In looking for such a criterion, we shall suppose it to be axiomatic that in any intrinsic comparison of differences between intervals, a subliminal or unnoticeable difference is always to be treated as less than a noticeable one. This being the case, if an interval be indistinguishable from some interval noticeably greater than another, it is necessarily greater than the other one—for, if the interval R be indistinguishable from the interval S, which is noticeably greater than T, either R is subliminally greater than S, or equal to it, or only subliminally less than it. In the first two cases, the naturalness of supposing R greater than T is obvious at once; in the latter case, the above principle will render it obvious. Another similar condition which determines that R is greater than T is that there should be an interval S, indistinguishable from T, but noticeably less than R. It is logically demonstrable that one of these two criteria determines that an interval R is greater than an interval T, or else that T is greater than R when, and only when, R and T are not genuinely equal, in accordance with our former definition. We may *define* R as greater than T, then, when one of the two conditions just stated is satisfied.

We have now a complete definition of the class of intensity-vectors belonging to any given range of sensations. In the next section of this paper, we shall cover the same ground we have just been covering in a stricter manner, with the aid of the symbolism of the *Principia Mathematica*. If $\phi(x, y, u, v)$ stand for, "The difference between x and y in a given respect seems less than that between u and v," then we shall use Id_ϕ to stand for the relation of genuine equality in the appropriate set between positive supraliminal intervals;* $\dot{s}$"D'$\overrightarrow{\mathrm{Id}}_\phi$ will be the class of

* We shall regard an interval as simply the ordered couple formed by its upper and lower extremity.

positive supraliminal vectors: Dc_ϕ stands for the relation, "less than," among positive supraliminal intervals, and Vc_ϕ is the class of all intensity-vectors. Vs_ϕ, which we shall later define in terms of Vc_ϕ is a class of relations which will form part of Vc_ϕ when ϕ has the particular kind of value just attributed to it, but which, unlike Vc_ϕ, will always consist of mutually exclusive vectors—*i.e.* of vectors such that no two distinct ones have a common beginning from which they reach to a common end—whatever properties ϕ may have.

We are now in a position to consider the problem of measurement itself. To this end, we define two vectors, R and S, as having the ratio μ/ν in a class of vectors κ (which in the case of the measurement of sensory-intensities, will be the Vs_ϕ just considered) if there is a vector T belonging to κ such that if we start from a member of the field of T and take successively μ steps belonging to T, we sometimes take one step belonging to R, while similarly, ν successive steps belonging to T sometimes cover the the same ground as one step belonging to S. This method of defining the ratio of two vectors in terms of a common submultiple, instead of in terms of a common multiple, as in the *Principia*, is chosen for the reason that we do not wish, for example, the existence of a loudness $\frac{9,999,999,999}{10,000,000,000}$ as great as that of the falling of a pin to depend on that of a loudness 9,999,999,999 times as great as that of the falling of a pin.

Each vector will bear various ratios to other vectors: it will be ten times this vector, twice that, half the other, and so on. We have seen, however, that in the case of sensation-intensities, no non-zero interval can be multiplied by an arbitrarily great numerical factor. The class of ratios, that is, which a given vector bears to other vectors, has, in general, a lower limit or minimum in the scale of real numbers which will be distinct from zero. In case there is a maximum vector to which the vector to be measured bears a ratio, this ratio will be the minimum of the ratios which the vector to be measured bears to other vectors, but, in general, a vector will not bear to any vector a ratio which is the minimum or lower limit of the ratios which it bears to other vectors, or, as we shall call it, its index. This index we shall represent by Ind_κ 'R, where R is the vector to be measured, and κ the class of vectors in terms of which the measurement takes place. The index of a vector may roughly be taken to represent its measure in terms of the greatest possible vector of the set, but a vector can be measured by its index whether such a greatest possible vector exist or not. Since the series of real numbers is Dedekindian—*i.e.* since, whenever we divide it into two classes, so that every term of the one is greater than every term of the other, either the former

class will have a minimum, or the second a maximum—every vector will have an index, and it is easy to show that no vector can have more than one index.

We have now found a way to measure such things as sensation-intervals; our next task is to discover a way to measure such things as sensations. Now, the natural measure of a sensation is the index of an interval stretching to it from a sensation of zero intensity. There may not be, however, a sensation of zero intensity. In such a case, the natural thing to do would seem to be to approximate to what the interval between this non-existent sense-datum of zero intensity and the given sensation would be, supposing we were wrong in judging data of zero intensity not to exist, by taking successively less and less intense sensations, measuring the intervals between these and the given sensation, and taking the upper limit or maximum of the values so obtained. Now, a sensation is less intense than a given sensation when the interval from it to the given sensation is positive, and the degree of their difference in intensity is measured by the index of their interval, so the natural measure of the intensity of a given sensation is the upper limit of the indices of intervals having it as their upper boundary. This we shall represent by $\text{Meas}_\kappa{}'x$, where x is the sensation to be measured, and κ is the class of vectors by which it is to be measured. It should be noted that in *any* system of measurement bounded at both ends, Meas_κ will enable us to measure the position of any given term—it is not confined in its application to sensation-intensities. It should also be noted that the measure of any term in an unbounded system of measurement is zero, as every vector can be repeated an infinite number of times, and hence the lower limit of the ratios it bears to other vectors is zero. Therefore, since the measure of a term is the limit of a set of indices of vectors, it also must be zero. In any system each term will have one, and only one measure, but the same measure may belong to different terms. In such a case, if κ is of the form Vc_ϕ or Vs_ϕ, where ϕ is the relation between x, y, u, and v, when the intensity-interval between x and y seems algebraically less than that between u and v, we shall say that two different terms having the same measure are of the same intensity, and that a sensation-intensity is the class of all terms having some given measure. The measure of an intensity is the measure of its terms. The relation between an intensity and its measure is one-one.

It will not be a necessary consequence of the definition of the measure of a vector that the measure of a vector containing an interval formed by taking steps belonging to two vectors R and S, successively will be the sum of the measures of R and S. We can easily obtain a new definition

of the measure of a vector, however, which will always have this desired property. This requires first the formation of a new class of vectors, which we shall term Reg_κ, as a function of the class of vectors κ with which we started. The measure of one of these, say R, will be designated $\overset{\smile}{\mathrm{Dist}}_\kappa$ 'R. We shall also define μ_κ as the relation between a member of Reg_κ and an μ-th submultiple of it (μ being any real number). We shall prove that it follows from our definitions that any vector which is the μ-th multiple of a ν-th multiple of another vector is the $(\mu.\nu)$-th multiple of the second vector. The details of the definitions of these various notions are, however, of no special interest to the general reader, so I shall reserve them for the technically logistical portion of this paper.

Finally, we shall show how it is possible to remove the limitation of our system of measurement to systems of measurement with definite maxima or upper limits, and at the same time construct a method of measurement of sensation-qualities and intensities which is in certain cases more natural than that just given. Since all ranges of sensation-qualities and intensities are bounded above, it is always possible to make the "maximum possible interval" our standard of measurement, and in the case of certain sensation-*qualities*, such as chroma, this seems the most natural standard to take—for example, we say that this patch of colour is of the highest possible degree of saturation, that one is only half saturated, and so on. In the case of most ranges of sensation-intensity, such as the scale of loudnesses, such a method of measurement seems highly artificial. It does not seem natural to measure the ticking of a watch in millionths of a boiler-factory-power. The interval which most psychologists have taken as a standard in such instances as these is the just-noticeable interval. We do not know, however, that all just-noticeable intervals are equal, nor yet that there are any just-noticeable intervals. We shall choose the first just-noticeable interval as our standard of measurement, and we shall avoid the assumption that there is some single definite just-noticeable interval, just as we avoided the assumption that there was any single, definite, greatest possible interval, by making the class of intervals which are less than just-noticeable—that is, are subliminal—our real standard of measurement. Now, all intervals which relate sense-data not noticeably more intense than any other sense data* are subliminal. We have now on our hands, therefore, the task of finding a way to measure terms of the fields of a class of vectors in terms of a cer-

* It may be necessary in some cases to exclude sensations of zero intensity from consideration here and elsewhere for reasons analogous to those which previously led us to defer our consideration of subliminal intervals till after supraliminal intervals had been discussed.

tain portion of these fields—namely, in the case of sense-intensities, in terms of the class of subliminal sense-data.

Now, we can use the system of measurement already developed to find the indices of all those portions of vectors which relate subliminal data to subliminal data, or, in general, data of a given portion of the fields of the class of vectors to data of the given portion of the fields of the class of vectors, in terms of the class of all such vectors. Let us regard the index of such a portion of the vector as a property of the whole vector, and not merely of the part for which it is primarily defined. Let us measure any vector R of the original set by finding some μ-th part of it to which an index has already been given by the method just indicated, and associating with R a quantity μ times the value of this index. This quantity will, in general, depend on μ. Let us call the relation of any such quantity to R, $\mathrm{Inx}_{\kappa, \alpha}$, where κ is the class of vectors originally taken, and α is the class of members of the fields of these vectors in terms of which all the members of the fields of these vectors are to be measured. Just as we previously defined the measure of a term to be the maximum or upper limit of the indices of intervals leading up to it, so we now define the measure of a term in terms of κ and α as the maximum or upper limit of the values of the $\mathrm{Inx}_{\kappa, \alpha}$'s of intervals leading up to it. This we shall call the $\mathrm{Meas}_{\kappa, \alpha}$ of the term in question. From this point on, the development of this theory of measurement runs precisely parallel to that of our previous theory.

At the end of this paper, we shall give a proof that under certain conditions which we there state, our method of measurement gives substantially the same results as that of the *Principia*. This portion of the paper is of a merely technical interest, in that it correlates this work with what has previously been done on the subject. It had better be omitted on a first reading of this paper.

1. We shall now cover the same ground we have already covered in the introduction in a strictly rigorous and logical manner, with the help of the symbolism of the *Principia Mathematica*. As we saw in the introduction, an experience which, as far as we are concerned, may be regarded as lying at the foundation of all our measurements of, for example, brightnesses, is, "The interval between x and y, considered with reference to their brightness, seems algebraically less than that between u and v," where all ascending supraliminal intervals—*i.e.* intervals between a sense-datum and another sense-datum of noticeably greater intensity, are regarded as positive. Let us call the above proposition $\phi(x, y, u, v)$. The thing that we should normally call the interval between x and y is the

ordinal couple $x \downarrow y$. Now, we shall have occasion to regard ϕ as a dyadic relation between intervals rather than as a tetradic relation among brightness-sensations. On this account we shall make the following definition :

$$(1) \quad \mathrm{Cp}_\phi = \hat{\mathrm{R}}\hat{\mathrm{S}}\{(\exists x, y, u, v).\mathrm{R} = x \downarrow y.\mathrm{S} = u \downarrow v.\phi(x, y, u, v)\} \quad \mathrm{Df}$$

If ϕ is the relation mentioned above, Cp_ϕ is the relation between a given brightness-interval and a brightness-interval which seems greater than it—*algebraically* greater, that is. We wish, however, to limit ourselves to the discussion of positive intervals. We obtain this result as follows : we first limit Cp_ϕ to positive *supraliminal* intervals—that is, to intervals which bear the relation $\mathrm{Cnv}\,\text{`}\,\mathrm{Cp}_\phi$ to some interval of the form $x \downarrow x$. Then we form from this relation the one defined by means of the following definitions as Id_ϕ

$$(2)^* \qquad \mathrm{P}_{se} = (\dot{-}\,\mathrm{P} \dot{-} \breve{\mathrm{P}}) \restriction \mathrm{C}\text{`}\mathrm{P} \qquad \mathrm{Df}$$

$$(3)^* \qquad \mathrm{P}_s = (\overrightarrow{\breve{\mathrm{P}}_{se}} \mid \overrightarrow{\mathrm{P}_{se}}) \restriction \mathrm{C}\text{`}\mathrm{P} \qquad \mathrm{Df}$$

$$(4) \qquad \mathrm{Id}_\phi = \{\mathrm{Cp}_\phi \restriction \hat{\mathrm{R}}[(\exists x).(x \downarrow x)\,\mathrm{Cp}_\phi\,\mathrm{R}]\}_s \qquad \mathrm{Df}$$

Id_ϕ is, then, the relation between two supraliminal positive brightness-intervals when they agree in every respect when they are compared with other brightness-intervals in respect to their magnitude. $\mathrm{D}\text{`}\overrightarrow{\mathrm{Id}}_\phi$ will, therefore, be the result of sorting out the class of all positive supraliminal brightness-intervals into classes each containing all the intervals of a given magnitude. $\dot{s}\text{``}\mathrm{D}\text{`}\overrightarrow{\mathrm{Id}}_\phi$ will be the class of all positive supraliminal vectors—of all relations, that is, which connect pairs of sensations such that the brightness-interval between the members of such a pair is of a given fixed size for each member of $\dot{s}\text{``}\mathrm{D}\text{`}\overrightarrow{\mathrm{Id}}_\phi$ chosen, and is positive and supraliminal.

Now, one brightness-interval is less than another if it is either noticeably less than some brightness-interval indistinguishable from (*i.e.* neither noticeably greater than nor noticeably less than) the other, or is

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* Cf. the discussion of indistinguishability and genuine identity in the introduction and *Studies in Synthetic Logic.*

indistinguishable from something noticeably less than the other.* Let us define this relation, as it applies to supraliminal positive brightness-intervals, as follows:—

$$(5) \qquad P_{dc} = P \mid P_{se} \cup P_{se} \mid P \qquad \text{Df}$$

$$(6) \qquad Dc_\phi = \{Cp_\phi \upharpoonright \hat{R}[(\exists x).\ (x \downarrow x)\ Cp_\phi R]\}_{dc} \qquad \text{Df}$$

Dc_ϕ is the desired relation. Now, one criterion of the equality of two subliminal or supraliminal positive intervals is, as we have seen, that they both can be formed by going up from some sensation by a step of a certain determinate size, and then down by a smaller step of another determinate size, or else by first taking a downward step of the smaller size, and by then ascending from the point just reached by a step of the larger size: that is, the relation between two terms connected by such an interval will be of the form $\breve{S} \mid R \cup R \mid \breve{S}$ where R and S are both members of $\dot{s}\text{``}D\text{`}\overrightarrow{Id}_\phi$, and S bears to R the relation $\breve{\epsilon} \mid Dc_\phi$. In symbols, we have the following definition:

$$(7) \qquad Vc_\phi = \hat{T}\ \{(\exists R, S),\ S[(\breve{\epsilon} \mid Dc_\phi) \upharpoonright \dot{s}\text{``}D\text{`}\overrightarrow{Id}_\phi]\, R \,.\, T = \breve{S} \mid R \cup R \mid \breve{S}\} \qquad \text{Df}$$

Vc_ϕ is, then, the class of brightness-vectors. We need to write

$$T = \breve{S} \mid R \cup R \mid \breve{S},$$

and not $\qquad T = \breve{S} \mid R \quad \text{nor} \quad T = R \mid \breve{S},$

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* These two criteria are mutually irreducible. A sensation-interval of maximal magnitude will not be indistinguishable from something noticeably greater than a subliminally smaller interval, while it will be noticeably greater than something indistinguishable from the latter interval. In the same way, a sensation-interval of minimal magnitude will not be indistinguishable from any interval noticeably less than a subliminally greater interval, but will be noticeably less than something indistinguishable from the latter interval. The same sort of a statement may be made with reference to sensations, considered as to their intensity. The statement to the contrary in the *Studies in Synthetic Logic* already referred to is simply false, and arose from the author's not considering the fact that all ranges of sensory qualities or intensities have, roughly speaking, a maximum, If the int'R of that paper be defined as $\{\breve{\epsilon};(R \mid R_{se} \cup R_{se} \mid R)\} \upharpoonright \lambda_R$ instead of as $\{\breve{\epsilon};(R_{se} \mid R)\} \upharpoonright \lambda_R$, the hypothesis

$$R \mid R_{se} \cup R_{se} \mid R\epsilon\ \text{Trans}$$

be substituted for $\qquad R_{se} \mid R\epsilon\ \text{Trans} \quad \text{or} \quad R \mid R_{se} \mid R \subseteq R,$

and everything written about the hypothesis $R \mid R_{se} \subseteq R_{se} \mid R$ be struck out as unnecessary, since int'R will, as now defined, always be connected, the paper will be correct. The hypothesis $R \mid R_{se} \subseteq R_{se} \mid R$ will be satisfied everywhere except, so to speak, in the neighbourhood of the top of the scale of sensation-intensities, if R be the relation, "noticeably more intense than."

since, to put it crudely, we may be able to take a step of the size $a-b$ downwards or upwards when we are unable to take a step of the size a in the given direction, where a is less than half the length of the whole sensation-range, while we are always able to take a step of size a either upwards or downwards from any given sense-datum. But even so, Vc_ϕ is not yet quite the class of sensation-intensity vectors we are looking for in order to apply our later theory of measurement, although it could, it is true, be used for that purpose as it stands. It will not be made up of mutually exclusive vectors in the case where ϕ is the relation which we have assumed it to be in the case of sensation-intensities. That is, it will be possible to go from one sensation to another, in general, by several distinct vectors. We would naturally suppose, however, that as the vectors which form the basis of our further theory of measurement represent, roughly, distances, that it would be impossible for two vectors to overlap. We get around this difficulty by defining in terms of Vc_ϕ a class of vectors which would naturally be called intensity-vectors, or, in the case now under consideration, brightness-vectors, which will, by its very definition, be made up of mutually exclusive vectors. To this end, we first define the overlapping of two relations as the relation which holds between them when they both relate a given pair of terms together. In symbols, this becomes

$$\mathrm{Ov} = \hat{P}\hat{Q}\{\dot{\exists}!\, P \dot{\frown} Q\} \quad \mathrm{Df} \tag{8}$$

Then we fuse together any members of Vc_ϕ which can be connected by a chain of overlappings, by the process indicated in the following definition:

$$\mathrm{Vs}_\phi = \dot{s}\ \text{“}\mathrm{D}\ \text{‘}\overrightarrow{(\mathrm{Ov} \upharpoonright \mathrm{Vc}_\phi)*} \quad \mathrm{Df} \tag{9}$$

Vs_ϕ is called the class of separated vectors (*i.e.* in the present case, of separated brightness-vectors). It is useful on account of such conditions as the following: it is possible under all circumstances to go from a given sensation to one whose intensity is greater by a units,* by going up $a+b$ units in the scale of sensations, arranged as to their intensity, and then descending b units, or else by going down b units, and then ascending $a+b$ units, if $a+b$ is less than $\frac{1}{2}c$, where c is the greatest number of units by which two sensations can differ in intensity in the appropriate respect, while, if $a+b > \frac{1}{2}c$, this is only possible under certain condi-

* I speak as if we were able to measure sensations at this stage, although in fact we are not, since to express this point in strictly exact language would involve an intolerable and confusing prolixity.

tions. Nevertheless, if it sometimes is possible, with a certain determination of b, to make a step a units in length by going b units down and $a+b$ units up, and it always is possible to make a step a units in length by the same process when b is given another determination, it would be most unnatural to regard these two processes as determining distinct vectors.

2. We now come to that part of our discussion which has most directly to do with measurement. We shall first define the ratio between two relations belonging to a given class (*i.e.* in the case of sensation-intensities, between two intensity-vectors) in the sense which will be relevant in this paper, and derive a few allied classes and relations from it. I shall say that if R and S both belong to κ, they have the κ-ratio $(\mu/\nu)_\kappa$, if there is a relation P which belongs to κ, such that

$$\exists!\, P^\mu \cap R \quad \text{and} \quad \dot{\exists}!\, P^\nu \cap S$$

In symbols, we have

$$(10) \qquad (\mu/\nu)_\kappa = \hat{R}\hat{S}\{(\exists P) : P, R, S \epsilon \kappa . \exists!\, P^\mu \cap R . \dot{\exists}!\, P^\nu \cap S\} \quad \text{Df}^*$$

The definition of μ/ν in the *Principia* will not do for us, because there, if R is to have to S the relation μ/ν, where μ/ν is expressed in its lowest terms, we must have $\dot{\exists}!\, R^\nu \cap S^\mu$, so that it would be impossible for us to say, for example, that a fog-horn sounds two-thirds as loud as a boiler-factory, without assuming that there is something sounding twice as loud as a boiler-factory. On the other hand, it will be seen that, on our definition, it may very well be that a fog-horn makes a noise two-thirds as loud as a boiler-factory, without there being any noise twice as loud as that of a boiler-factory.

An important notion connected with $(\mu/\nu)_\kappa$ is that of the class of κ-ratios which a given relation R bears to other relations. I shall define Rt_κ as follows:

$$(11) \qquad \mathrm{Rt}_\kappa = \hat{S}\hat{R}\{(\exists M, \mu, \nu) . S = (\mu/\nu)_\kappa . \mu, \nu \neq 0 . R (\mu/\nu)_\kappa M\} \quad \text{Df}$$

Then $\overrightarrow{\mathrm{Rt}}_\kappa$ 'R is this class. In a vector family,† the analogue of $\overrightarrow{\mathrm{Rt}}_\kappa$ 'R —*i.e.* in a submultipliable connected vector-family, $\overrightarrow{\mathrm{Rt}}_\kappa$ 'R itself—contains ratios less than any given ratio. In a class of relations which serve as

* This is not the $(\mu/\nu)_\kappa$ used in the discussion of cyclical systems in the *Principia*.

† Cf. *Principia Mathematica*, Part VI, Section B.

the basis of a system of measurement with a maximum, the ratios, taken in the sense of the *Principia*, which are the analogues of $\overrightarrow{\mathrm{Rt}}_\kappa$ 'R, determine as their "lower limit" in the series of real numbers a number which, in general, will not be zero, which we shall call the κ-*index* of R, or Ind_κ 'R. To define this, we need first to determine the analogue of one of our ratios in the system of the *Principia*. If $(\mu/\nu)_\kappa$ be one of our ratios, its analogue will be Eq_κ '$(\mu/\nu)_\kappa$, where Eq_κ is defined by

$$(12) \qquad \mathrm{Eq}_\kappa = \hat{\mathrm{R}}\hat{\mathrm{S}}\{(\exists\mu,\nu).\,\mathrm{R} = \mu/\nu.\,\mathrm{S} = (\mu/\nu)_\kappa\} \quad \mathrm{Df}$$

Then Ind_κ will be defined by

$$(13) \qquad \mathrm{Ind}_\kappa = p\,|\,(\overrightarrow{\mathrm{H}}_\epsilon)\,|\,(\mathrm{Eq}_\kappa)_\epsilon\,|\,\overrightarrow{\mathrm{Rt}}_\kappa \quad \mathrm{Df}$$

This will always make Ind_κ 'R a real number or Λ, by the definition of a real number given in $*310^*$ of the *Principia*. It follows from $*72-12-15-16$, $*71-25$ that Ind_κ is a one-many relation: that is, no relation has more than one κ-index. We are enabled to compare incommensurable members of κ by their κ-indices, for their κ-indices express, roughly speaking, their magnitude in terms of that of the maximum of all the κ-vectors. It should be noted, however, that we have nowhere assumed that κ contains any greatest κ-vector: indeed, we have assumed nothing at all about κ except that it is a class of relations.

Ind_κ enables us to assign a real number to every member of κ—that is, for example, to every vector between sensations considered with regard to their intensities or qualities. We do not merely wish, however, to measure vectors between sensations considered with regard to their intensities and qualities, but also to measure these intensities and qualities themselves. The measure of the intensity of a sensation is, roughly speaking, the index of its difference from a sensation of zero intensity—that is, it is the upper limit or maximum (taken in the series of real numbers, in their natural order) of the indices of the vectors leading up to it from sense-data of smaller intensity. If κ stands for the class of separated intensity-vectors, and Meas_κ is to stand for the relation between the measure of the intensity of a sense-datum and a sense-datum possessing this measure, we get, in symbols

$$(14) \qquad \mathrm{Meas}_\kappa = \{\mathrm{limax}_{\Theta'}\,|\,(\mathrm{Ind}_\kappa)_\epsilon\,|\,(\overleftarrow{\mathrm{C}})_\epsilon\,|\,\overleftarrow{\varepsilon}\}\ \text{`}s\text{`C``}\kappa \quad \mathrm{Df}$$

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* References beginning with a $*$ are to paragraphs and theorems in the *Principia Mathematica*.

By $*310-1$, $*207-41$, $*204-1$, $*72-12-15$, $*71-25-26$ of the *Principia*, Meas_κ is a one-many relation. It obviously correlates with the whole of $s`C``\kappa$ a part of $C`\Theta'$, or the class of all positive real numbers and zero. Meas_κ will not, in general, be one-one.*

From Meas_κ we may derive a class of vectors or relations connecting members of $s`C``\kappa$ which will always have certain very important properties, quite independent of those of κ. I shall define this class, which I shall call Reg_κ, as follows

$$\text{(15)} \qquad \text{Dist}_\kappa = \hat{R}\hat{\mu}\{R = \breve{\text{Meas}}_\kappa{}^{\iota}\mu +_\alpha\} \quad \text{Df}$$

$$\text{(16)} \qquad \text{Reg}_\kappa = D`\text{Dist}_\kappa - \iota`\dot{\Lambda}$$

The measure of a member of Reg_κ is the value of μ for which this member is $\breve{\text{Dist}}_\kappa`R$, since $\text{Reg}_\kappa \upharpoonright \breve{\text{Dist}}_\kappa \in 1 \to 1$. That this is so may be proved as follows: it results from the form of the definition of Dist_κ that $\text{Dist}_\kappa \in 1 \to \text{Cls}$, and hence that $\text{Reg}_\kappa \upharpoonright \text{Dist}_\kappa \in 1 \to \text{Cls}$. Now let us suppose that $\text{Dist}_\kappa`\mu = \text{Dist}_\kappa`\nu \neq \dot{\Lambda}$: *i.e.* that $\text{Dist}_\kappa`\mu$ and $\text{Dist}_\kappa`\nu$ are identical, and belong to Reg_κ. This gives, by definition,

$$\breve{\text{Meas}}_\kappa{}^{\iota}\mu +_\alpha = \breve{\text{Meas}}_\kappa{}^{\iota}\nu +_\alpha \neq \dot{\Lambda}$$

From this we deduce

$$(\exists R, S) : (\exists \varpi, \rho) . R \breve{\text{Meas}}_\kappa \varpi . \varpi (\mu +_\alpha) \rho . \rho \text{ Meas}_\kappa S :$$

$$(\exists \sigma, \tau) . R \breve{\text{Meas}}_\kappa \sigma . \sigma (\nu +_\alpha) \tau . \tau \text{ Meas}_\kappa S,$$

whence we obtain

$$(\exists R, S, \varpi, \rho, \sigma, \tau) . \varpi \text{ Meas}_\kappa R . \sigma \text{ Meas}_\kappa R . \rho \text{ Meas}_\kappa S . \tau \text{ Meas}_\kappa S .$$

$$\varpi = \mu +_\alpha \rho . \sigma = \nu +_\alpha \tau .$$

Since $\text{Meas}_\kappa \in 1 \to \text{Cls}$, and since $D`\text{Meas}_\kappa \subset C`\Theta'$, by definition, it follows from $*310-123$, $*312-55-41$ of the *Principia* that $\mu = \nu$. Hence

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* A condition which will make Meas_κ one-one is

$$\text{Ind}_\kappa \upharpoonright \kappa \in 1 \to 1 : : (\exists x) : . y \in s`C``\kappa . \supset_y : (\exists R) : xRy : zSy . S \in \kappa . \supset_{z,S} . S (\breve{\text{Ind}}_\kappa{}^{\iota}\Theta') R$$

$\text{Ind}_\kappa \upharpoonright \kappa \in 1 \to 1$ is true if, for example, the members of κ, arranged in the natural order of their indices, form a series, as we should naturally expect when they are *e.g.* intervals between sensation-intensities defined as in the "Studies in Synthetic Logic" (not between sensations). The second part of this condition amounts to the assumption that there is a member of $s`C``\kappa$ such that if $y$ is another member of $s`C``\kappa$, there is a member of κ which relates this term to y, whose κ-index is less than that of y.

$\mathrm{Reg}_\kappa \upharpoonleft \mathrm{Dist}_\kappa \in 1-1$. Since Reg_κ is the domain of this relation we can always speak of $\overset{\smile}{\mathrm{Dist}_\kappa}\text{`}\mathrm{R}$ if $\mathrm{R} \in \mathrm{Reg}_\kappa$.

It is easy to prove that if R, S and R|S are all members of Reg_κ, the measure of R|S is the sum of the measures of R and of S, or, in symbols,

$$(17) \quad \vdash : \mathrm{R}, \mathrm{S}, \mathrm{R}|\mathrm{S} \in \mathrm{Reg}_\kappa . \supset_{\mathrm{R}, \mathrm{S}, \kappa} . \overset{\smile}{\mathrm{Dist}_\kappa}\text{`}(\mathrm{R}\,|\,\mathrm{S}) = \overset{\smile}{\mathrm{Dist}_\kappa}\text{`}\mathrm{R} +_a \overset{\smile}{\mathrm{Dist}_\kappa}\text{`}\mathrm{S}$$

As a consequence of this, since $\mathrm{Reg}_\kappa \upharpoonleft \mathrm{Dist}_\kappa \in 1 \rightarrow 1$ we get

$$(18) \qquad \vdash : \mathrm{R}, \mathrm{S}, \mathrm{R}|\mathrm{S}, \mathrm{S}|\mathrm{R} \in \mathrm{Reg}_\kappa . \supset . \mathrm{R}|\mathrm{S} = \mathrm{S}|\mathrm{R}$$

Relative multiplication, then, applied to members of Reg_κ, is not only an associative operation, but a commutative operation, and may be regarded, roughly, as a kind of addition. This shows that the old opinion that only extensive magnitudes are subject to a commutative, associative operation of addition is erroneous. We can not only get a commutative, associative operation of addition which will apply to vectors such as the members of Reg_κ, but also one which will apply to sensation-intensities, as we may now call the members of $\mathrm{D}\text{`}\overleftarrow{\mathrm{Meas}_\kappa}$, where κ is a class of separated intensity-vectors, since they are classes of all the sense-data whose intensity has a given measure. This operation is simply the operation on α and β which gives the γ such that the measure of the members of γ is the sum of the measure of the members of α and the measure of the members of β.

Another consequence of (17) is that if we define μ_κ as the relation which holds between R and S when they both belong to Reg_κ and

$$\overset{\smile}{\mathrm{Dist}_\kappa}\text{`}\mathrm{R} = \mu \times_a \overset{\smile}{\mathrm{Dist}_\kappa}\text{`}\mathrm{S}$$

—that is, if we put

$$(19) \qquad \mu_\kappa = \{\mathrm{Dist}_\kappa ; \mu \times_a\} \upharpoonright \mathrm{Reg}_\kappa \quad \mathrm{Df}$$

we shall have

$$(20) \qquad \vdash : (\mu_\kappa\text{`}\mathrm{R})\,|\,(\nu_\kappa\text{`}\mathrm{R}) = \varpi_\kappa\text{`}\mathrm{R} . \supset . \mu +_a \nu = \varpi$$

or, more generally,

$$(21) \qquad \vdash : \exists ! [(\mu_\kappa\text{`}\mathrm{R})\,|\,(\nu_\kappa\text{`}\mathrm{R})] \dot{\cap} \varpi_\kappa\text{`}\mathrm{R} . \supset . \mu +_a \nu = \varpi$$

Moreover, since $\mathrm{Reg}_\kappa \upharpoonleft \mathrm{Dist}_\kappa \in 1 \rightarrow 1$, and $(\mu \times_a)\,|\,(\nu \times_a) = (\mu \times_a \nu) \times_a$, we get

$$(22) \qquad \vdash : \exists\, \mu_\kappa\,|\,\nu_\kappa \dot{\cap} \varpi_\kappa . \supset . \mu \times_a \nu = \varpi$$

These propositions are the rough analogues of $*\,356-33-54$ in the *Principia*.

Let us next bring the discussion of measurement that we have just finished into correlation with that part of our theory that derived Vs_ϕ from ϕ. If ϕ be such a relation as, "The interval of saturation in color between x and y seems less than that between u and v," the natural unit of measurement, as we said in the introduction, would seem to be a complete saturation, and the method of measurement which we have just elaborated would seem fairly natural and appropriate. The measure of the saturation of a given sensation x, will then be $\mathrm{Meas}_{\mathrm{Vs}_\phi}\text{`}x$. It will be natural to say that two sensations have the same degree of chroma if they bear to one another the relation $\overset{\smile}{\mathrm{Meas}}_{\mathrm{Vs}^\phi} | \mathrm{Meas}_{\mathrm{Vs}_\phi}$. This leads to the following definitions

(23) $$\mathrm{Qual}_\phi = \mathrm{D}\text{`}\overleftarrow{\mathrm{Meas}}_{\mathrm{Vs}_\phi} \quad \mathrm{Df}$$

(24) $$\mathrm{Mag}_\phi = \mathrm{Meas}_{\mathrm{Vs}_\phi} | \epsilon \upharpoonright \mathrm{Qual}_\phi \quad \mathrm{Df}$$

Qual_ϕ is the class of all degrees of saturation, in the particular case just discussed. It consists, however, of degrees of saturation considered as quantities, and though it may *de facto* coincide with the λ_{R} of the *Studies in Synthetic Logic*, if R be taken to be the relation, "noticeably more saturated than," or with $\mathrm{D}\text{`}\mathrm{sg}\text{`}(\dot{s}\text{`}\mathrm{Vs}_\phi)_{\mathrm{se}}$, it cannot, without the aid of complicated logical hypotheses, be proved identical with either.

Mag_ϕ is the relation of the magnitude of a degree of saturation to the degree of saturation, and is one-one. If we so desire, we can define Dist_ϕ as $\hat{\mathrm{R}}\hat{\mu}\{\mathrm{R} = \overset{\smile}{\mathrm{Mag}}_\phi ; \mu +_a\} \mathrm{Reg}_\phi$ as $\mathrm{D}\text{`}\mathrm{Dist}_\phi - \iota\text{`}\dot{\Lambda}$, and μ_ϕ as

$$\{\mathrm{Dist}_\phi ; \mu \times_a\} \upharpoonright \mathrm{Reg}_\phi$$

and the propositions

$$\vdash : \dot{\exists}! [(\mu_\phi\text{`}\mathrm{R}) | (\nu_\phi\text{`}\mathrm{R})] \dot{\cap} \varpi_\phi\text{`}\mathrm{R} . \supset . \mu +_a \nu = \varpi,$$

and $$\vdash : \dot{\exists}! \mu_\phi | \nu_\phi \dot{\cap} \varpi_\phi . \supset . \mu \times_a \nu = \varpi,$$

will be universally valid. Reg_ϕ will consist of all intervals, positive, zero, and negative, between degrees of saturation, and if R be any such interval, its measure will be $\overset{\smile}{\mathrm{Dist}}_\phi\text{`}\mathrm{R}$. μ_ϕ will be the proportion between a member of Reg_ϕ and another, one μ-th of its size.

3. In our previous work we have, roughly speaking, made the maximum of the magnitudes of all members of κ our real unit of measurement: let us see how this measurement may be adapted to measurement by smaller units. The way to do this is to select a certain region of our

scale of measurement which has a definite maximum, to measure the intervals lying in this region by the method given above, and to measure all intervals in general by finding fractions of them lying in this interval. To this end, we make the following definitions

(25) $\mu \div \lambda = \mu \times_a \mathrm{H}\text{``}(1/\lambda)$ Df

(26) $\mathrm{Inx}_{\kappa, a} = \hat{\mu}\hat{\mathrm{R}}\{(\exists\lambda, \mathrm{S}).(\mu \div \lambda)\,\mathrm{Ind}_{\restriction a\text{``}\kappa - \iota\text{`}\dot{\Lambda}}\,\mathrm{S}\restriction a.\,\mathrm{S}^{\lambda} = \mathrm{R}\}$ Df

Here we use as our standard of measurement the greatest distance between members of a, measured by members of κ, and not the greatest distance between members of $s\text{`}\mathrm{C}\text{``}\kappa$. $\mathrm{Inx}_{\kappa, a}$ is not, in general, a one-many relation: its converse domain is $s\text{`}\mathrm{Pot}\text{``}\{\kappa \cap \hat{\mathrm{R}}\{\exists!\,\mathrm{R}\restriction a\}\}$ and its domain is included in the class of real numbers. We next define $\mathrm{Meas}_{\kappa, a}$ in terms of $\mathrm{Inx}_{\kappa, a}$, as we defined Meas_{κ} in terms of Ind_{κ}, in the following manner

(27) $\mathrm{Meas}_{\kappa, a} = \{\mathrm{limax}_{\Theta'} \mid (\mathrm{Inx}_{\kappa, a})_{\epsilon} \mid (\breve{\mathrm{C}})_{\epsilon} \mid \overleftarrow{\epsilon}\}\restriction$

$s\text{`}\mathrm{C}\text{``}\mathrm{Pot}\text{``}\{\kappa \cap \hat{\mathrm{R}}\{\dot{\exists}!\,\mathrm{R}\restriction a\}\}$ Df

Notwithstanding the fact that $\mathrm{Inx}_{\kappa, a}$ is not, in general, one-many, $\mathrm{Meas}_{\kappa, a}$ is always necessarily one-many.

We are now in a position to define $\mathrm{Dist}_{\kappa, a}$, $\mathrm{Reg}_{\kappa, a}$, and $\mu_{\kappa, a}$ just as we defined Dist_{κ}, Reg_{κ}, and μ_{κ}: that is, as follows

(28) $\mathrm{Dist}_{\kappa, a} = \hat{\mathrm{R}}\hat{\mu}\{\mathrm{R} = \breve{\mathrm{Meas}}_{\kappa, a}\text{`}\mu +_a\}$ Df

(29) $\mathrm{Reg}_{\kappa, a} = \mathrm{D}\text{`}\mathrm{Dist}_{\kappa, a} - \iota\text{`}\dot{\Lambda}$ Df

(30) $\mu_{\kappa, a} = \{\mathrm{Dist}_{\kappa, a}\text{`}\mu x_a\}\,\mathrm{Reg}_{\kappa, a}$ Df

All the theorems which we proved concerning Meas_{κ}, Dist_{κ}, Reg_{κ}, and μ_{κ} will remain valid if we alter κ every time that it occurs as a subscript to κ, a.

Now, if $\phi(x, y, u, v)$ stands for "the brightness-interval between x and y seems less than that between u and v," the theory of *sensation*-measurement developed in § 2, though it is still applicable, does not seem appropriate nor natural: we do not usually take the greatest possible brightness, say, that of the sun, as our standard of brightness. Our usual way of measuring brightness seems to be, perhaps, to count the number of "thresholds" which lie between it and utter darkness. In all ordinary psychological work, this method of measurement is adequate, but it will

not enable us to subdivide the step from one limen to the next into equal parts. Nevertheless, the limen is a very natural standard to use in the measurement of brightnesses. Since we do not wish to assume that all steps from a given sense-datum to a just brighter one are equal, we shall make the first interliminal difference our standard of measurement. We do this by taking Vs_ϕ as the κ and the class of sense-data lying in the first interliminal space as the α of the theory of measurement which we have just developed. Now, a sense-datum lies in the first interliminal space with respect to brightness, or is of subliminal brightness, if no interval leading to it from below is supraliminal: that is, if it belongs to the field of $\dot{s}\,\hat{\mathrm{R}}\{(\exists x).(x\downarrow x)\,\mathrm{Cp}_\phi\,\mathrm{R}\}$, but not to its converse domain. Let us define this class, and certain functions of it, as follows

$$\mathrm{Sbl}_\phi = \overrightarrow{\mathrm{B}}\,\dot{s}\,\hat{\mathrm{R}}\{(\exists x).(x\downarrow x)\,\mathrm{Cp}_\phi\,\mathrm{R}\} \quad \mathrm{Df} \tag{31}$$

$$\mathrm{Quant}_\phi = \mathrm{D}\,\mathrm{Gs}\,\mathrm{Meas}_{\mathrm{Vs}_\phi,\ \mathrm{Sbl}_\phi} \quad \mathrm{Df} \tag{32}$$

$$\mathrm{Mgn}_\phi = \mathrm{Meas}_{\mathrm{Vs}_\phi,\ \mathrm{Sbl}_\phi}\,|\,\epsilon\,\restriction\,\mathrm{Quant}_\phi \quad \mathrm{Df} \tag{33}$$

Sbl_ϕ is, in the case just considered, the class of all sensations of subliminal brightness. Qnant_ϕ is the class of all degrees of brightness. $\mathrm{Mgn}_\phi\,\dot{\xi}$ is the quantitative measure of $\dot{\xi}$ in terms of the brightness-limen, if $\dot{\xi}$ is a brightness. The further development of this theory of measurement runs precisely parallel to that of measurement in terms of the maximum interval.

4. The question which we have next to settle is that of the relation of our theory of measurement to that of the *Principia*. It is obvious that since the theory of the *Principia* applies only to kinds of measurement without definite maxima, while our theory of measurement, in the form it takes in § 2, only applies to kinds of measurement with definite maxima, the form of our theory with which we must compare that of the *Principia* is that developed in § 3, which need not only apply to kinds of measurement with definite maxima. To carry out this comparison, we need to set some standard problem, and compare the answers which the two systems give to it. Now, the theory in the *Principia* starts from a certain sort of class of relations or vectors κ, and culminates in the definition of the real proportion X among members of κ as X_κ. Our definition of the real proportion μ, among members of $\mathrm{Reg}_{\kappa,\alpha}$ is $\mu_{\kappa,\alpha}$, and therefore a natural definition of the real proportion μ, among members of κ will be $\mu_{\kappa,\alpha}\,\restriction\,\kappa$. The fact that the real proportion in the *Principia* appears in

the form of a function of a relation, while ours appears as a function of a class, is due to the fact that the real numbers which are applied there are the $\dot{s}$'s of the real numbers we apply. The natural way to put the question, "When does the theory of measurement developed in the *Principia* give the same results as that developed in § 3 of the paper?" is "When is the formula $\mu_{\kappa,\alpha} \restriction \kappa = (\dot{s}`\mu)_\kappa$ valid?"

Now, if κ is a connected family, containing all the powers of its members, it follows from $*331-24-42$ that

$$(\mu/\nu)_\kappa = \hat{R}\hat{S}\{(\exists P) \cdot P \epsilon \kappa \cdot R = P^\mu \cdot S = P^\nu\},$$

or, *in extenso*,

(34) $\quad \vdash : \kappa \epsilon \mathrm{FM}\,\mathrm{conx} \cdot s`\mathrm{Pot}``\kappa \subset \kappa \cdot \supset \cdot$

$$(\mu/\nu)_\kappa = \hat{R}\hat{S}\{(\exists P) . P \epsilon \kappa . R = P^\mu . S = P^\nu\}.$$

Since, if $R = P^\mu$ and $S = P^\nu$, $R^\nu = P^{\mu \times_c \nu} = S^\mu$, and since, under the above conditions, R, S, and R^ν, being powers of a member of κ, are members of κ, and hence not Λ, we derive from $*302-02$ and $*303-01$ the conclusion,

(35) $\quad \vdash : \mathrm{Hp}\,(34) . \supset . (\mu/\nu)_\kappa \subseteq (\mu/\nu) \restriction \kappa$

From (34), (35), and $*333-42$, $*351-1$, we get

(36) $\quad \vdash : \mathrm{Hp}\,(34) \cdot \kappa \epsilon \mathrm{FM}\,\mathrm{ap}\,\mathrm{submult} . \supset . (\mu/\nu)_\kappa = (\mu/\nu) \restriction \kappa$

Furthermore, it results from (34) and $*336-41$ that

(37) $\quad \vdash : \mathrm{Hp}\,(34) \cdot R \epsilon \kappa_\partial . \supset . (\mu/\nu)_{\overleftarrow{U_\kappa}`R} = (\mu/\nu)_\kappa \restriction U_\kappa`R$

Moreover, it is easy to prove that if we call $p`\overleftarrow{\dot{s}}`\kappa``\mathrm{D}`R$ by the name $\alpha$, and if $P \restriction \alpha$ and $Q \restriction \alpha$ exist, then to say that P bears the relation $(\mu/\nu)_\kappa \restriction \overleftarrow{U_\kappa}`R$ to Q is equivalent to saying that $P \restriction \alpha$ bears the relation $(\mu/\nu)_{\restriction \alpha `` \overleftarrow{U_\kappa}`R}$ to $Q \restriction \alpha$, under the hypotheses that κ is a connected family containing all the powers of its members, and that R belongs to κ_∂. In symbols we have

(38) $\quad \vdash :. \mathrm{Hp}\,(34) . R \epsilon \kappa_\partial \cdot \alpha = p`\overleftarrow{\dot{s}}`\kappa_\partial``\mathrm{D}`R . \dot{\exists}!\, P \restriction \alpha \cdot \dot{\exists}!\, Q \restriction \alpha : \supset :$

$$P\{(\mu/\nu)_\kappa \restriction \overleftarrow{U^\kappa}`R\}\,Q . \equiv . (P \restriction \alpha)(\mu/\nu)_{\restriction \alpha `` \overleftarrow{U^\kappa}`R}(Q \restriction \alpha).$$

Putting (38) and (36) together, we get

$$(39)\quad \vdash :. \mathrm{Hp}(38).\mathrm{Hp}(36) : \supset : P\,\{(\mu/\nu)\restriction \overleftarrow{U_\kappa}\,`R\}\,Q.$$
$$\equiv .(P\restriction\alpha)(\mu/\nu)_{\restriction \alpha``\overleftarrow{U_\kappa}R}\,(Q\restriction\alpha).$$

With the end of (39), (11), and (12), we obtain

$$(40)\quad \vdash :. \mathrm{Hp}(39) : \supset : P\,\{[\mathrm{Eq}_{\restriction \alpha``\overleftarrow{U_\kappa}R}`(\mu/\nu)_{\restriction \alpha``\overleftarrow{U_\kappa}R}]\restriction \overleftarrow{U_\kappa}`R\}\,Q.$$
$$\equiv .P\,\{\restriction \alpha`^{\dagger}(\mu/\nu)_{\restriction \alpha``\overleftarrow{U_\kappa}`R}\}\,Q$$

whence we deduce

$$(41)\quad \vdash : \mathrm{Hp}(36).R\epsilon\kappa_\partial .\alpha = p`\overleftarrow{\dot{s}}`\kappa_\partial``D`R.\supset .$$
$$(\mathrm{Eq}_{\restriction \alpha``\overleftarrow{U_\kappa}`R}`S)\restriction\{\overleftarrow{U_\kappa}`R\cap \hat{X}\{\dot\exists !\,X\restriction\alpha\}\} = \{(\restriction\alpha)^{\dagger}\restriction D`\mathrm{Rt}_{\restriction \alpha``\overleftarrow{U_\kappa}`R}\}`S$$

The question now arises, when will it be true that if $RU_\kappa T$,

$$\dot\exists !\,T\restriction p`\overleftarrow{\dot{s}}`\kappa_\partial``D`R?$$

Now, if κ is an initial family, it is not difficult to deduce from $*335-15$ and the definition of a vector-family that $p`\overleftarrow{\dot{s}}`\kappa_\partial``D`R$ is made up of all those things which bear such a relation as $\breve{P}$ to everything bearing such a relation as $R|Q$ to $\mathrm{init}`\kappa$, where $P\epsilon\kappa_\partial$ and $Q\epsilon\kappa$. Now, by $*335-17$, the relative product of two members of an initial family is a member of the same family (see also $*331-23$), and hence, since

$$x\breve{P}|R\,\mathrm{init}`\kappa .\supset .x\breve{P}|\breve{Q}|R|Q\,\mathrm{init}`\kappa,$$

if $x\breve{P}|R\,\mathrm{init}`\kappa$ and if whenever $Q\epsilon\kappa$, $P|Q\subset J$, then $x\,\epsilon\, p`\overleftarrow{\dot{s}}`\kappa_\partial``D`R$. Therefore, by $*335-21$, $\overrightarrow{\breve{P}|R}`\mathrm{init}`\kappa\subset p`\overleftarrow{\dot{s}}`\kappa_\partial``D`R$. Therefore, since $R\epsilon\kappa_\partial$, $\mathrm{init}`\kappa\,\epsilon\, p`\overleftarrow{\dot{s}}`\kappa``D`R$. Moreover, since all initial families are connected by $*336-41$, if $RU_\kappa T$ there is a member of $\kappa_\partial$, say S, such that $R = S|T$. Since $\kappa\subset 1\to 1$ and all the members of $\kappa$ have a common converse domain, $T = \breve{S}|R$ and hence $T`\mathrm{init}`\kappa$ is the same as $\breve{S}`R`\mathrm{init}\,\kappa$, and consequently belongs to $p`\overleftarrow{\dot{s}}`\kappa_\partial``D`R$. Therefore $\dot\exists !\,T\restriction p`\overleftarrow{\dot{s}}`\kappa_\partial``D`R$. This gives us, by (41), $*335-17$, and the definition of FM init,

$$(42)\quad \vdash : \kappa\epsilon\mathrm{FM\ ap\ init\ submult}.R\epsilon\kappa_\partial .\alpha = p`\overleftarrow{\dot{s}}`\kappa_\partial``D`R.\supset .$$
$$\restriction \overleftarrow{U_\kappa}`R|\mathrm{Eq}_{\restriction \alpha``\overleftarrow{U_\kappa}`R} = (\restriction\alpha)^{\dagger}\restriction D`\mathrm{Rt}_{\restriction \alpha``\overleftarrow{U_\kappa}`R}$$

If $\dot{s}`\kappa_\partial$ is a series, then no term can at once belong to D`R and to $p`\overleftarrow{\dot{s}}`\kappa_\partial``$D`R, by $*202-503$. It may be deduced from $*336-41$, moreover, that if $P\epsilon\overleftarrow{U}_\kappa`R$, $D`P \subset D`R$. From this and what we have said above, we may conclude that $\lceil\alpha``\overleftarrow{U}_\kappa`R = \lceil\alpha``\kappa-\iota`\dot{\Lambda}$. From this, (42), and $*334-32$, we get

$$(43) \qquad \vdash : \kappa\epsilon FM \text{ sr init submult} \cdot R\epsilon\kappa_\partial \cdot \alpha = p`\overleftarrow{\dot{s}}\kappa``D`R . \supset .$$

$$\lceil\overleftarrow{U}_\kappa`R \mid Eq_{\lceil\alpha``\kappa-\iota`\dot{\Lambda}} = (\lceil\alpha)^{\dagger}\lceil D`Rt_{\lceil\alpha`\kappa-\iota`\dot{\Lambda}}$$

Now, it follows from this and (13) that $Ind_{\lceil\alpha``\kappa-\iota`\dot{\Lambda}}`S$ is the class of all those ratios which are less than any ratio which the member of $\overleftarrow{U}_\kappa`R$ of which S forms a part bears to any member of $\overleftarrow{U}_\kappa`R$: by $*352-72$, then, if κ be a serial family, as we have supposed in (43), the member of $\overleftarrow{U}_\kappa`R$ containing S bears to R (or to any member of $\overleftarrow{U}_\kappa`R$) no ratio which is a member of $Ind_{\lceil\alpha``\kappa-\iota`\dot{\Lambda}}`S$, and no member of $\kappa$ preceding the one containing S bears R such a ratio, while, by the definition of $Ind_{\lceil\alpha``\kappa-\iota`\dot{\Lambda}}$ and $*352-72$, any member of $\kappa$ following the one containing S does bear R such a ratio, if any. Hence it is easy to prove that if the rational multiples of R form a median class ($*271$) of $U_\kappa$, $(\dot{s}`Ind_{\lceil\alpha``\kappa-\iota`\dot{\Lambda}})_\kappa$ is a relation which holds between the member of κ containing S and R, if we understand the use of κ as a subscript in the sense defined in $*356-01$. With the aid of $*351-11$, $*336-44$, $*270-4$, $*271-15$, we arrive at the conclusion,

$$(44) \qquad \vdash : \kappa\epsilon FM \text{ sr init} \cdot (C`R_\kappa) \text{ med } U_\kappa \cdot Cnv`\dot{s}`\kappa_\partial \epsilon \text{ semi Ded} .$$

$$\alpha = p`\overleftarrow{\dot{s}}`\kappa_\partial``D`R . Ind_{\lceil\alpha``\kappa-\iota`\dot{\Lambda}}`S \neq \dot{\Lambda} . T\epsilon\kappa_\partial . S \subset T . \supset .$$

$$T\,(\dot{s}`Ind_{\lceil\alpha``\kappa-\iota`\dot{\Lambda}}`S)\,R$$

Now, if it is true that by repeating any vector belonging to $\kappa_\partial \lceil\alpha$ a sufficient number of times, we can get beyond any given vector belonging to κ, we may replace $Ind_{\lceil\alpha``\kappa-\iota`\dot{\Lambda}}`S \neq \dot{\Lambda} . S \subset T$ in the above formula by

$$S = T\lceil\alpha \neq \dot{\Lambda},$$

as follows readily enough from $*352-72$, $*336-64$. That this repetition is possible is a consequence of $*337-13$ and $*336-01-011$. We

thus obtain the following theorem:

(45) $\vdash : \kappa \epsilon \mathrm{FM}\ \mathrm{sr}\ \mathrm{init} \cdot (\mathrm{C}`\mathrm{R}_\kappa)\ \mathrm{med}\ \mathrm{U}_\kappa \cdot \mathrm{Cnv}`\dot{s}`\kappa_\partial \epsilon\ \mathrm{semi\ Ded} \cdot$

$$\alpha = p`\overleftarrow{\dot{s}}`\kappa_\partial``\mathrm{D}`\mathrm{R} \cdot \mathrm{T} \epsilon \kappa_\partial \cdot \mathrm{S} = \mathrm{T} \restriction \alpha \neq \dot{\Lambda} \cdot \supset \cdot$$

$$\mathrm{T}\,(\dot{s}`\mathrm{Ind}^{\restriction \alpha}``_{\kappa - \iota}`\dot{\Lambda}`\mathrm{S})_\kappa\,\mathrm{R}$$

Now, we have seen that $\mathrm{T} \restriction \alpha \neq \dot{\Lambda} \cdot \equiv \cdot \mathrm{RU}_\kappa \mathrm{T}$, under the above conditions. I wish to show that, given any vector belonging to κ, there is some member of $\overleftarrow{\mathrm{U}}_\kappa`\mathrm{R}$, say Q, such that $\mathrm{P} \epsilon \mathrm{Pot}`\mathrm{Q}$. We prove this as follows: by $*337-13$ and $*336-01-011$, there is some power of R, say R^ν, which bears the relation U_κ to P: By $*337-27$, there is a member of κ, say M, such that $\mathrm{M}^\nu = \mathrm{P}$. This M is the Q we want, since by $*336$, if κ is a serial family, and ν a positive integer, $\mathrm{RU}_\kappa \mathrm{S}$ is equivalent to $\mathrm{R}^\nu \mathrm{U}_\kappa \mathrm{S}^\nu$. Hence, if $\mathrm{P} \epsilon \kappa_\partial$, it follows from $*356-33$ and a little elementary arithmetic that the class $\overrightarrow{\mathrm{Inx}}_{\kappa, \alpha}`\mathrm{P}$ contains just two members—the real number which, when applied, relates P to R—and that if $\mathrm{P} \sim \epsilon \kappa_\partial$, $\mathrm{Inx}_{\kappa, \alpha}`\mathrm{P}$ contains Λ alone, $\mathrm{Meas}_{\kappa, \alpha}`x$ will then be the upper limit or maximum of the measures in terms of R of the vectors leading up to x. Since these measures, by $*356-63$, are themselves each the limit or maxima of the class applied rational numbers which the rational multiples of R less than some vector leading up to x bear to R, $\mathrm{Meas}_{\kappa, \alpha}`x$ may easily be shown to be the class of all the ratios which vectors which connect members of $\overleftarrow{\dot{s}}`\kappa_\partial`x$ bear to R, and consequently to be a real number (cf. $*336-41$, $*352-72$), such that, when we apply it to $\kappa$, the vector leading from $x$ to $\mathrm{init}`\kappa$ bears it to R. In symbols, we have

(46) $\vdash : \mathrm{Hp}\,(45) \cdot x \epsilon s`\mathrm{C}``\kappa \cdot \supset \cdot$

$$[(\kappa \restriction \breve{\subset})`(x \downarrow \mathrm{init}`\kappa)]\,(\dot{s}`\mathrm{Meas}_{\kappa, \alpha}`x)_\kappa\,\mathrm{R}$$

Now, it follows from the assumption $(\mathrm{C}`\mathrm{R}_\kappa)\,\mathrm{med}\,\mathrm{U}_\kappa$ that every member of $\kappa$ has a real measure in terms of R. Since $\dot{s}`\kappa_\partial$ is serial, and since, by $*331-22$, $\mathrm{I} \restriction s`\square``\kappa \epsilon \kappa$, it follows from the fact that $s`\square``\kappa = s`\mathrm{C}``\kappa$ in every vector family that any two members of $s`\mathrm{C}``\kappa$ are connected by a member of κ. Hence we may deduce from $*356-26-54$ that if μ is positive, $\mathrm{Dist}_{\kappa, \alpha}`\mu \epsilon \kappa_\partial$ if $\mu$ is zero, $\mathrm{Dist}_{\kappa, \alpha}`\mu = \mathrm{I} \restriction s`\square``\kappa$, while if $\mu$ is negative, $\mathrm{Dist}_{\kappa, \alpha}`\mu \epsilon \mathrm{Cnv}``\kappa_\partial$. From this, by $*356-26$, and the fact that vectors standing in the relation x_κ to R always exist if κ is initial, serial, submultipliable, and semi-Dedekindian, and x is a non-negative relational

real number, we may deduce the conclusion

$$(47) \qquad \vdash : \mathrm{Hp}\,(45) . \supset . \mathrm{Reg}_{\kappa,\alpha} = \kappa \cup \mathrm{Cnv}\,\text{``}\,\kappa$$

Another easily proved theorem is

$$(48) \qquad \vdash : \mathrm{Hp}\,(45) . \supset . \mathrm{P}(\dot{s}\,\text{`}\overbrace{\mathrm{Dist}_{\kappa,\alpha}}\,\text{`}\,\mathrm{P})_\kappa\,\mathrm{R}$$

By (30) and $*356-33-26$, if μ is a positive or zero real number, and $\mathrm{P}\,(\mu_{\kappa,\alpha} \restriction \kappa)\,\mathrm{Q}$, then we have, in the sense of $*356$, $\mathrm{P}\,(\dot{s}\,\text{`}\,\mu)_\kappa\,\mathrm{Q}$. The converse of this is easily proved by the same theorems, since P and Q always bear some applied real number to R: that is, we have

$$(49) \qquad \vdash : \mathrm{Hp}\,(45) . \mu \epsilon \mathrm{C}\,\text{`}\,\Theta' . \supset . \mu_{\kappa,\alpha} \restriction \kappa = (\dot{s}\,\text{`}\,\mu)_\kappa$$

This is the theorem we set out to prove, and it establishes the fact that in an initial, serial, semi-Dedekindian family, if the rational multiples of a given vector form a median class of the series of vectors, the system of measurement defined in the *Principia* gives substantially the same results as the system defined in this paper, if, to put it crudely, we take this given vector whose rational multiples form a median class of the series of vectors as a unit, by making the class of all the things which follow every member of its domain in the series generated by the vectors the α of our previous work.*

5. In conclusion, let us consider what bearing all this work of ours can have on experimental psychology. One of the great defects under which the latter science at present labours is its propensity to try to answer questions without first trying to find out just what they ask. The experimental investigation of Weber's law is a case in point: what most experimenters do take for granted before they begin their experiments is infinitely more important and interesting than any results to which their experiments lead. One of these unconscious assumptions is that sensations or sensation-intervals can be measured, and that this process of measurement can be carried out in one way only. As a result, each new experimenter would seem to have devoted his whole energies to the invention of a method of procedure logically irrelevant to everything that had gone before: one man asks his subject to state when two intervals between sensations of a given kind appear different; another bases his

* It will be noticed that in such a vector family, the series of vectors, arranged in order of magnitude, is of the form $(\overbrace{+1})\,\delta$.

whole work on an experiment where the observer's only problem is to divide a given colour-interval into two equal parts, and so on indefinitely, while even where the experiments are exactly alike, no two people choose quite the same method for working up their results. Now, if we make a large number of comparisons of sensation-intervals of a given sort with reference merely to whether one seems larger than another, the methods of measurement given in this paper indicate perfectly unambiguous ways of working up the results so as to obtain some quantitative law such as that of Weber, without introducing such bits of mathematical stupidity as treating a "just noticeable difference" as an "infinitesimal," and have the further merit of always indicating *some* tangible mathematical conclusion, no matter what the outcome of the comparisons may be.

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ON STEADY FLUID MOTIONS WITH FREE SURFACES

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When a thin rigid plane is fixed in a fluid in motion, and normal to the direction of flow, we observe, near to its edges, the formation of a "free" surface, or surface of slip, which remains well defined for a certain distance from the plane, but degenerates and thereafter vanishes in a whirling and chaotic movement. Experience also shows that the pressure behind the plane is, in this case, distributed in an approximately uniform manner over its surface, and is less than the pressure in the undisturbed current.

The theory of Kirchhoff, which assumes a free surface extending to an infinite distance, neglects the suction behind the plane. This suction, causing an increment of the velocity on the free surface, must evidently modify the distribution of the velocities on the front face of the plane.

In the present paper we study the case in which (the movement of the fluid being supposed to be parallel to a fixed plane) two planes are disposed, one behind the other, at right angles to the stream, and their edges connected by free surfaces (see Fig. 1). The velocity at the free surface is, in this case, as we shall show further on, greater than the velocity of the stream. The two planes mutually attract each other, but the resultant pressure of the fluid on the system of the two planes is evidently zero.

The case studied by Kirchhoff (which may be considered as a limiting case, in which the two planes are infinitely removed one from the other) should not be inconsistent with the paradox of d'Alembert.

In a real fluid, a fictitious plane representing the cumulative effect of friction and of the shocks which, by causing a dissipation of the energy, produce the resistance, may be supposed to correspond to the second plane.

Borrowing from experience one single contribution, viz. the intensity of the suction behind the plane, we may calculate the distribution of velocity over the front face, and the total pressure exerted by the stream

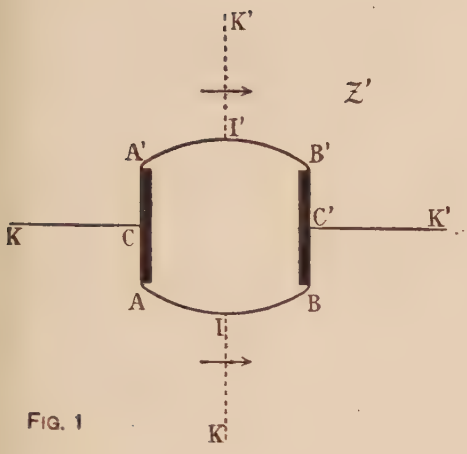

FIG. 1

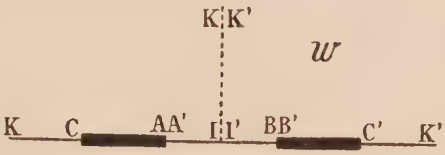

FIG. 4

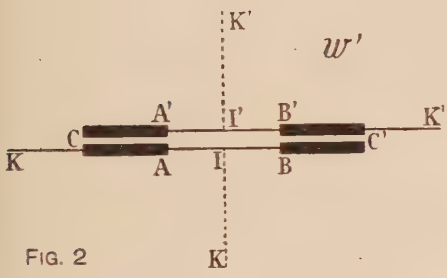

FIG. 2

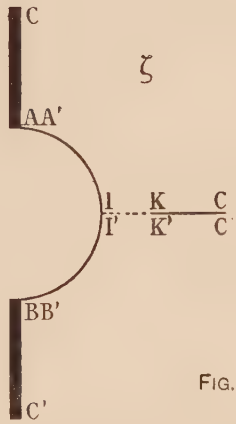

FIG. 5

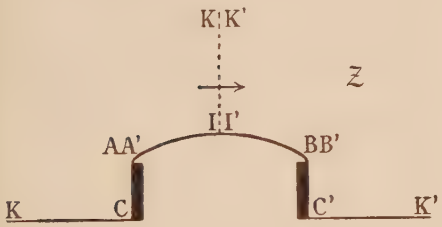

FIG. 3

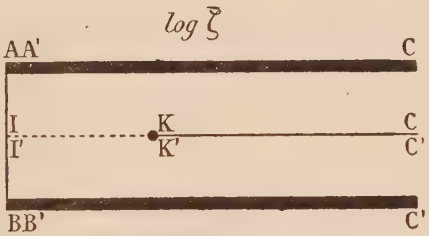

FIG. 6

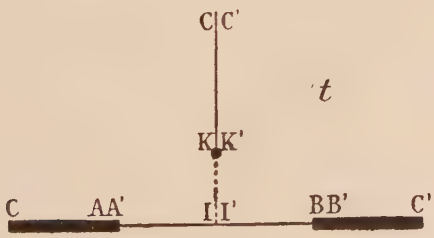

FIG. 7

on the plane. Fig. 8 shows the distance which ought to exist between the two planes, in order that they may be mutually attracted by a force approximately equal to that which a stream of real fluid exerts on a thin plane.

Independently of the application to the problem of the resistance of real fluids, the problem which we propose to study has a certain interest from the point of view of pure analysis.

Let z' and w' stand for the complex variables $x+iy$ and $\phi+i\psi$. The planes of these variables are represented in Figs. 1 and 2. Corresponding points are denoted by the same letters.

Let us make the following conventions: — We shall designate the part $KCAIBC'$ of the stream line $\psi=0$ by $\psi=-0$, and the part $CA'I'B'C'K'$ by $\psi=+0$, and assume that

$$\left(\frac{\psi}{|\psi|}\right)_{\psi=+0}=1 \quad \text{and} \quad \left(\frac{\psi}{|\psi|}\right)_{\psi=-0}=-1.$$

These conventions being made, let us consider the half-planes (see Figs. 3 and 4) of the complex variables

$$z=x+i\,|y|=x+i\,\frac{\psi}{|\psi|}\,y,$$

$$w=\phi+i\,|\psi|=\phi+i\,\frac{\psi}{|\psi|}\,\psi.$$

It is easy to see that we can interpret these half-planes as the planes of the variables z' and w' bent double.

We must find the relation of dependence which exists between the variables z' and w', say

$$z'=\phi(w')=P(\phi,\psi)+iQ(\phi,\psi), \tag{1}$$

in order to be able afterwards to determine the functions

$$x=P(\phi,\psi),$$

$$y=Q(\phi,\psi).$$

Remarking that, n being any whole number,

$$\left(i\,\frac{\psi}{|\psi|}\right)^{4n}=1, \quad \left(i\,\frac{\psi}{|\psi|}\right)^{4n+1}=i\,\frac{\psi}{|\psi|}, \quad \left(i\,\frac{\psi}{|\psi|}\right)^{4n+2}=-1,$$

$$\left(i\,\frac{\psi}{|\psi|}\right)^{4n+3}=-i\,\frac{\psi}{|\psi|},$$

we may write

$$x+i\frac{\psi}{|\psi|}y = P(\phi,\psi)+i\frac{\psi}{|\psi|}Q(\phi,\psi) = F\left(\phi+i\frac{\psi}{|\psi|}\psi\right),$$

and consequently,

$$z = F(w). \tag{2}$$

Conversely, if we find an expression of the form (2), it will suffice, in passing to the formula (1), to replace $\frac{\psi}{|\psi|}$ by the unit.

We may apply to the complex variables

$$x+i\frac{\psi}{|\psi|}y \quad \text{and} \quad \phi+i\frac{\psi}{|\psi|}\psi,$$

the ordinary rules of the theory of the functions of a complex variable, but we must always take care to replace $\sqrt{(-1)}$ by $i\frac{\psi}{|\psi|}$.

One may write, *e.g.*, employing the usual notations,*

$$\xi = q_1\frac{dz}{dw} = \frac{q_1}{u-i\frac{\psi}{|\psi|}v} = \frac{q_1}{q}\left(\frac{u}{q}+i\frac{\psi}{|\psi|}\frac{v}{q}\right) = \frac{q_1}{q}e^{i(\psi/|\psi|)\theta},$$

$$\log\xi = \log\frac{q_1}{q}+i\frac{\psi}{|\psi|}\theta,$$

as is easy to verify directly.

The boundaries of corresponding areas in the planes of the variables ξ, $\log\xi$, and w respectively are indicated in the Figs. 5, 6, and 7. Employing the method of Schwarz and Christoffel, we may transform the area indicated in Fig. 6 into the upper half-plane of the auxiliary variable t (see Fig. 7) by the substitution

$$\log\xi = M\cosh^{-1}t+N,$$

$$t = \cosh\left(\frac{\log\xi-N}{M}\right).$$

In order to calculate the values of M and N, we assume that the points (A, A') and (B, B'), for which

$$\log\xi = \tfrac{1}{2}i\pi \quad \text{and} \quad \log\xi = -\tfrac{1}{2}i\pi,$$

* We employ the notations adopted in H. Lamb's *Hydrodynamics*, but we retain, in this paper, the primitive sign for the velocity-potential (φ), and we designate the velocity at the free surface by q_1.

respectively, correspond to the points $t = -1$ and $t = +1$. Thus we find that

$$N = -\tfrac{1}{2}i\pi, \quad M = 1;$$

and therefore

$$t = \cosh(\log\zeta + \tfrac{1}{2}i\pi) = \tfrac{1}{2}i\left(\zeta - \frac{1}{\zeta}\right). \tag{3}$$

At the free surface we have $q/q_1 = 1$, and consequently

$$t = -\sin\theta\,\frac{\psi}{|\psi|}. \tag{4}$$

It is easy to verify that, to transform the half-plane w into the half-plane t, we may put

$$t = \frac{w\tan\alpha}{\sqrt{(q_1^2 b^2 - w^2)}}, \tag{5}$$

$$w = \frac{q_1 bt}{\sqrt{(\tan^2\alpha + t^2)}}. \tag{6}$$

The constant $\tan\alpha$ is of zero dimension; it determines the relation of the breadth l of the planes to the distance h which separates them. The constant b determines the unit of length, and the constant q_1 the unit of velocity.

Equating the second parts of the expressions (3) and (5), replacing ζ by $q_1\,dz/dw$, integrating and replacing $\psi/|\psi|$ by unity, we find the relation sought for between the variables w' and z'.

We may also easily obtain the equation of the "free" stream-line by utilizing the argument which is ordinarily applied in these cases. On this stream-line we have $d\phi/ds = q_1$, and, consequently, measuring the length of the arc S from the point II', we may put $\phi = q_1 s$. Substituting this value of ϕ in the equation (6), putting $\psi = 0$, and taking into consideration (4), we obtain, as the intrinsic equation of the free stream-line,

$$S = -\frac{b\sin\theta}{\sqrt{(\tan^2\alpha + \sin^2\theta)}}.$$

Integrating the expressions

$$dx = \cos\theta\,ds,$$

$$dy = \sin\theta\,ds,$$

we find

$$\left.\begin{aligned}\frac{x}{b} &= -\tfrac{1}{2}\frac{\sin 2\theta}{\sqrt{(\tan^2\alpha+\sin^2\theta)}}+\frac{\theta}{|\theta|}\left\{\sec\alpha E\left(\cos\alpha,\ \frac{\pi}{2}-|\theta|\right)\right.\\ &\qquad\left.-\sec\alpha\sin^2\alpha F\left(\cos\alpha,\ \frac{\pi}{\alpha}-|\theta|\right)-\frac{h}{2b}\right\}\\ \frac{y}{b} &= \frac{\tan^2\alpha}{\sqrt{(\tan^2\alpha+\sin^2\theta)}}-\frac{\tan^2\alpha}{\sqrt{(\tan^2\alpha+1)}}+\frac{l}{2b}\end{aligned}\right\}. \quad (7)$$

The angle θ varies from $+\frac{1}{2}\pi$ to $-\frac{1}{2}\pi$. The functions

$$F\left(\cos\alpha,\ \frac{\pi}{2}-|\theta|\right) \quad \text{and} \quad E\left(\cos\alpha,\ \frac{\pi}{2}-|\theta|\right)$$

are the elliptic integrals of Legendre of the first and second kind, viz.

$$F\left(\cos\alpha,\ \frac{\pi}{2}-|\theta|\right)=\int_0^{\frac{1}{2}\pi-|\theta|}\frac{d\phi}{\sqrt{(1-\cos^2\alpha\sin^2\phi)}},$$

$$E\left(\cos\alpha,\ \frac{\pi}{2}-|\theta|\right)=\int_0^{\frac{1}{2}\pi-|\theta|}\sqrt{(1-\cos^2\alpha\sin^2\phi)}\,d\phi,$$

and
$$\frac{h}{2b}=\sec\alpha\,(E'-\sin^2\alpha\,K'), \quad (8)$$

where
$$K'=\int_0^{\frac{1}{2}\pi}\frac{d\phi}{\sqrt{(1-\cos^2\alpha\sin^2\phi)}},$$

$$E'=\int_0^{\frac{1}{2}\pi}\sqrt{(1-\cos^2\alpha\sin^2\phi)}\,d\phi$$

are the complete complementary integrals.

In order to calculate the relation between $l/2b$ and α, we first observe that along CA' (see Fig. 5)

$$\zeta=\frac{q_1}{q}\,i\,\frac{\psi}{|\psi|},$$

and, consequently, in making this substitution in the formula (3), we have

$$t=-\tfrac{1}{2}\left(\frac{q_1}{q}+\frac{q}{q_1}\right),$$

$$\frac{q}{q_1}=-t-\sqrt{(t^2-1)}.$$

The minus sign is prefixed to the radical in order that we may have $q=0$

for $t = -\infty$. Now in integrating along CA' (see Fig. 3), we may write

$$l = 2\int_0^{\frac{1}{2}l} dy = 2\int_{-\infty}^{-1} \frac{dy}{d\phi}\frac{d\phi}{dt}\,dt = -2b\int_{-\infty}^{-1} \frac{[t-\sqrt{(t^2-1)}]\tan^2\alpha\,dt}{(\tan^2\alpha+t^2)^{\frac{3}{2}}},$$

and, completing the calculation, we find

$$\frac{l}{2b} = \sec\alpha(\sin^2\alpha+E-\cos^2\alpha\,K), \tag{9}$$

where E and K are complete elliptic integrals, viz.

$$E = \int_0^{\frac{1}{2}\pi} \sqrt{(1-\sin^2\alpha\sin^2\phi)}\,d\phi,$$

$$K = \int_0^{\frac{1}{2}\pi} \frac{d\phi}{\sqrt{(1-\sin^2\alpha\sin^2\phi)}}.$$

The ordinate y of the free stream-line is a maximum for $\theta = 0$. Denoting this maximum ordinate by $\frac{1}{2}l'$, we deduce from the formulæ (7) and (9) that

$$\frac{l'-l}{l} = \frac{\sin\alpha(1-\sin\alpha)}{\sin^2\alpha+E-\cos^2\alpha\,K}. \tag{10}$$

Let us now calculate the ratio of the velocity q_1 at the free surface to the velocity q_0 of the stream in terms of the parameter α. Putting in (5) $\psi = -0$, $\phi = -\infty$, we find

$$t = \left(\frac{\tan\alpha.\phi}{\sqrt{(q_1^2b^2-\phi^2)}}\right)_{\phi=-\infty} = \left(-\frac{i\frac{\psi}{|\psi|}\tan\alpha.\phi}{\sqrt{(\phi^2-q_1^2b^2)}}\right)_{\phi=-\infty} = i\tan\alpha.$$

Making this substitution in the formula (3) and noting that for $\psi = -0$, $\phi = -\infty$, we have $\theta = 0$ and $\xi = q_1/q_0$, we find

$$\frac{q_1}{q_0} = \tan\alpha+\sqrt{(\tan^2\alpha+1)} = \sqrt{\left(\frac{1+\sin\alpha}{1-\sin\alpha}\right)}. \tag{11}$$

The $+$ sign is taken for the radical, since q_1/q_0 is positive.

We may also calculate the suction behind the plane. We have

$$p_0+\tfrac{1}{2}\rho q_0^2 = p_1+\tfrac{1}{2}\rho q_1^2,$$

whence

$$\frac{p_0-p_1}{\rho q_0^2} = \frac{\sin\alpha}{1-\sin\alpha}. \tag{12}$$

The total pressure of the stream on the plane is

$$P = 2\int_0^{\frac{1}{2}l} (p-p_1)dy = \rho q_1^2 \int_0^{\frac{1}{2}l} \left(1-\frac{q^2}{q_1^2}\right) dy.$$

Taking into consideration the formulæ

$$\left(\frac{q}{q_1}\right)^2 = -\left(1+2t\frac{q}{q_1}\right),$$

$$t = \frac{\tan\alpha.\phi}{\sqrt{(q_1^2b^2-\phi^2)}},$$

$$\frac{dy}{d\phi} = \frac{1}{q},$$

and integrating along AC' (see Fig. 3), we obtain

$$\frac{P}{\rho q_0^2 l} = \frac{1+\sin\alpha}{1-\sin\alpha}\left(1+\frac{1}{l}\int_{-\infty}^{-1} 2t\frac{q}{q_1}\frac{dy}{d\phi}\frac{d\phi}{dt}\,dt\right)$$

$$= \frac{1+\sin\alpha}{1-\sin\alpha}\left(1-2\sin\alpha\tan\alpha\,\frac{b}{l}\right) = \frac{1+\sin\alpha}{1-\sin\alpha}\,\frac{E-\cos^2\alpha K}{\sin^2\alpha+E-\cos^2\alpha K}. \quad (13)$$

Noting that $$K = \frac{\pi}{2}\left[1+(\tfrac{1}{2})^2\sin^2\alpha+\left(\frac{1.3}{2.4}\right)^2\sin^4\alpha+\ldots\right],$$

$$E = \frac{\pi}{2}\left[1-(\tfrac{1}{2})^2\sin^2\alpha-\left(\frac{1.3}{2.4}\right)^2\frac{\sin^4\alpha}{3}-\ldots\right],$$

we can write $$\lim_{\alpha=0}\frac{E-\cos^2\alpha K}{\sin^2\alpha} = \tfrac{1}{4}\pi,$$

and consequently [see formula (9)]

$$\lim_{\alpha=0}\frac{l}{2b\sin^2\alpha} = \frac{\pi+4}{4},$$

and $$\lim_{\alpha=0}\frac{P}{\rho q_0^2 l} = \frac{\pi}{\pi+4}.$$

This ratio coincides with that of Kirchhoff. It corresponds to the zero value of the parameter α, which determines an infinite mutual receding of

the planes. To prove this, the formulæ (8) and (9) give

$$\frac{h}{l} = \frac{E' - \sin^2 \alpha K'}{\sin^2 \alpha + E - \cos^2 \alpha K}. \tag{14}$$

When α is sufficiently small, we write, omitting terms of a higher order,

$$\sin^2 \alpha + E - \cos^2 \alpha K = \frac{\pi+4}{4} \sin^2 \alpha,$$

$$E - K = -\frac{\pi}{4} \sin^2 \alpha.$$

From the latter of these expressions and Legendre's formula*

$$KE' + K'E - KK' = \frac{\pi}{2},$$

it results that $$\lim_{\alpha=0} \sin^2 \alpha K' = 0, \tag{15}$$

while the value of E' for $\alpha = 0$ is equal to unity. Thus, in fact, the value of the ratio h/l for $\alpha = 0$ is infinite.

Let us consider the second limiting case, where the two planes approach each other indefinitely. This case has only theoretical interest, for the velocities being infinitely great, we must assume the pressure at a distance from the plane also to be infinite.

Putting in the formula (15) $\alpha = \frac{1}{2}\pi - \alpha'$, we obtain

$$\lim_{\alpha'=\frac{1}{2}\pi} \cos^2 \alpha' K = 0.$$

It results from this remark and the formulæ (13) and (14) that

$$\lim_{\alpha=\frac{1}{2}\pi} \frac{P}{\rho q_0^2 l} = \infty, \qquad \lim_{\alpha=\frac{1}{2}\pi} \frac{h}{l} = 0.$$

We may also note that for $\alpha = \frac{1}{2}\pi$,

$$\lim_{\alpha=\frac{1}{2}\pi} \left(\frac{P}{\rho q_0^2 l} : \frac{p_0 - p_1}{\rho q_0^2} \right) = 1.$$

In the table below we have calculated for various values of the parameter α, the corresponding values of the ratio h/l from the formula (14), of the ratio $(l_1 - l)\, l$ from the formula (10), of the ratio q_1/q_0 from the

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* *Exercices de calcul intégral*, Paris, 1811, t. 1, p. 61.

formula (11), of the coefficients $P/\rho q_0^2 l$ and $(p_0-p_1)/\rho q_0^2$ from the formulæ (12) and (13).

α^0	$\frac{h}{l}$	$\frac{l'-l}{l}$	$\frac{q_1}{q_0}$	$\frac{P}{\rho q_0^2 l}$	$\frac{p_0-p_1}{\rho q_0^2}$
0	∞	∞	1·00	0·44	0
10	17·58	2·660	1·19	0·62	0·21
20	3·956	1·082	1·43	0·90	0·52
30	1·490	0·555	1·73	1·33	1·00
40	0·668	0·307	2·14	2·07	1·80
50	0·314	0·164	2·75	3·51	3·27
60	0·141	0·082	3·73	6·61	6·46
70	0·054	0·033	5·68	15·6	15·6
80	0·012	0·008	11·4	64·4	64·8
90	0	0	∞	∞	∞

When the parameter $\alpha = 10°$, the value of the ratio $[l(p_0-p_1) : P]$ is 0·339. An approximately similar relation exists between the coefficients of the suction and the total resistance for a thin plane held normally in a

FIG. 8.

moving fluid. Putting in the formula (7), $\alpha = 10°$, we have calculated the corresponding free stream-lines. These curves are shown in Fig. 8.

PERMUTATIONS, LATTICE PERMUTATIONS, AND THE HYPERGEOMETRIC SERIES

By Major P. A. MacMahon.

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1. An assemblage of letters

$$x_1^{m_1+m_2+\ldots+m_n}\, x_2^{m_2+m_3+\ldots+m_n} \ldots x_{n-1}^{m_{n-1}+m_n}\, x_n^{m_n}$$

in which the quantities $m_1, m_2, \ldots, m_n$ may each be zero or any positive integer is termed a Lattice Assemblage with respect to the ordered system of letters $x_1, x_2, \ldots, x_n$. The reason for this nomenclature is that the assemblage may be denoted graphically by a regular lattice of nodes.

A permutation of this lattice assemblage is defined to be a Lattice Permutation if on drawing a line between *any two* letters of the permutation the assemblage to the left of the line is a lattice assemblage.

The essential characteristic of a lattice assemblage is that the repetitional exponents are in descending order of magnitude. Lattice permutations are of great importance in certain theories in Combinatory Analysis and have been studied by the writer.*

It is convenient to denote by

$$P\,(m_1+m_2+\ldots+m_n,\ m_2+m_3+\ldots+m_n,\ \ldots,\ m_{n-1}+m_n,\ m_n),$$

$$LP\,(m_1+m_2+\ldots+m_n,\ m_2+m_3+\ldots+m_n,\ \ldots,\ m_{n-1}+m_n,\ m_n),$$

the number of permutations and the number of lattice permutations of the assemblage, respectively.

The general hypergeometric series

$$1+\frac{a_1 a_2 \ldots a_k}{1.b_2 \ldots b_k}\,x+\frac{a_1(a_1+1)\,a_2(a_2+1)\ldots a_k(a_k+1)}{1.2.b_2(b_2+1)\ldots b_k(b_k+1)}\,x^2+\ldots$$

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* *Phil. Trans. Roy. Soc.*, "Combinatory Analysis."

has been usually written

$$F\begin{pmatrix} a_1\, a_2\, a_3 \dots a_k \\ b_2\, b_3 \dots b_k \end{pmatrix} x\Big),$$

but from considerations of symmetry I prefer to write it in the present communication in the notation

$$F\begin{pmatrix} a_1\, a_2\, a_3 \dots a_k \\ 1\; b_2\, b_3 \dots b_k \end{pmatrix} x\Big),$$

which, in any case, appears to be in no need of justification.

Permutations and the Hypergeometric Series.

2. To establish a connexion between the permutations of

$$x_1^{m_1+m_2+\dots+m_n}\, x_2^{m_2+m_3+\dots+m_n} \dots x_{n-1}^{m_{n-1}+m_n}\, x_n^{m_n},$$

and the hypergeometric series, we may first note the easily verifiable relations

$$1+\binom{2}{1} x_1x_2+\binom{4}{2} x_1^2x_2^2+\dots+\binom{2s}{s} x_1^s x_2^s+\dots$$

$$= F\left\{\begin{matrix} \frac{1}{2}, & \frac{2}{2}, \\ 1, & 1, \end{matrix}\; 2^2x_1x_2\right\},$$

$$1+\frac{3!}{(1!)^3}\, x_1x_2x_3+\frac{6!}{(2!)^3}(x_1x_2x_3)^2+\dots+\binom{3s}{s}(x_1x_2x_3)^s+\dots$$

$$= F\left\{\begin{matrix} \frac{1}{3}, & \frac{2}{3}, & \frac{3}{3}, \\ 1, & 1, & 1, \end{matrix}\; 3^3x_1x_2x_3\right\},$$

and, in general, the relation

$$1+\frac{n!}{(1!)^n}\, x_1x_2\dots x_n+\frac{(2n)!}{(2!)^3}\,(x_1x_2\dots x_n)^2+\dots+\frac{(sn)!}{(s!)^n}(x_1x_2\dots x_n)^s+\dots$$

$$= F\left\{\begin{matrix} \frac{1}{n}, & \frac{2}{n}, & \frac{3}{n}, & \dots, & \frac{n-1}{n}, & \frac{n}{n}, \\ 1, & 1, & 1, & \dots, & 1, & 1, \end{matrix}\; n^n x_1x_2x_3\dots x_n\right\},$$

which yields the hypergeometric series which generates the numbers which enumerate the permutations of the assemblage

$$(x_1 x_2 \dots x_n)^s$$

for all values of s.

It may be written

$$\sum_{s=0}^{\infty} P(s, s, \dots, s)(x_1 x_2 \dots x_n)^s = F\left\{\begin{matrix} \frac{1}{n}, & \frac{2}{n}, & \dots, & \frac{n}{n}, \\ 1, & 1, & \dots, & 1, \end{matrix} \; n^n x_1 x_2 \dots x_n\right\}.$$

To generalise by finding an expression for

$$\Sigma P(m_1+m_2+\dots+m_n,\ m_2+m_3+\dots+m_n,\ \dots,\ m_{n-1}+m_n,\ m_n) x_1^{m_1+\dots+m_n} \dots x_n^{m_n},$$

we note the relations

$$\sum_{m_2=0}^{\infty} P(m_1+m_2,\ m_2)\, x_1^{m_1+m_2} x_2^{m_2} = x_1^{m_1} F\left\{\begin{matrix} \frac{m_1+1}{2}, & \frac{m_1+2}{2}, \\ 1, & m_1+1, \end{matrix} \; 2^2 x_1 x_2\right\}$$

$$\sum_{m_3=0}^{\infty} P(m_1+m_2,\ m_2+m_3,\ m_3)\, x_1^{m_1+m_2+m_3} x_2^{m_2+m_3} x_3^{m_3}$$

$$= P(m_1+m_2,\ m_2) x_1^{m_1+m_2} x_2^{m_2}$$

$$\times F\left\{\begin{matrix} \frac{m_1+2m_2+1}{3}, & \frac{m_1+2m_2+2}{3}, & \frac{m_1+2m_2+3}{3}, \\ 1, & m_2+1, & m_1+m_2+1, \end{matrix} \; 3^3 x_1 x_2 x_3\right\},$$

which are verifiable without difficulty.

In fact the coefficient of $x_1^{m_1+m_2+m_3} x_2^{m_2+m_3} x_3^{m_3}$ on the right-hand side is

$$\frac{(m_1+2m_2)!}{(m_1+m_2)!\, m_2!} \cdot \frac{(m_1+2m_2+3m_3)!}{(m_1+2m_2)!} \div \frac{m_3!\,(m_2+m_3)!\,(m_1+m_2+m_3)!}{m_2!\,(m_1+m_2)!},$$

which is $P(m_1+m_2+m_3,\ m_2+m_3,\ m_3)$,

in agreement with the left-hand side.

Guided by these relations we may conjecture a general formula and proceed to its verification.

Write
$$m_1+m_2+\ldots+m_n = m_{1,n},$$
$$m_2+\ldots+m_n = m_{2,n},$$
$$\ldots \quad \ldots \quad \ldots$$
$$m_{n-1}+m_n = m_{n-1,n},$$
$$m_n = m_{n,n},$$
$$m_1+2m_2+3m_3+\ldots+(n-1)\,m_{n-1} = M_{n-1},$$
and
$$\sum_{m_{n,n}=0}^{\infty} P(m_{1,n}, m_{2,n}, \ldots, m_{n-1,n}, m_{n,n})\, x_1^{m_{1,n}} x_2^{m_{2,n}} \ldots x_{n-1}^{m_{n-1,n}} x_n^{m_{n,n}}$$
$$= P(m_{1,n-1}, m_{2,n-1}, \ldots, m_{n-2,n-1}, m_{n-1,n-1})\, x_1^{m_{1,n-1}} x_2^{m_{2,n-1}} \ldots x_{n-2}^{m_{n-2,n-1}} x_{n-1}^{m_{n-1,n-1}}$$
$$F\left\{\begin{matrix} \frac{1}{n}(M_{n-1}+1), & \frac{1}{n}(M_{n-1}+2), & \frac{1}{n}(M_{n-1}+3), & \ldots, & \frac{1}{n}(M_{n-1}+n), \\ 1, & m_{n-1,n-1}+1, & m_{n-2,n-1}+1, & \ldots, & m_{1,n-1}+1 \end{matrix}\; n^n x_1 x_2 \ldots x_n\right\}$$
wherein $m_1, m_2, \ldots, m_{n-1}$ may be each zero or any positive integer, and the summation on the left is in regard to m_n ($\equiv m_{n,n}$) from zero to infinity. It suffices to show that the coefficients of
$$x_1^{m_{1,n}} x_2^{m_{2,n}} \ldots x_{n-1}^{m_{n-1,n}} x_n^{m_{n,n}}$$
agree on the two sides.

On the right-hand side we have to pick out the coefficients of
$$(x_1 x_2 \ldots x_n)^{m_{n,n}}$$
in the hypergeometric series. This is
$$\frac{M_n!}{M_{n-1}!}\;\frac{m_{1,n-1}!\, m_{2,n-1}! \ldots m_{n-1,n-1}!}{m_{1,n}!\, m_{2,n}! \ldots m_{n,n}!},$$
and now multiplication by
$$P(m_{1,n-1}, m_{2,n-1}, \ldots, m_{n-1,n-1}) \equiv \frac{M_{n-1}!}{m_{1,n-1}!\, m_{2,n-1}! \ldots m_{n-1,n-1}!},$$
gives
$$\frac{M_n!}{m_{1,n}!\, m_{2,n}! \ldots m_{n,n}!} \equiv P(m_{1,n}, m_{2,n}, \ldots, m_{n,n}).$$

Hence the conjectured theorem is established.

Lattice Permutations and the Hypergeometric Series.

3. It has been established (*loc. cit.*) that the number of lattice permutations of the assemblage

$$x_1^{m_1+m_2+\ldots+m_n}\, x_2^{m_2+m_3+\ldots+m_n} \ldots x_{n-1}^{m_{n-1}+m_n}\, x_n^{m_n},$$

which, in the above notation, may be written

$$x_1^{m_{1,n}}\, x_2^{m_{2,n}} \ldots x_{n-1}^{m_{n-1,n}}\, x_n^{m_{n,n}},$$

is $LP(m_{1,n}, m_{2,n}, \ldots, m_{n-1,n}, m_{n,n})$

$$= \frac{M_n!\, \Pi_{a,b}\,(m_a+m_{a+1}+\ldots+m_{b-1}+b-a)}{(m_{1,n}+n-1)!\,(m_{2,n}+n-2)! \ldots (m_{n-1,n})!\, m_{n,n}!},$$

where the product Π is for every pair of integers drawn from $1, 2, 3, \ldots, n$ such that $b > a$.

From this formula we find by putting

$$m_1 = m_2 = \ldots = m_{n-1} = 0 \quad \text{and} \quad m_n = s,$$

$$LP(s) = 1,$$

$$LP(s, s) = \frac{(2s)!\ 1!}{(s+1)!\, s!},$$

$$LP(s, s, s) = \frac{(3s)!\quad 2!\quad 1!}{(s+2)!\,(s+1)!\, s!},$$

$$\ldots \qquad \ldots \qquad \ldots \qquad \ldots \qquad \ldots$$

$$LP(s, s, \ldots, s) = \frac{(ns)!\quad (n-1)!\quad (n-2)!\quad \ldots\quad 2!\quad 1!}{(s+n-1)!\,(s+n-2)!\,(s+n-3)! \ldots (s+1)!\, s!},$$

and at once verify the relations

$$\sum_{s=0}^{\infty} LP(s, s)(x_1x_2)^s = F\left\{\begin{matrix}\frac{1}{2}, & \frac{2}{2}, & \\ 1, & 2, & \end{matrix}\ 2^2x_1x_2\right\},$$

$$\sum_{s=0}^{\infty} LP(s, s, s)(x_1x_2x_3)^s = F\left\{\begin{matrix}\frac{1}{3}, & \frac{2}{3}, & \frac{3}{3}, & \\ 1, & 2, & 3, & \end{matrix}\ 3^3x_1x_2x_3\right\},$$

$$\sum_{s=0}^{\infty} LP(s, s, \ldots, s)(x_1x_2 \ldots x_n)^s = F\left\{\begin{matrix}\frac{1}{n}, & \frac{2}{n}, & \ldots, & \frac{n}{n}, & \\ 1, & 2, & \ldots, & n, & \end{matrix}\ n^n x_1x_2 \ldots x_n\right\},$$

which are for comparison with the analogous relations of § 2.

We further verify the relations

$$\sum_{m_2=0}^{\infty} LP(m_1+m_2,\ m_2)\, x_1^{m_1+m_2} x_2^{m_2}$$

$$= x^{m_1} F\left\{\begin{matrix} \tfrac{1}{2}(m_1+1),\ \tfrac{1}{2}(m_1+2), \\ 1, \qquad m_1+2, \end{matrix}\ 2^2 x_1 x_2\right\}$$

$$\sum_{m_3=0}^{\infty} LP\,(m_1+m_2+m_3,\ m_2+m_3,\ m_3)\, x_1^{m_1+m_2-m_3} x_2^{m_2+m_3} x_3^{m_3}$$

$$= LP\,(m_1+m_2,\ m_3)\, x_1^{m_1+m_2} x_2^{m_2}$$

$$\times F\left\{\begin{matrix} \tfrac{1}{3}(m_1+2m_2+1),\ \tfrac{1}{3}(m_1+2m_2+2),\ \tfrac{1}{3}(m_1+2m_2+3), \\ 1, \qquad\qquad m_2+2, \qquad\qquad m_1+m_2+3, \end{matrix}\ 3^3 x_1 x_2 x_3\right\},$$

suggesting the general relation

$$\sum_{m_n=0}^{\infty} LP(m_{1,n},\ m_{2,n},\ \ldots,\ m_{n,n})\, x_1^{m_{1,n}} x_2^{m_{2,n}} \ldots x_n^{m_{n,n}}$$

$$= LP\,(m_{1,n-1},\ m_{2,n-1},\ \ldots,\ m_{n-1,n-1})\, x_1^{m_{1,n-1}} x_2^{m_{2,n-1}} \ldots x_{n-1}^{m_{n-1,n-1}}$$

$$\times F\left\{\begin{matrix} \frac{1}{n}(M_{n-1}+1),\ \frac{1}{n}(M_{n-1}+2),\ \ldots,\ \frac{1}{n}(M_{n-1}+n), \\ 1, \qquad m_{n-1,n-1}+2,\ \ldots,\ m_{1,n-1}+n, \end{matrix}\ n^n x_1 x_2 \ldots x_n\right\},$$

where the lower row of terms in the series is

$$1,\ m_{n-1,n-1}+2,\ m_{n-2,n-1}+3,\ m_{n-3,n-1}+4,\ \ldots,\ m_{2,n-1}+n-1,\ m_{1,n-1}+n.$$

To prove this we find that the coefficient of $(x_1 x_2 \ldots x_n)^{m_n}$ in the hypergeometric series is

$$\frac{M_n!}{M_{n-1}!}\ \frac{(m_{n-1,n-1}+1)!\,(m_{n-2,n-1}+2)! \ldots (m_{1,n-1}+n-1)!}{m_{n,n}!\,(m_{n-1,n}+1)!\,(m_{n-2,n}+2)! \ldots (m_{1,n}+n-1)!},$$

and

$$\frac{LP\,(m_{1,n},\ m_{2,n},\ \ldots,\ m_{n,n})}{LP\,(m_{1,n-1},\ m_{2,n-1},\ \ldots,\ m_{n-1,n-1})}$$

$$= \frac{M_n!\ \Pi_{a,b}(m_a+\ldots+m_{b-1}+b-a)}{(m_{1,n}+n-1)!\,(m_{2,n}+n-2)! \ldots m_{n,n}!}$$

$$\times \frac{(m_{1,n-1}+n-2)!\,(m_{2,n-1}+n-2)! \ldots m_{n-1,n-1}!}{M_{n-1}!\ \Pi'_{a,b}(m_a+\ldots+m_{b-1}+b-a)},$$

where in the product Π' in the denominator the numbers a, b are selected from the series 1, 2, 3, ..., $n-1$, and not from 1, 2, 3, ..., n.

Hence

$$\frac{\Pi_{a,\,b}(m_a+\ldots+m_{b-1}+b-a)}{\Pi'_{a,\,b}(m_a+\ldots+m_{b-1}+b-a)}$$

$$= (m_{1,\,n-1}+n-1)(m_{2,\,n-1}+n-2)\ \ldots\ (m_{n-1,\,n-1}+1),$$

leading to

$$\frac{LP(m_{1,\,n},\ m_{2,\,n},\ \ldots,\ m_{n,\,n})}{LP(m_{1,\,n-1},\ m_{2,\,n-1},\ \ldots,\ m_{n-1,\,n-1})}$$

$$= \frac{M_n!}{M_{n-1}!}\,\frac{(m_{n-1,\,n-1}+1)!\,(m_{n-2,\,n-1}+2)!\ \ldots\ (m_{1,\,n-1}+n-1)!}{m_{n,\,n}!\,(m_{n-1,\,n}+1)!\,(m_{n-2,\,n}+2)!\ \ldots\ (m_{1,\,n}+n-1)!}$$

$$= \text{coefficient of } (x_1x_2\ \ldots\ x_n)^{m_{n,\,n}}$$

in the hypergeometric series.

This establishes the relation.

The result of the investigation is the two theorems :—

$$\sum_{m_{n,\,n}=0}^{\infty} P(m_{1,\,n},\ m_{2,\,n},\ \ldots,\ m_{n,\,n})\ x_1^{m_{1,\,n}}\,x_2^{m_{2,\,n}}\ \ldots\ x_n^{m_{n,\,n}}$$

$$= P(m_{1,\,n-1},\ m_{2,\,n-1},\ \ldots,\ m_{n-1,\,n-1})\ x_1^{m_{1,\,n-1}}\,x_2^{m_{2,\,n-1}}\ \ldots\ x_{n-1}^{m_{n-1,\,n-1}}$$

$$\times F\left\{\begin{matrix}\frac{1}{n}(M_{n-1}+1),\ \frac{1}{n}(M_{n-1}+2),\ \frac{1}{n}(M_{n-1}+3),\ \ldots,\ \frac{1}{n}(M_{n-1}+n), \\ 1,\qquad m_{n-1,\,n-1}+1,\ m_{n-2,\,n-1}+1,\ \ldots,\ m_{1,\,n-1}+1,\end{matrix}\quad n^n x_1x_2\ldots x_n\right\}$$

$$\sum_{m_{n,\,n}=0}^{\infty} LP(m_{1,\,n},\ m_{2,\,n},\ \ldots,\ m_{n,\,n})\ x_1^{m_{1,\,n}}\,x_2^{m_{2,\,n}}\ \ldots\ x_n^{m_{n,\,n}}$$

$$= LP(m_{1,\,n-1},\ m_{2,\,n-1},\ \ldots,\ m_{n-1,\,n-1})\ x_1^{m_{1,\,n-1}}\,x_2^{m_{2,\,n-1}}\ \ldots\ x_{n-1}^{m_{n-1,\,n-1}}$$

$$\times F\left\{\begin{matrix}\frac{1}{n}(M_{n-1}+1),\ \frac{1}{n}(M_{n-1}+2),\ \frac{1}{n}(M_{n-1}+3),\ \ldots,\ \frac{1}{n}(M_{n-1}+n), \\ 1,\qquad m_{n-1,\,n-1}+2,\ m_{n-2,\,n-1}+3,\ \ldots,\ m_{1,\,n-1}+n,\end{matrix}\quad n^n x_1x_2\ldots x_n\right\}$$

where

$$m_{1,\,n} = m_1+m_2+\ldots+m_n,$$

$$m_{2,\,n} = m_2+m_3+\ldots+m_n,$$

$$\ldots\qquad\ldots\qquad\ldots\qquad\ldots$$

$$m_{n,\,n} = m_n,$$

$$M_{n-1} = m_1+2m_2+3m_3+\ldots+(n-1)\,m_{n-1},$$

and $P(m_{1,n}, m_{2,n}, \ldots, m_{n,n}), \quad LP(m_{1,n}, m_{2,n}, \ldots, m_{n,n}),$

denote the numbers of permutations and lattice permutations, respectively, of the assemblage of letters

$$x_1^{m_{1,n}} x_2^{m_{2,n}} \ldots x_n^{m_{n,n}}.$$

The two relations only differ in the lower rows of parameters of the hypergeometric series, and then only in a simple and interesting manner. In both $m_1, m_2, \ldots, m_{n-1}$ may be each zero or any positive integer.

4. If $$(1-x_1-x_2-\ldots-x_n)^{-1}$$

be expanded a certain portion will involve lattice assemblages of the letters *quâ* the ordered letters

$$x_1, \ x_2, \ \ldots, \ x_n.$$

This portion we will denote by

$$L\,(1-x_1-x_2-\ldots-x_n)^{-1}.$$

The connexion between this lattice portion of the function and the hypergeometric series is manifest from the investigation of § 2.

Thus for $n = 2$,

$$L(1-x_1-x_2)^{-1}$$

$$= F\left\{\begin{matrix}\frac{1}{2}, & \frac{2}{2}, \\ 1, & 1,\end{matrix}\ 2^2 x_1 x_2\right\}$$

$$+x_1 F\left\{\begin{matrix}\frac{2}{2}, & \frac{3}{2}, \\ 1, & 2,\end{matrix}\ 2^2 x_1 x_2\right\}$$

$$+\ldots$$

$$+x_1^{m_1} F\left\{\begin{matrix}\frac{m_1+1}{2}, & \frac{m_1+2}{2}, \\ 1, & m_1+1,\end{matrix}\ 2^2 x_1 x_2\right\}$$

$$+\ldots,$$

and since $(1-x_1-x_2)^{-1}$ is a symmetric function we may further deduce that

$$(1-x_1-x_2)^{-1}$$

$$= F\left\{\begin{matrix}\frac{1}{2}, & \frac{2}{2}, \\ 1, & 1,\end{matrix}\ 2^2x_1x_2\right\}$$

$$+(x_1+x_2)F\left\{\begin{matrix}\frac{2}{2}, & \frac{3}{2}, \\ 1, & 2,\end{matrix}\ 2^2x_1x_2\right\}$$

$$+\ldots$$

$$+(x_1^{m_1}+x_2^{m_2})F\left\{\begin{matrix}\frac{m_1+1}{2}, & \frac{m_1+2}{2}, \\ 1, & m_1+1,\end{matrix}\ 2^2x_1x_2\right\}$$

$$+\ldots.$$

Again, for $n=3$,

$$L(1-x_1-x_2-x_3)^{-1}$$

$$= \sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty} P(m_1+m_2,\ m_2)\, x_1^{m_1+m_2}x_2^{m_2}$$

$$\times F\left\{\begin{matrix}\frac{1}{3}(m_1+2m_2+1), & \frac{1}{3}(m_1+2m_2+2), & \frac{1}{3}(m_1+2m_2+3), \\ 1, & m_2+1, & m_1+m_2+1,\end{matrix}\ 3^3x_1x_2x_3\right\},$$

$$(1-x_1-x_2-x_3)^{-1}$$

$$= \sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty}\Big[P(m_1+m_2,\ m_2)\,(\Sigma x_1^{m_1+m_2}x_2^{m_2})$$

$$\times F\left\{\begin{matrix}\frac{1}{3}(m_1+2m_2+1), & \frac{1}{3}(m_1+2m_2+2, & \frac{1}{3}(m_1+2m_2+3), \\ 1, & m_2+1, & m_1+m_2+1,\end{matrix}\ 3^3x_1x_2x_3\right\}\Big].$$

The general formulæ are

$$L(1-x_1-x_2-\ldots-x_n)^{-1}$$

$$= \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \ldots \sum_{m_{n-1}=0}^{\infty} P(m_{1,n-1}, m_{2,n-1}, \ldots, m_{n-1,n-1}) x_1^{m_{1,n-1}} x_2^{m_{2,n-1}} \ldots x_{n-1}^{m_{n-1,n-1}}$$

$$F\left\{\begin{matrix} \frac{1}{n}(M_{n-1}+1), & \frac{1}{n}(M_{n-1}+2), & \frac{1}{n}(M_{n-1}+3), & \ldots, & \frac{1}{n}(M_{n-1}+n), \\ 1, & m_{n-1,n-1}+1, & m_{n-2,n-1}+1, & \ldots, & m_{1,n-1}+1, \end{matrix} \quad n^n x_1 x_2 \ldots x_n \right\},$$

$$(1-x_1-x_2-\ldots-x_n)^{-1}$$

$$= \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \ldots \sum_{m_{n-1}=0}^{\infty} \left[P(m_{1,n-1}, m_{2,n-1}, \ldots, m_{n-1,n-1}) (\Sigma x_1^{m_{1,n-1}} x_2^{m_{2,n-1}} \ldots x_{n-1}^{m_{n-1,n-1}}) \right.$$

$$\left. \times F\left\{\begin{matrix} \frac{1}{n}(M_{n-1}+1), & \frac{1}{n}(M_{n-1}+2), & \frac{1}{n}(M_{n-1}+3), & \ldots, & \frac{1}{n}(M_{n-1}+n), \\ 1, & m_{n-1,n-1}+1, & m_{n-2,n-1}+1, & \ldots, & m_{1,n-1}+1, \end{matrix} \quad n^n x_1 x_2 \ldots x_n \right\} \right].$$

5. Similarly if we expand

$$(1-x_1-x_2-\ldots-x_n)^{-1} \Pi_{a,b} \left(1-\frac{x_b}{x_a}\right),$$

where the product has reference to every pair of letters x_a, x_b subject to the condition $b > a$, the coefficient of a lattice assemblage of letters enumerates the lattice permutations of such assemblage (*loc. cit.*).

The portion of the expansion which involves lattice assemblages *quâ* the ordered letters $x_1, x_2, \ldots, x_n$ is here denoted by

$$L(1-x_1-x_2-\ldots-x_n)^{-1} \Pi_{a,b} \left(1-\frac{x_b}{x_a}\right).$$

The investigation of § 3 shows that

$$L\frac{1-\frac{x_2}{x_1}}{1-x_1-x_2}$$

$$= F\left\{\begin{matrix}\frac{1}{2}, & \frac{2}{2}, \\ 1, & 2,\end{matrix}\ 2^2x_1x_2\right\}$$

$$+x_1F\left\{\begin{matrix}\frac{2}{2}, & \frac{3}{2}, \\ 1, & 3,\end{matrix}\ 2^2x_1x_2\right\}$$

$$+x_1^2F\left\{\begin{matrix}\frac{3}{2}, & \frac{4}{2}, \\ 1, & 4,\end{matrix}\ 2^2x_1x_2\right\}$$

$$+\ldots$$

$$+x_1^{m_1}F\left\{\begin{matrix}\frac{m_1+1}{2}, & \frac{m_1+2}{2}, \\ 1, & m_1+2,\end{matrix}\ 2^2x_1x_2\right\}$$

$$+\ldots;$$

but we cannot obtain an expression for

$$(1-x_1-x_2)^{-1}\left(1-\frac{x_2}{x_1}\right),$$

because it is neither symmetrical nor integral.

Similarly
$$L\frac{\left(1-\frac{x_2}{x_1}\right)\left(1-\frac{x_3}{x_1}\right)\left(1-\frac{x_3}{x_2}\right)}{1-x_1-x_2-x_3}$$

$$=\sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty}LP(m_1+m_2,\ m_2)\,x_1^{m_1+m_2}x_2^{m_2}$$

$$\times F\left\{\begin{matrix}\frac{1}{3}(m_1+2m_2+1), & \frac{1}{3}(m_1+2m_2+2), & \frac{1}{3}(m_1+2m_2+3), \\ 1, & m_2+2, & m_1+m_2+3,\end{matrix}\ 3^3x_1x_2x_3\right\}.$$

The general theorem is

$$L\,\frac{\Pi_{a,\,b}\left(1-\dfrac{x_b}{x_a}\right)}{1-x_1-x_2-\ldots-x_n}$$

$$=\Sigma\Sigma\ldots\Sigma\,LP(m_{1,\,n-1},\ m_{2,\,n-1},\ \ldots,\ m_{n-1,\,n-1})\,x_1^{m_{1,\,n-1}}x_2^{m_{2,\,n-1}}\ldots x_{n-1}^{m_{n-1,\,n-1}}$$

$$F\left\{\begin{matrix}\frac{1}{n}(M_{n-1}+1),\ \frac{1}{n}(M_{n-1}+2),\ \frac{1}{n}(M_{n-1}+3),\ \ldots,\ \frac{1}{n}(M_{n-1}+n), & \\ 1,\quad m_{n-1,\,n-1}+2,\ m_{n-2,\,n-1}+3,\ \ldots,\ m_{1,\,n-1}+n, & n^n x_1x_2\ldots x_n\end{matrix}\right\}.$$

SOME CONSIDERATIONS ON THE GENERAL THEORY OF RULED SURFACES

By C. V. Hanumanta Rao.

[Read June 12th, 1919.]

One of the principal problems in the study of ruled surfaces is the problem of classification, and the classification by Schwarz* of the possible types of quintic scrolls is the model. His method may be said to consist in a combination of the principle of correspondence between two plane curves and the principle of the irreducibility of the plane section of a scroll. But this method is extremely laborious when the order of the surface is fairly high; for instance, there appear to be more than sixty types of rational sextic scrolls alone. On the other hand, the writings of Segre† on the rational and elliptic scrolls suggest a different method. Here we first analyse the nature of the general scroll of given genus and order when it is situated in its normal space, and consider the various ways of generating it by means of a (1, 1)-correspondence between two hyperplanar sections. We then project the surface from the proper number of arbitrary points into a space of three dimensions. By varying the origin of projection we thereby obtain all possible scrolls of the given genus and given order in ordinary space. Two scrolls in ordinary space that are both obtainable from the same scroll in higher space by means of uniform projections are in homographic correspondence; we therefore obtain by the above method all possible types of scrolls in space of three dimensions that are distinct so far as a homographic transformation is concerned.

The present paper begins with some elementary properties of ruled surfaces, and I proceed to prove that any scroll normal in ordinary space necessarily has two directrices, distinct or coincident, a directrix being a line met by every generator of the surface. From this I deduce some properties of the general scroll situated in its normal space; for instance,

* *Complete Works*, Vol. 2 (1890), pp. 25–64.

† *Atti Torino*, Vol. 19 (1883), pp. 355–372; Vol. 21 (1885), pp. 868–891.

such a surface has no singular curve properly so called. I proceed to give a short treatment of the rational and elliptic scrolls following Segre's work. This is followed by a discussion, based on the same lines, of the general scroll of genus 2; and I thus give a classification of the possible types of sextic scrolls from the point of view mentioned above.

The possibility of extending the above method to scrolls of higher genus or greater order would necessarily depend on an examination of the normal space of a given scroll in terms of its order and genus. Segre* has, in fact, obtained by means of the Zeuthen principle of correspondence a lower limit for the normal space in terms of the order and the genus of the surface. A straightforward discussion of the question with the aid of adjoint surfaces leads to a result which is practically the same as Segre's result. Then follows a result which I owe to Prof. Baker which determines the normal space of a scroll with two directrices in terms of the characteristic numbers of an arbitrary plane section.

Indicating by S_ρ the normal space for the scroll, and by S_r the normal space of its hyperplane section, the difference

$$\delta = r-(\rho-1),$$

is called the deficiency in freedom of the characteristic series of the system of hyperplane sections. It is known that δ is always less than or equal to p the genus.† I here prove that it is not greater than $p-i$, where i is the index of speciality of the plane section. Moreover, when p is not greater than $n-3$, where n is the order of the ruled surface, we can always construct a scroll for which δ is as low as one, provided p is not zero.

There is a further question of interest connected with ruled surfaces. Given any twisted curve, under what circumstances can we transform it birationally into a plane curve of the same order which can be made the base of a ruled surface? The interest as well as the difficulty of the question arises from the condition that the curve is to be of the same order. I have not been able to give a complete answer, but I discuss some particular classes of curves, and prove that such a transformation is possible in the case of all twisted curves of order less than or equal to eight.

Again I thank Prof. Baker for the help and criticism I have received from him.

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* *Math. Annalen*, Vol. 34 (1890), p. 4.

† Castelnuovo, *Annali di Math.*, Vol. 25 (1897), p. 292.

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1. A singly infinite aggregate of straight lines is said to constitute a ruled surface, and the lines are called generators. Cones and cylinders are degenerate forms and their study differs in no way from that of plane curves. When every two consecutive generators intersect, the surface becomes a developable surface. The term scroll is limited to such ruled surfaces as have only one generator passing through each general point of an arbitrary plane section. Any two plane sections of a scroll have then the same genus, for the generators effect a (1, 1)-correspondence between them. The genus of the scroll is defined as the genus of an arbitrary plane section. This affords a means of classifying the scrolls of any given order according to their genus.

The tangent plane at any point of a generator passes through the generator, and varies as the point moves along the generator.* Conversely every plane passing through a generator touches the surface at some point on the generator. Thus the number of tangent planes through an arbitrary line is the same as the number of generators meeting the given line. Hence the class of the scroll equals its order; in this respect a scroll is like a quadric. It is also evident that the reciprocal surface is likewise a scroll, and has the same genus as the original surface.

2. The following is a well known result in the theory of correspondence. Given in space two curves of orders m and n, of the same genus,

* Cf. Salmon, *Solid Geometry*, Vol. 2 (1915), §§ 456, 463.

and having the same moduli, the lines joining the corresponding points of a (1, 1)-correspondence between them generate a ruled surface of order $m+n$. If the curves have k self-corresponding points, the order of the surface is diminished by k.

Any plane passing through a generator of a scroll F_n^p* meets the surface in a residual curve of order $n-1$ met by the generator in $n-1$ points. Each of these is in a sense a double point for the plane section, but only one of them is the point of contact of the plane with the surface. The remaining $n-2$ points are fixed points on the generator not varying as the plane moves about the line. For† consider any two planes through the generator. Each of these contains a residual C_{n-1}. These two curves are placed in (1, 1)-correspondence by the generators, and the generators constitute a scroll of order n. Hence in virtue of the result cited above, the two curves must have $n-2$ common self-corresponding points. But the common points of the two curves must lie on the line common to the two planes, and that is the generator. These are then $n-2$ fixed points on the generator, not varying with the plane. Through each of these $n-2$ points there passes another generator, and they are points of a double curve on the surface. The double curve is, in fact, the locus of points on the scroll through which pass two generators. A scroll F_n^p has thus in general a double curve met by every generator in $n-2$ points. This singular curve on the scroll may be of higher multiplicity, and it may also split up into distinct and separate curves each with its own order of multiplicity. Since any scroll must have at least a double curve,‡ it follows on reciprocation that for any scroll there exists in general an infinity of planes passing through two generators. Hence, finally, the upper limit to the number of generators which may lie in a plane depends on the multiplicity of the singular curve of the reciprocal scroll, and the multiplicity of any multiple points that curve may possess.

3. The equations to any scroll may be expressed in the form

$$x = a+pz, \quad y = b+qz, \quad \phi(\lambda, \mu) = 0,$$

where a, b, p, q are algebraic single-valued functions of two variables λ, μ and ϕ represents a curve of the same order and with the same moduli as the plane section of the scroll. The generators are thus in (1, 1)-correspondence with the generators of a cone of the same order; the points of

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* We indicate a scroll of order n and genus p by F_n^p; and so also a curve by C_n^p.

† This remark is due to Cayley, *Collected Papers*, Vol. 2, p. 33.

‡ We are just now concerned only with scrolls in ordinary space.

the scroll may now be placed in (1, 1)-correspondence with the points of the cone in an infinite number of ways. In this correspondence any curve on the scroll meeting each generator in k points is transformed into a curve on the cone which meets each generator in k points. In particular, a unisecant curve (that is, a curve which meets each generator in one point only) is transformed into a unisecant curve. Now since on a cone the only singularities possible are multiple generators, it follows that on a scroll a point which is a singularity for one plane section is also a singularity of the same nature for the curve of intersection of every plane passing through the point. It also follows, that on the scroll we cannot have a singularity which does not lie on a singular curve every point of which is a singularity of the same nature. In other words, a scroll can have no isolated singularities.

The singularities of a scroll may therefore be analysed as follows. There are multiple generators: and there are singular curves,* met by every generator, through each point of which there passes a fixed number of generators, this number being the multiplicity of the singular curve. The multiple generator is met as a generator by a finite number of other generators, but it is not met by every generator nor does a fixed number of variable generators pass through each of its points. Thus the multiple generator is of essentially a different nature from the singular curve: through each point of it there do pass a fixed number of generators, but whatever the point chosen, all the generators coincide with itself.

We come now to a third type of singularity. There may also exist on the scroll a straight line met by every generator. Such a line is called a directrix. It may be thought of as a piece of the singular curve thrown out and standing by itself as a line. Through each point of the directrix there pass a fixed number of variable generators, this number being the multiplicity of the directrix. Obviously no scroll can have more than two directrices. A directrix is not in general a generator, in the sense that it does not form part of the mechanism effecting the (1, 1)-correspondence between any two plane sections. On the other hand, we may have one or more generators coinciding with the directrix. If k variable generators pass through each point of a directrix, and in any plane through the directrix it counts as a $(k+h)$-fold line, then we call it a k-fold directrix which is also a h-fold generator. Such a degeneration cannot take place when there are two directrices, for the two directrices cannot intersect. Again, there cannot occur a directrix on a cone or a

* It is believed that we cannot have more than three distinct singular curves in any case.

developable surface. Thus, whereas the multiple generator is essentially of a different nature from the singular curve, the directrix is of precisely the same nature as the singular curve; and hence arises its importance.

4. We consider now the nature of a plane section of the scroll F_n^p. The following is a principle of great importance due to Schwarz,* and may be called the principle of irreducibility. Given a scroll F_n^p, every plane section either is an irreducible C_n^p or else consists of a number $n-\mu$ of generators together with an irreducible curve C_μ^p, a lower limit for μ being conditioned by the fact that μ must be sufficiently high to allow the curve to have p for its genus.

The proof is extremely simple. There certainly exists at least one plane meeting the scroll in an irreducible C_n^p; and there cannot exist a (1, 1)-correspondence between this irreducible curve and a composite curve, unless this latter consists of an irreducible curve together with a number of the lines which effect the correspondence. There are thus on any scroll F_n planes containing an irreducible C_n, planes passing through one generator and containing a residual C_{n-1}, and there are also in general planes through two generators containing a residual C_{n-2}. In the general case then we can always suppose μ to be $n-2$, and it is frequently much less. The importance of the principle arises from the fact that in conjunction with the principle of correspondence it can be made the basis of an effective and exhaustive classification of scrolls of any given order.

An immediate consequence is the following. When a scroll F_n possesses a μ-fold directrix, the section by any plane through the directrix consists, besides the directrix, entirely of generators, their number being $n-\mu$. The truth of this follows from the fact that since every generator meets the directrix, the section by any such plane consists, besides the directrix, entirely of the lines which serve to effect the correspondence.

Consider now the various methods of generating a given scroll F_n^p by means of a (1, 1)-correspondence between two plane curves. We can always generate the scroll by means of the (1, 1)-correspondence between two plane sections C_n^p having n common self-corresponding points. We may again replace either (or both) of these curves by a C_μ^p where C_μ is the irreducible plane curve of minimum order, with μ (or $2\mu-n$) common self-corresponding points. It follows that μ is not less than $n/2$. Suppose in particular, that we take a C_n^p and a C_μ^p with μ self-corresponding points. Then the line common to the two planes meets the C_n in $n-\mu$ further points. Through these points pass $n-\mu$ generators lying in the

* *Loc. cit.*, p. 28.

plane of C_μ and making up together with it the complete plane section. By specialising the behaviour as to intersections and contact of these generators in relation to the C_μ we obtain all possible sub-varieties of the scroll, but the specialisation must be in conformity with the nature of the general plane section C_n^p as to singularities.

Thus the general procedure in the classification of the scrolls of given order n and given genus p would be as follows. We have on the one hand various types for the general C_n^p so far as the singularities of the curve are concerned. We have again various possibilities for the irreducible C_μ^p, where μ itself is subject to the relation

$$n/2 \leqslant \mu \leqslant n-2.$$

We associate each of the former group with each of the latter, so that the two curves have not less than μ points in common. Now we effect a (1, 1)-correspondence between the two curves so as to have μ self-corresponding points. Any minor types can then be obtained by specialising the nature of the correspondence. A number of types not possessing a plane curve of order $n-2$ or less may have escaped the above classification. Such a thing can only happen when the scroll possesses two directrices and has no multiple generators whatsoever. If the directrices are α-fold and β-fold, we have

$$\alpha+\beta = n,$$

and we may consider the scroll as generated by an (α, β)-correspondence between two non-intersecting lines.

5. From the fact that each generator meets the double curve in $n-2$ points, Cayley deduced that the order of the curve must be at least $n-2$. Since the singular curve may break up into several parts each with its own genus and multiplicity, it is obvious that a limiting value for p in terms of n would be much more useful.

Segre* has proved that when the genus is not zero, any ruled surface in any space is necessarily a cone if its hyperplane section is a normal curve. Thus, indicating by S_ρ the normal space for a scroll, and by S_r the normal space for its plane section, we have

$$\rho \leqslant r.$$

We may call this result Segre's lemma.

Consider in particular a scroll F_n^p of the type† $[\alpha, \beta]$. Besides the

* *Annali di Math.*, Vol. 22, p. 139.

† We indicate by this notation that the scroll has an α-fold and a β-fold directrix.

two directrices, the scroll may possess some multiple generators. Any arbitrary plane section C_n^p has thus an α-fold point and a β-fold point besides other possible singularities. We have thus

$$p \leqslant \frac{(n-1)(n-2)}{2} - \frac{\alpha(\alpha-1)}{2} - \frac{\beta(\beta-1)}{2},$$

where $$\alpha+\beta = n.$$

Thus we have $$p \leqslant \alpha\beta - n + 1,$$

and the expression on the right attains its maximum when α and β are as near equality as possible. So also for a scroll F_n^p with a single α-fold directrix, any plane through the directrix contains $n-\alpha$ generators which yield by their intersections

$$\frac{(n-\alpha)(n-\alpha-1)}{2}$$

double points. Since a scroll does not permit of any singularities that are not situated on singular curves, it follows that on an arbitrary plane we have an α-fold point besides

$$\frac{(n-\alpha)(n-\alpha-1)}{2}$$

double points. Thus we have

$$p = \frac{(n-1)(n-2)}{2} - \frac{\alpha(\alpha-1)}{2} - \frac{(n-\alpha)(n-\alpha-1)}{2} = \alpha(n-\alpha) - n + 1.$$

Thus in the case of scrolls F_n^p with one or two directrices, we have

$$p \leqslant \frac{(n-2)^2}{4},$$

when n is even; and $$p \leqslant \frac{(n-1)(n-3)}{4},$$

when n is odd.

It is obvious that these limits are reached. It is known* that these are the upper limits for the genus of a twisted curve of order n; and Segre's lemma states that these are the limits for the genus of scrolls of order n.

6. Consider now a scroll with two directrices. *There cannot exist on the scroll any singular curve.* For, if O be a point on the singular curve,

* Castelnuovo, *Atti Torino*, Vol. 24 (1888), pp. 346-373.

the plane of the two generators through O would contain both the directrices, which is absurd. Again, in the case of a scroll with a single α-fold directrix the $n-\alpha$ generators in any plane through the directrix may all intersect on the directrix itself, or some of these intersections may lie outside the directrix. In the former case the scroll has no singular curve; and we shall agree to speak of such a scroll as a scroll with two coincident directrices. In the latter case, not all of the multiple points thus arising outside the directrix can be due to multiple generators. There exists thus a singular curve. Further, it is easy to see that on a scroll without a directrix there exists a singular curve, for if this is not so, we should have only multiple generators; and since these have to account for all the singularities in every plane, it follows that every generator must meet them, which is absurd. In particular we cannot have a scroll all of whose singularities are multiple generators. Thus on a scroll with two directrices, distinct or coincident, there does not exist a singular curve; on all other scrolls there does exist a singular curve. Hence conversely, if a singular curve does not exist, the scroll must necessarily have two directrices, distinct or coincident.

7. We now propose to prove that *every scroll F_n^p which is normal in S_3 must necessarily have two directrices.**

We require the following lemma. Given in S_3 a F_n^p and a F_{n-1}^p which have a plane curve C_{n-1} common, the scroll F_n^p cannot be normal in S_3. The proof of the lemma is as follows. Using for clearness x, y, z with reference to F_n, and x', y', z' with reference to F_{n-1}, we suppose the axes chosen in such manner that the plane of the common C_{n-1} is the z, z' plane. The two surfaces are evidently in (1, 1)-correspondence, and the equations connecting them may be taken in the form

$$x' = x, \quad y' = y, \quad z' = R(x, y, z),$$

where R is a rational function. There is on F_n an ∞^3 system (α) of curves of grade n given by the plane sections. On residuating with respect to a point O on the surface, we obtain an ∞^2 system (β) of curves of grade $n-1$. If O be taken to lie on the z axis, we find that the system (β) includes as part of it the ∞^1 system (γ) given by

$$x+\lambda y = F_n = 0.$$

* The following proof requires that p should not be zero. But the result stated holds in that case also, as we shall see later.

On transformation, the system (β) is changed into an ∞^2 system (β') of curves of grade $n-1$ on F_{n-1}; and the system (γ) is changed into an ∞^1 system (γ') of curves of grade $n-1$ on F_{n-1}; and (γ') is included in (β'). Further, it follows from the equations of transformation that the system (γ') is given by

$$x'+\lambda y' = F_{n-1}(x', y', z') = 0.$$

Moreover there exists on F_{n-1} an ∞^3 system (α_1) of curves of grade $n-1$ given by the plane sections. The system (γ') is common to the systems (α_1) and (β'). We shall show that (β') cannot be complete; for let (β') be given by

$$(x'+\lambda y')P+\lambda_1 Q = 0,$$

where P and Q are surfaces adjoint to F_{n-1}. Any surface of the form

$$\lambda P y'+\lambda_1 Q = 0 \qquad (1)$$

meets F_{n-1} in a free curve which has $n-1$ intersections on the locus

$$Px' = 0,$$

and therefore $n-1$ intersections on the plane x'. In other words any surface of the system (1) cuts out on F_{n-1} a free curve of order $n-1$. The degrees of P and Q therefore differ by one, and Q passes through the free intersection of P with F_{n-1}. But then such an expression as Q/P belongs to the completed system individuated by (α_1). Thus (β') is contained in the complete system of which (α_1) is a part. (β') is therefore not complete.

Lastly, we observe that the principles of residuation and birational transformation are such that the completeness or otherwise of (β') depends on the completeness or otherwise of (β), and so also the completeness of (β) is dependent on the completeness of (α). We have proved that (β') is not complete. It follows that (α) is not complete. Hence the scroll F_n cannot be normal in S_3.

8. We come now to the proof of the theorem that every scroll normal in S_3 necessarily has two directrices. In virtue of the above lemma, it is enough to prove that when F_n is not a scroll with two directrices, we can always construct a ruled surface F_{n-1} which is not a cone and which has a plane C_{n-1} common with the scroll F_n.

Suppose first that the scroll F_n has no directrix at all. Then any generator g_0 is met by $n-2$ other generators, of which there are at least two which are non-intersecting; for if every two of them intersect, it follows that there is a plane through g_0 which contains $n-1$ generators and therefore a directrix. Let g_1, g_2 be two non-int secting genera-

tors, both of which meet g_0. Let g_0 meet g_2 in Q, and let QR be any line through Q meeting g_1; and suppose Q is a k^*-fold point for the scroll. Then the plane of g_1, g_0 contains a further C_{n-2} which has Q for a $(k-1)$-fold point and meets QR in $(n-k-1)$ further points. On the other hand, the plane of QR and g_2 contains a further C_{n-1} which has Q for a $(k-1)$-fold point, passes through R and has $(n-k-1)$ further points on QR. Thus the two curves C_{n-1} and C_{n-2} have $n-2$ points common. The given scroll F_n is the result of a (1, 1)-correspondence between these two curves; but this correspondence has only $n-3$ self-corresponding points. We can now construct a new (1, 1)-correspondence between the two curves, so as to have all the $n-2$ common points for self-corresponding points. The result is a ruled surface F_{n-1} which is certainly not a cone, since it contains an irreducible plane C_{n-2}.

Similarly, suppose that the scroll has only one directrix; and we shall assume† that its multiplicity μ is greater than one. In virtue of what was said in § 6, it follows that there exists a singular curve, and that not all the generators in any plane through the directrix meet on the directrix. Let g_0, g_1 be two generators meeting on the directrix, and let g_2 be a third generator meeting g_0 and not g_1. We proceed as before and construct a ruled surface F_{n-2}. The theorem is therefore proved, that every scroll normal in S_3 must necessarily have two directrices distinct or coincident.

9. In virtue of Segre's lemma it follows that when p is too high with respect to n to allow a C_n^p to exist in S_4, the scroll F_n^p is necessarily normal in S_3, and therefore has two directrices. Castelnuovo‡ has proved that the genus of a C_n^p existing in S_r is subject to the inequality

$$p \leqslant \chi\left[(n-r)-\frac{(\chi-1)(r-1)}{2}\right],$$

where

$$\frac{n-1}{r-1}-1 \leqslant \chi < \frac{n-1}{r-1}.$$

In particular for a curve C_n^p situated in S_4 we have an inequality which is effectively the same as

$$p \leqslant \frac{(n-2)(n-3)}{6}.$$

Therefore, when p is greater than the above limit, the scroll F_n^p must

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* We have $k \geqslant 2$.

† This amounts to saying that the enus of the scroll is not zero.

‡ *Atti Torino*, Vol. 24 (1888), p.

necessarily be normal in S_3 and therefore have two directrices. Conversely, *the genus of a scroll F_n^p which has no directrix at all, but which may have any other singularities whatever, is subject to the above inequality.**

10. We now propose to prove that *a scroll situated in its normal space S_ρ cannot have a singular curve.* This result may usefully be compared with similar results in the theory of curves. We have seen above that this holds when the normal space is S_3; it is thus enough to prove it when $\rho \geqslant 4$.

The following lemma is necessary. If a scroll F_n is situated normally in S_ρ ($\rho \geqslant 4$), its projection from an arbitrary point on itself yields a scroll F_{n-1} situated normally in $S_{\rho-1}$. The proof is precisely the same as in the case of curves. Suppose now if possible that the scroll F_n situated normally in S_ρ has a singular curve. Projecting it from $\rho-3$ arbitrary points on itself, we obtain a scroll $F_{n-\rho+3}$ situated normally in S_3; this latter scroll has by what was proved in §§ 6–8 no singular curve, and its only singularities are two directrices and multiple generators. Now for the purposes of the lemma above we can choose the $\rho-3$ points anywhere we like on F_n, provided only they are independent and determine a $S_{\rho-4}$. Hence if the normal scroll does have a singular curve, it must lie in either of the two $S_{\rho-2}$ joining the two directrices to any and every arbitrary $S_{\rho-4}$. But this is absurd. Hence a normal scroll can have no singular curve apart from directrices and multiple generators. In particular any given scroll in S_3 which does not possess a directrix yields on being projected upwards into its normal space a scroll with only multiple generators; and even these will not exist if the given scroll has none. A simple example† is the F_5 with a twisted quintic for double curve. It is known that the scroll is normal in S_4 and in S_4 it has no singularities at all.

Hence the singular curve of any scroll in ordinary space is not an essential feature of the scroll but the necessary consequence of the collapse due to projection from the normal space downwards.

This suggests a different method of classifying scrolls from that indicated in § 4. In order to classify all possible scrolls F_n^p of given order and genus in ordinary space, we first find out what the normal space for such a scroll is. It may be that ρ is not completely determined by n and p alone; yet it must be capable of being determined as lying between

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* Cf. Wiman, *Acta Math.*, Vol. 19 (1895), pp. 63–72, where the same result is proved by analytic methods to hold in the case of a severely restricted class of scrolls.

† Baker, *Proc. London Math. Soc.*, Ser. 2, Vol. 12 (1913), p. 12.

definite limits ρ_1, ρ_2. We then analyse the nature of a scroll F_n^p situated normally in S_ρ where ρ varies from ρ_1 up to ρ_2. Either it has a directrix or it has not; and the nature as to multiple generators of the scroll is dependent on the nature as to multiple points of the hyperplane section C_n^p situated in $S_{\rho-1}$. We then analyse the various possibilities as to the minimum irreducible curve that can exist on the normal scroll. This enables us to determine the various ways of generating the scroll by means of a (1, 1)-correspondence between two curves with or without self-corresponding points. We then project the normal scroll from the proper number of points taken arbitrarily in its space on to a S_3. We thereby obtain all the possible types of F_n^p in S_3, provided we choose the points of projection as arbitrarily as possible. Two scrolls in S_3 that are both obtainable by means of uniform projections from the same scroll in higher space are in homographic correspondencce. Thus of the scrolls F_n^p in S_3 which are normal in a given space S_ρ there are as many independent types, not transformable into one another homographically, as there are ways of generating the normal scroll F_n^p in S_ρ.

11. We come now to a discussion of rational scrolls. Consider first the case of cubic scrolls. Since every plane section has a double point, it follows that there is a double line. The double line may be either a double directrix, or a simple directrix and a simple generator put together, thus giving rise to two distinct types. Consider the former type. The plane through the two generators through an arbitrary point of the directrix contains a third line which cannot be a generator, but must be a unisecant curve of order 1. There is thus a simple directrix, besides the double directrix, and the scroll is of the [1, 2] type. Any plane through a generator cuts out a conic which does not meet the simple directrix. Hence the scroll may be generated by a (1, 1)-correspondence between a conic and a line not meeting it. Consider then in S_4 a conic and a line not situated in the same hyperplane. The result of a (1, 1)-correspondence between them is a cubic surface, which is necessarily a scroll because it possesses a directrix. Projecting from an arbitrary point lying in the plane of the conic, we obtain in S_3 a cubic scroll of the [1, 2] type.

Consider now the second type. Since it has a line which is a directrix as well as a generator, it follows that it cannot have a second directrix. We can see easily that this surface also is not normal in S_3, for consider the cubic scroll in S_4 constructed above, and take the plane passing through the directrix and an arbitrary generator. On projecting from an

arbitrary point in this plane, we obtain a scroll of the second type with a line which is a directrix as well as a generator.

Further we cannot have a cubic scroll in space higher than S_4, for a cubic curve is normal in S_3; hence every cubic scroll is normal in S_4. Again two cubic scrolls, one of either type, are connected by a simple relation, since both are derivable from the same configuration in S_4. The two scrolls may be represented by

$$x_1^2 x_3 = x_2^2 x_4,$$

and

$$y_2^3 = y_1(y_1 y_3 + y_2 y_4);$$

these are transformable into one another with the aid of the relations

$$\frac{x_1}{y_1} = \frac{x_2}{y_2} = \frac{x_3}{y_3} = \frac{x_4}{y_2 - \dfrac{y_1 y_4}{y_2}}.$$

12. We shall now prove that no rational scroll can be normal in S_3. For let

$$x_1 = P_1, \quad x_2 = P_2, \quad x_3 = 0, \quad x_4 = P_4,$$

and

$$x_1 = Q_1, \quad x_2 = 0, \quad x_3 = P_2, \quad x_4 = Q_4,$$

represent two arbitrary plane sections of such a scroll of order n. The P's and the Q's are polynomials in a parameter λ, the highest power occuring being λ^n. We have then

$$\frac{P_1}{Q_1} = \frac{P_4}{Q_4}$$

for each of the n zeros of P_2. Consider now whether we can find two polynomials P_5, Q_5 of order n in λ such that

$$\frac{P_5}{Q_5} = \frac{P_1}{Q_1}$$

for each of the zeros of P_2, and such that P_1, P_2, P_4, P_5 are linearly independent, as also Q_1, Q_2, Q_4, Q_5. When n equals two, there are only three independent polynomials. In general there are $n+1$ homogeneous constants available in each of P_5, Q_5, and the zeros of P_2 are only n. Thus the problem is always soluble provided n is greater than two.

If now we add to the two sets of equations above

$$x_5 = P_5, \quad x_5 = Q_5,$$

respectively, we obtain in S_4 a scroll of order n which projects from the

x_5 vertex into the given scroll. Thus no rational scroll F_n^0 can be normal in S_3 unless it be a quadric surface.

We can now prove that every F_n^0 is normal in S_{n+1}. For assume that every F_{n-1}^0 is normal in S_n, and suppose if possible that there exists one F_n^0 which is normal in $S_{n+1-\epsilon}$. In virtue of the above result we must have

$$n+1-\epsilon \geqslant 4.$$

Hence on projecting the scroll from a point of itself, we obtain a F_{n-1}^0 normal in $S_{n-\epsilon}$, which contradicts the assumption unless ϵ equals zero. We have already shown that every F_3^0 is normal in S_4. It follows therefore that every F_n^0 is normal in S_{n+1}.

13. Consider now the nature of a F_n^0 situated normally in S_{n+1}.* On such a surface we cannot have two intersecting generators, for on projecting from the point of intersection we should obtain a rational surface of order $n-2$ in S_n, which is absurd. For the same reason we cannot have a multiple generator, nor a double line which is a simple directrix as well as a simple generator. Thus for a F_n^0 in S_{n+1} we can only have a simple directrix, and even that may not always exist.

Again any C_α ($\alpha \leqslant n$) situated on the scroll must be a unisecant curve; for if it were bisecant (or higher) every generator would lie in the space in which the C_α is situated, and this is not greater than S_n, whereas the scroll is situated in S_{n+1}. Next every C_α^0 ($\alpha \leqslant n$) on the scroll must be situated in its normal space S_α. For suppose if possible it is situated in $S_{\alpha-\epsilon}$; taking any $(n-\alpha+\epsilon)$ arbitrary points on the scroll, these together with the $S_{\alpha-\epsilon}$ fix a S_n which contains a composite curve of order $n+\epsilon$, viz. the curve C_α besides the $(n-\alpha+\epsilon)$ generators through the selected points: thus ϵ must equal zero.

The following is then a method of generating a F_n^0 in S_{n+1}. Consider two curves C_α^0, $C_{n-\alpha}^0$ situated normally in two spaces S_α, $S_{n-\alpha}$ respectively, these two spaces being situated independently in a S_{n+1}. A (1, 1)-correspondence leads to a surface of order n, which cannot but be a scroll since no two generators of a ruled surface of order n in S_{n+1} can intersect. As we vary α from 1 up to and including $\frac{n-1}{2}$ or $\frac{n}{2}$ according as n is odd or even, we obtain various types of F_n^0 in S_{n+1}.

We shall now show that every F_n^0 in S_{n+1} can be obtained by the above method. In order that a S_n may contain an assigned generator, it is

* Most of the results in this and the two following articles are due to Segre; cf. his memoirs in the *Atti Torino*.

necessary to subject it to two conditions. Hence we can always have a S_n determined so as to pass through* $\left[\frac{n+1}{2}\right]$ arbitrary generators. The residual curve of intersection may then be reducible or irreducible. Let us say that it is a curve of order

$$\alpha = n - \left[\frac{n+1}{2}\right] - \epsilon.$$

Then we can easily see that there exists on the scroll a curve of order

$$\beta = \left[\frac{n+1}{2}\right] + \epsilon.$$

For take $n-\beta$ arbitrary generators. Through two points on each of them we can pass a S_n, since

$$n+1 \geqslant 2(n-\beta).$$

The residual curve is of order β, and cannot degenerate into a $C_{\beta-\epsilon'}$ and ϵ' generators, for then the unisecant curves C_α and $C_{\beta-\epsilon'}$ would yield a scroll of order $n-\epsilon'$ which is not so.

Thus the above method of generating scrolls F_n^0 in S_{n+1} yields all such scrolls as are possible. On projecting from an arbitrary S_{n-3}, we obtain all possible scrolls F_n^0 in S_3. Consider in particular scrolls F_4^0 in S_5. There are only two ways of generation. Either we take two conics situated in two independent planes, or we take a line and a twisted cubic such that the line does not meet the space of the cubic. We then effect a (1, 1)-correspondence and obtain the two types of rational quartics in S_5. It is obvious that the various types of scrolls F_4^0 in S_3 are transformable into one or other of the two types [1, 3] and [2, 2]. So also every F_5^0 in S_6 is the result of a (1, 1)-correspondence between either a C_1 and a C_4, or a C_2 and a C_3, the two curves in either case being situated in independent spaces. It follows that in S_3 every F_5^0 is transformable into one or other of the two types [1, 4] and [2, 3]. There are thus $\frac{n}{2}$ or $\frac{n-1}{2}$ types of scrolls F_n^0 in S_{n+1} according as n is even or odd.

14. We pass now to a discussion of elliptic scrolls. Consider first the scroll F_4^1. The minimum plane curve is a C_3^1 only. Hence the scroll has two directrices and no multiple generators; and is thus of the type [2, 2].

* By $[x]$ we mean the integral part of x.

In virtue of Segre's lemma its normal space is S_3. We can now prove that every F_n^1 is normal in S_{n-1}, either on lines similar to the proof in § 12 or by anticipating a result to be proved later.

Consider the nature of a F_n^1 situated normally in S_{n-1}. It cannot have any multiple generators for a series of simple projections yield a F_4^1 in S_3 which is a scroll without any multiple generators. Again suppose if possible two generators intersect. The projection from their point of intersection on to a S_{n-2} is necessarily uniform since the condition for this is

$$\frac{n-2}{2}+1 < n-2,$$

which is satisfied when n is greater than four. Thus on projection we obtain in S_{n-2} a ruled surface of order $n-2$ and genus 1; and this surface must necessarily be a cone in virtue of Segre's lemma. Thus if two generators meet, their point of intersection must lie on a double directrix; for then only can a scroll be projected into a cone. Again it is easy to see that three generators cannot meet in a point, for a projection from that point will necessarily be uniform and lead to a ruled surface of order $n-3$ and genus 1 in S_{n-2}, which is absurd. Thus a scroll F_n^1 in S_{n-1} may have a double directrix, but it cannot have any multiple generators.

We prove as before that every C_α^1 ($\alpha \leqslant n-1$) on the scroll is a unisecant curve. Further every such curve is situated in its normal space $S_{\alpha-1}$. For suppose it lies in $S_{\alpha-1-\epsilon}$. Take $(n-2+\epsilon-\alpha)$ further points. Together with the $S_{\alpha-1-\epsilon}$ they determine a S_{n-3} containing not only the C_α but also the generators through the selected points. We have therefore a S_{n-3} containing a composite curve of order $(n-2+\epsilon)$. The system of S_{n-2} passing through the S_{n-3} cut out a linear system of sets of $(2-\epsilon)$ generators. Now ϵ cannot be two for otherwise we should have a S_{n-3} which is not a hyperplane cutting out a curve of order n. Nor can ϵ be one, for then the scroll would be rational. Hence ϵ is zero and every C_α^1 ($\alpha \leqslant n-1$) is situated in its normal space.

15. Consider now a C_α^1 and a $C_{n-\alpha}^1$ situated normally in spaces independent of each other. A (1, 1)-correspondence between them, assuming they have the same moduli, yields a ruled surface F_n^1 in S_{n-1}. We now propose to analyse all such scrolls according to their minimum hyperplanar curves. When there is a directrix, it is a double directrix and the unisecant curve of minimum order on the surface is a C_{n-2}. A (1, 2)-correspondence between a line and a C_{n-2}^1 such that the line does not meet the space S_{n-3} in which the curve is situated yields the scroll.

Now suppose there is no directrix, and that n is odd. Since a S_{n-2}

can be made to pass through $n-1$ arbitrary points, we can have a S_{n-2} through $\frac{n-1}{2}$ arbitrary generators. We thus obtain a residual irreducible unisecant curve of order

$$\alpha = n - \frac{n-1}{2} - \epsilon = \frac{n+1}{2} - \epsilon,$$

and of genus one. We shall prove that there exists a C_β^1 where

$$\beta = n - \alpha.$$

For taking α generators, a S_{n-2} can be had to pass through them provided

$$n-1 \geqslant 2\alpha,$$

that is, provided $$\epsilon \geqslant 1.$$

We have thus types where a (1, 1)-correspondence is set up between two curves C_α and C_β, with no self-corresponding points. When ϵ equals zero, we can only take two curves of order $\frac{n+1}{2}$ and there is one self-corresponding point.

Now suppose n is even. A S_{n-2} can be had to pass through $\frac{1}{2}n-1$ arbitrary generators, and there is thus a residual unisecant irreducible curve of order α where

$$\alpha = n - \left(\frac{n}{2} - 1\right) - \epsilon = \frac{n}{2} + 1 - \epsilon.$$

We can again have a C_β where

$$\beta = n - \alpha,$$

provided $$n-2 \geqslant 2\alpha,$$

that is, provided $$\epsilon \geqslant 2.$$

When ϵ equals one, we set up a (1, 1)-correspondence between two curves of order $\frac{1}{2}n$. When ϵ is zero, the unisecant curve of minimum order, of which there is an infinity, is a curve of order $\frac{1}{2}n+1$.

Thus the scroll F_n^1 in S_{n-1} has either a double directrix or belongs to one of the following types.

A. When n is odd:

1. We take two $C_{\frac{1}{2}(n+1)}$ with one self-corresponding point.
2. We take a $C_{\frac{1}{2}(n-1)+\epsilon}$ and a $C_{\frac{1}{2}(n+1)-\epsilon}$ ($\epsilon \geqslant 0$), and there are no self-corresponding points.

B. When n is even :

1. We take a $C_{\frac{1}{2}n-1-\epsilon}$ and a $C_{\frac{1}{2}n+1+\epsilon}$ ($\epsilon \geqslant 0$), and there are no self-corresponding points.

2. We take two $C_{\frac{1}{2}n}$.

3. We take two $C_{\frac{1}{2}n+1}$ with two self-corresponding points.

16. We found that every F_n^0 is normal in S_{n+1}, and every F_n^1 is normal in S_{n-1}. In other words, the normal space is uniquely determined in terms of the order and the genus. But this is by no means the case when the genus is greater than one. Consider then some particular types of scrolls of genus 2. A F_5^2 has for its minimum plane curve a C_4^2; hence it must have two directrices, and can have no multiple generators. It is thus of the [2, 3] type. And further in virtue of Segre's lemma it is normal in S_3. Consider now a F_6^2 of the [2, 4] type with one double generator. The system of quadrics through the fourfold directrix and the double generator is an ∞^4 system, and obviously includes the system of plane sections. Since the normal space for a C_6^2 is S_4, it follows that the scroll is normal in S_4. Now consider a F_6^2 of the [3, 3] type with two double generators. We observe that a scroll in any space higher than S_3 can at the most have one directrix. If now the given scroll is normal in S_4, we should have to replace one of the threefold directrices by a unisecant cubic curve. But this is impossible, since a cubic curve cannot be of genus 2. Thus the scroll is normal in S_3 only. Or again two planes through a double generator yield two plane curves C_4^2 with two distinct points common, viz., the points where the double generator meets the two directrices. Obviously, it is impossible to have in S_4 two planes with two distinct points common without their lying in a S_3.

Consider now a F_7^2 of the [2, 5] type with two double generators. The two systems of quadrics passing through the fivefold directrix and either of the two double generators are ∞^4 systems; and each of them contains the ∞^3 system of plane sections. Thus the complete system containing the system of plane sections is at least ∞^5; and since the normal space for a C_7^2 is S_5, it follows that the normal space for the scroll is S_5.

Consider now a F_7^2 of the [3, 4] type with a triple generator and a double generator, or with four double generators. We shall show that for both these types the normal space is S_4. But first we shall show that the normal space is at least as high as S_4. Consider in S_4 a line and a plane which do not meet. Let O be any point in the plane and take any curve C_4^2 in the plane. We can obtain a g_3^1 on the quartic by means of lines

passing through P_0, a fixed point on the quartic. A (1, 3)-correspondence between the line and the quartic yields a F_7^2 in S_4 with the line for triple directrix. The three points in which the line OP_0 meets the quartic form one set of the g_3^1, and therefore correspond to a single point Q_0 on the directrix. The plane OP_0Q_0 thus meets the S_3 of projection in a line which is a triple generator, and the quartic itself is projected into a fourfold directrix. This is therefore a scroll of the first of the two types mentioned above. On the other hand, we may obtain a g_3 on the quartic by means of conics passing through the double point and three fixed points on the curve. Since there is no single set of the g_3^1 of which the three points lie in a line through O, it follows that on projection we obtain in S_3 a scroll of the second of the two types mentioned. Thus both the types are normal at least in S_4. Further two planes through the triple generator yield two plane quartics of genus 2 with one point common, viz. the point where the triple generator meets the fourfold directrix. And obviously two planes with a common point fix a S_4 and not a higher space. Thus the normal space for the first type is precisely S_4. Passing to the second type, we see easily that such a scroll cannot exist in S_5. For supposing this were possible, we should still have a threefold directrix; and an arbitrary hyperplane section leads to a C_7^2 in S_4 with a triple point. This is obviously absurd. Thus for both the scrolls F_7^2 of the type [3, 4] the normal space is precisely S_4. Moreover, since the above three types are the only types of F_7^2 with two directrices, it follows that every other F_7^2 is normal in at least S_4. Thus every scroll F_7^2 is normal either in S_4 or in S_5.

It appears from the above examples that every F_n^2 is normal in either S_{n-2} or in S_{n-3}. We shall indeed prove this result later, but for the present we assume its truth.

17. We now propose to examine the nature of a F_n^2 situated in its normal space. We consider first a F_n^2 situated normally in S_{n-2}. Suppose there is a point through which ϵ ($\geqslant 2$) generators, distinct or coincident, pass. The projection from this point into a S_{n-3} is necessarily uniform provided

$$\frac{n-\epsilon}{2}+1<n-3,$$

which is certainly so when $n>6$.

We thus obtain in S_{n-3} a ruled surface of order $n-\epsilon$ and genus 2. This surface cannot be a scroll, for that would require in virtue of Segre's lemma, that

$$n-3\leqslant n-\epsilon-2,$$

which is not so. The surface is therefore a cone, and since the normal space for a cone of order $n-\epsilon$ and genus 2 is $S_{n-\epsilon-1}$, it follows that we must have

$$n-\epsilon-1 = n-3,$$

and thus

$$\epsilon = 2.$$

Thus the scroll F_n^2 normal in S_{n-2} cannot have a multiple generator, but may have a double directrix. Further, we can show that it must have a double directrix. A F_5^2 is normal in S_3 and has a double directrix. Hence a F_6^2 normal in S_4 must also have a double directrix, since a cubic curve cannot be of genus 2. So also a F_7^2 normal in S_5, and more generally a F_n^2 normal in S_{n-2} must have a double directrix. An arbitrary S_{n-3} through two generators meeting the directrix in the same point yields a residual C_{n-2}^2, and this is the unisecant curve of minimum order. It is situated in its normal space; for if it were situated in $S_{n-4-\epsilon}$, the system of hyperplanes through this space would yield a $g_2^{1+\epsilon}$ of sets of two generators. Since the scroll is not rational, it follows that ϵ is necessarily zero. The only method of generation is thus by setting up a (1, 2)-correspondence between a line and a C_{n-2}^2 situated normally in S_{n-4} and such that the line does not meet the S_{n-4}.

Consider now a F_n^2 situated normally in S_{n-3}. Suppose there exists a point through which ϵ ($\geqslant 2$) generators, distinct or coincident, pass. It can be shown that the projection from this point is uniform when n is fairly high. On projection, we obtain then in S_{n-4} a ruled surface of order $n-\epsilon$ and genus 2. An application of Segre's lemma shows that when ϵ is two, the new surface is a scroll: and that when ϵ is three, the new surface is a cone. Further, it is not possible to assign a greater value to ϵ. Thus the scroll may have a triple directrix but cannot have a double directrix. On the other hand, it may have double generators but annot have generators of higher multiplicity. That a double directrix is not possible can also be seen thus. If a F_7^2 normal in S_4 has a double directrix, then a F_6^2 normal in S_3 must also have one; but we saw in § 16 that this is not so. Thus a F_7^2 normal in S_4 cannot have a double directrix, and repeating the argument, we see that a F_n^2 normal in S_{n-3} cannot have a double directrix. So also since a F_6^2 normal in S_3 can have only two double generators, it follows that even of double generators the scroll F_n^2 normal in S_{n-3} can at the most have two only. Combining the results regarding the directrix, we see that *the necessary and sufficient condition that any given F_n^2 should be normal in S_{n-2} is that it should have a double directrix.*

Consider now the methods of generation of a F_n^2 normal in S_{n-3}. When

the scroll has a triple directrix, an arbitrary hyperplane through the three generators issuing from a point of the directrix yields a residual C^2_{n-3} situated normally in S_{n-5}. We do not consider the possibility of the triple directrix degenerating into a double directrix and a simple generator put together, for such a degeneration cannot take place in the case of a scroll situated in its normal space. Otherwise, we should obtain by a series of projections a scroll normal in S_3 having two directrices, one of which degenerates; and we have already seen that this is not possible.

18. We proceed now to an analysis of sextic scrolls. A F^0_6 is normal in S_7 and can be obtained by a (1, 1)-correspondence between a C_1 and a C^0_5, a C_2 and a C^0_4, or two rational cubics, the two curves in each case being situated in independent spaces. On projection we see that all rational sextic scrolls in S_3 are reducible by transformation to one or other of the three types

$$[1, 5], \quad [2, 4], \quad [3, 3].$$

Passing to scrolls F^1_6, such a scroll situated normally in S_5 may have a double directrix; or it may have two elliptic cubics on it; or again it may only be capable of being generated by means of a (1, 1)-correspondence between two elliptic quartics with two self-corresponding points. There are therefore three fundamental types of elliptic sextic scrolls in S_3, to which all others can be reduced by transformation. We may take them as a scroll of the [2, 4] type with two double generators, a scroll of the [3, 3] type with a triple generator, or a scroll of the [3, 3] type with three double generators.

When the genus is two, we have seen that there is only one type which is normal in S_3, viz. the [3, 3] type with two double generators. All other scrolls F^2_6 are normal in S_4 and have a double directrix as well as a C^2_4 for the unisecant curve of minimum order. They can thus be reduced to the type [2, 4] with one double generator.

When the genus is three, the sextic scroll is necessarily normal in S_3 in virtue of Segre's lemma. There are only two types, the type [3, 3] with one double generator, and the type [2, 4].

Lastly, when the genus is four there is only the one type with two triple directrices.

It may be pointed out that a scroll with two directrices can always be made to yield a second type where the two directrices coincide. Further, we are concerned throughout with transformations which consist of two series of uniform projections from simple points. Whereas such a trans-

formation is homographic, it does not follow that every homographic transformation can be reduced to such a transformation. Thus the types given above may not all be independent for homographic transformations.

19. We now proceed to a discussion of the general problem as to the normal space of a given scroll, and we limit ourselves for the present to the case of scrolls with two directrices.

Any scroll of order n with two directrices may be represented by

$$[(x, y)_\alpha (z, t)_\beta] = 0,$$

where $$\alpha+\beta = n,$$

and the expression on the left stands for

$$\Sigma x^{\alpha-\epsilon} y^{\epsilon} z^{\beta-\epsilon'} t^{\epsilon'},$$

the summation being taken subject to

$$0 \leqslant \epsilon \leqslant \alpha, \qquad 0 \leqslant \epsilon' \leqslant \beta.$$

Changing the coordinate planes we can write the equation in the form

$$F(x, y, z, t) = [(x, y)_\alpha (ax+by+t, z)_\beta] = 0.$$

The section by the t plane is

$$f(x, y, z) = [(x, y)_\alpha (ax+by, z)_\beta] = 0.$$

Again, the scroll may also be represented by

$$y = \frac{y_0}{x_0} x, \quad ax+by+t = \frac{ax_0+by_0}{z_0} z, \quad f(x_0, y_0, z_0) = 0.$$

For the result of eliminating x_0, y_0, z_0 between the above equations is precisely the function F. This is because we have

$$\frac{x_0}{x} = \frac{y_0}{y} = \frac{z_0}{z(ax+by)/(ax+by+t)},$$

and $$f\left(x, y, \frac{ax+by}{ax+by+t} z\right) = \left(\frac{ax+by}{ax+by+t}\right)^\beta F(x, y, z, t).$$

The latter form of the equations to the scroll has the advantage of giving the equation of the generator through any assigned point of the section f. In particular, the α generators through the foot of the α directrix are given by

$$ax+by = 0, \qquad h(x, y) = 0,$$

where h is the coefficient of z^β in the expression f. Further, any k-fold

point x_1, y_1, z_1 of f gives rise to a k-fold generator whose equation is

$$y = \frac{y_1}{x_1} x, \qquad ax+by+t = \frac{ax_1+by_1}{z_1} z. \tag{1}$$

Thus apart from the two directrices the only other singularities of the scroll are multiple generators.

Let now $$\phi(x, y, z) = 0$$

be any adjoint of order $n-2$ to the curve f. Then the equations

$$y = \frac{y_0}{x_0} x, \qquad ax+by+t = \frac{ax_0+by_0}{z_0} z, \qquad \phi(x_0, y_0, z_0) = 0,$$

represent a surface whose equation is

$$\phi\left(x, y, \frac{ax+by}{ax+by+t} z\right) = 0. \tag{2}$$

Since ϕ is an adjoint of order $n-2$ its expression is necessarily of the form

$$\phi(x, y, z) = [(x, y)_{\alpha-1} (ax+by, z)_{\beta-1}] = 0.$$

Hence the equation (2) is the same as

$$[(x, y)_{\alpha-1} (ax+by+t, z)_{\beta-1}] = 0,$$

and the expression on the left may be denoted by

$$\Phi(x, y, z, t).$$

From the form of the expression it is obvious that Φ is again a scroll of order $n-2$ with two directrices. We observe that Φ is obtained from ϕ in precisely the same way as F was obtainable from f. Further, it is evident that the surface Φ is an adjoint to the scroll F. For, if x_1, y, z_1 is a k-fold point of f, the k-fold generator through it of the scroll F is given by the equations (1). The point x_1, y_1, z_1 is again a $(k-1)$-fold point for ϕ, and the surface Φ has a $(k-1)$-fold generator through it whose equations are precisely the equations (1). Hence on every adjoint ϕ_{n-2} to the curve f we can build up a scroll Φ_{n-2} which is adjoint to F.

Conversely, it is obvious that every adjoint surface of order $n-2$ is met by the t plane in a curve of order $n-2$ which is adjoint to the curve f. Thus the freedom of the system of surfaces Φ_{n-2} is precisely the same as the freedom of the system of curves ϕ_{n-2} which are adjoint to the plane section f; and the freedom of the latter system is $(n+p-2)$, for the freedom of the system of adjoints ϕ_{n-3} is $(p-1)$, and a curve of order $n-2$

has $(n-1)$ more disposable constants than a curve of order $n-3$. In other words, we have*

$$\omega_{n-2} = 0.$$

Further, every adjoint Φ_{n-2} meets the scroll in a free curve consisting of $(n+2p-2)$ generators; these are the generators through the $(n+2p-2)$ free points of intersection of ϕ, the plane section of Φ, with f the section of F by the same plane. Thus the adjoints Φ_{n-2} from a system of freedom $n+p-2$, and each such adjoint cuts out a set of $n+2p-2$ free generators.

20. We now proceed to the problem mentioned at the beginning of § 19.

A general surface of order $n-2$ is one of a system of freedom

$$\frac{n(n-1)(n+1)}{6} - 1;$$

and we have seen above that the adjoint surfaces of order $n-2$ form a system whose freedom is only $(n+p-2)$. Defining K_{n-2}, the cost of adjointness to a surface of order $n-2$, as the number of conditions to be satisfied by a general surface of order $n-2$ before it becomes adjoint, we have therefore the equality

$$K_{n-2} = \frac{n(n-1)(n+1)}{6} - 1 - (n+p-2).$$

Further, we cannot take $n+2p-2$ generators arbitrarily to form one of the sets mentioned. In order that a Φ_{n-2} should pass through a prescribed generator, it is necessary and sufficient to make it pass through one point of the generator. For the generator has already an $(\alpha-1)$-fold point and a $(\beta-1)$-fold point where it meets the directrices, on the surface. Since then the system of Φ_{n-2} has a freedom $(n+p-2)$, we can choose $(n+p-2)$ generators only at random; they serve to fix a Φ_{n-2} and p further generators are thereby determined.

Consider now any one particular adjoint Φ_{n-2} cutting out a set of $(n+2p-2)$ generators. Let ρ be the freedom of the system of adjoints Φ_{n-1} passing through this set of generators. Then the given scroll is normal in S_ρ. Now the freedom of the general system of adjoints Φ_{n-1} subjected to no further conditions as to passing through prescribed

* Picard and Simart, *Théorie des fonctions algébriques de deux variables*, Vol. 2 (1906), p. 157. The introduction of the numbers ω in the theory of surfaces is due to Castelnuovo.

generators is

$$\frac{n(n+1)(n+2)}{6} - 1 - K_{n-1},$$

where K_{n-1} is the cost of adjointness to a surface of order $n-1$. Hence we have

$$\rho = \frac{n(n+1)(n+2)}{6} - 1 - K_{n-1} - H_{n-1},$$

where H_{n-1} indicates the number of conditions to be satisfied in order that an adjoint Φ_{n-1} should pass through the set of generators cut out by the fixed Φ_{n-2}.

We proceed to evaluate K_{n-1}. In the case of plane curves it is known that the cost of adjointness k_m remains the same for all values of m greater than or equal to $n-3$. But this is not so in the case of surfaces. For instance, if a line is to lie entirely on a surface of order N, the number of conditions to be satisfied is $N+1$ and varies with N. Noether* has given a formula for the number of conditions to be satisfied in order that a surface of order m should have a given line as a λ-fold line. The number is

$$N_m = \frac{\lambda(\lambda+1)}{6}(3m-2\lambda+5).$$

Hence

$$N_{m+1} - N_m = \frac{\lambda(\lambda+1)}{2},$$

and this number does not depend on m. In our case the adjoint surface must have the two directrices as $(\alpha-1)$ and $(\beta-1)$-fold lines, and every δ-fold generator for a $(\delta-1)$-fold line. Hence it would appear that

$$K_{n-1} - K_{n-2} = \frac{(\alpha-1)\alpha}{2} + \frac{(\beta-1)\beta}{2} + \Sigma\frac{(\delta-1)\delta}{2}$$

$$= \frac{(n-1)(n-2)}{2} - p. \qquad (\gamma_1)$$

For the purpose of calculating the difference between the two K's, we have considered the various lines—directrices and multiple generators—as non-intersecting lines. But actually this is not so. The fact of their intersection thus accounts for a possible reduction in the difference as calculated above. Expressing this reduction by ϵ, we have

$$K_{n-1} - K_{n-2} = \frac{(n-1)(n-2)}{2} - p - \epsilon. \qquad (\gamma)$$

* *Math. Annalen*, Vol. 3 (1869), p. 177.

Hence combining (α), (β), (γ), we obtain

$$\rho = (2p+3n-3)+\epsilon-H_{n-1}.$$

There are two cases where ϵ equals zero. One is when there are no multiple generators at all, and the other is when we have only double generators.

21. It remains to evaluate H_{n-1}: we make a short digression to the theory of plane curves. The general curves C_{n-3} form a system of freedom

$$\frac{n(n-3)}{2},$$

but the adjoints ϕ_{n-3} are only a system of freedom $(p-1)$. Thus the cost of adjointness is

$$k_{n-3} = \frac{n(n-3)}{2}-(p-1),$$

and this remains the same for higher adjoints. The ϕ_{n-2} through the $2p-2$ free intersections of a fixed ϕ_{n-3} form a system of freedom r, where

$$r = n-p+i, \quad i \geqslant 0.$$

Now the freedom of the general system of adjoints ϕ_{n-2} subjected to no further conditions is

$$\frac{(n-2)(n+1)}{2}-\left[\frac{n(n-3)}{2}-(p-1)\right].$$

Hence the number of conditions imposed on a ϕ_{n-2} by the free intersections of a fixed ϕ_{n-3} is given by

$$\begin{aligned} h_{n-2} &= \frac{(n-2)(n+1)}{2}-\left[\frac{n(n-3)}{2}-(p-1)\right]-r \\ &= 2p-2-i. \end{aligned}$$

So also we have

$$\begin{aligned} h_{n-1} &= \frac{(n-1)(n+2)}{2}-\left[\frac{n(n-3)}{2}-(p-1)\right]-r \\ &= 2p-2+n-i. \end{aligned}$$

In other words, if of the free intersections of a fixed ϕ_{n-2} we compel an adjoint ϕ_{n-1} to pass through all but i, then the ϕ_{n-1} will automatically pass through the i points, however they may have been selected.

Now consider H_{n-1}. For an adjoint Φ_{n-1} to pass through a specified

generator, it is necessary and sufficient to make it pass through two arbitrary points of the generator; for the generator already meets the adjoint in

$$(\alpha-1)(\beta-1) = n-2$$

points, where it meets the directrices. We have therefore as a first approximation

$$H_{n-1} = 2\,(2p-2+n)-\epsilon', \quad (\epsilon' \geqslant 0),$$

and we obtain

$$\rho = n-2p+1+\epsilon+\epsilon', \quad (\epsilon, \epsilon' \geqslant 0).$$

This is the result obtained by Segre.

Further, the plane section of a Φ_{n-1} is ϕ_{n-1}. Select any group of $(2p-2+n-i)$ generators from the given set of $(2p-2+n)$ generators. Take two arbitrary planes, and make an adjoint Φ_{n-1} pass through the intersections by these planes of the selected group of generators. This implies

$$2\,(2p-2+n-i)$$

conditions at the most, and the Φ_{n-1} will automatically pass through the traces on the two planes of the remaining i generators. We have seen above that a Φ_{n-1} passing through two points of a generator contains that generator entirely. Thus Φ_{n-1} will contain entirely every one of the generators in the set. Thus

$$H_{n-1} = 2\,(2p-2+n-i)-\epsilon', \quad (\epsilon' \geqslant 0),$$

and we reach a second approximation

$$\rho = n-2p+1+2i+\epsilon+\epsilon', \quad (\epsilon, \epsilon' \geqslant 0).$$

It does not appear possible to go any further with the above argument. But in particular cases where the multiplicities of the multiple generators are known, we should be able to find a lower limit for ϵ. It may be added that though the above result has been proved only for scrolls with two directrices, it holds in more general cases. For consider any given scroll F_n^r normal in S_ρ with its plane section normal in S_r. If the given scroll does not possess two directrices, assume that S_ρ is higher than S_3. Consider now the scroll F_n^r situated normally in S_ρ. Project it from $\rho-3$ points on itself on to a S_3. Assume that the new scroll thus obtained will be normal in S_3; it is of order $(n-\rho+3)$, and its plane section will be normal in $S_{r-\rho+3}$. Hence on applying the above formula to the latter scroll, we find that

$$3 \geqslant (n-\rho+3)-2p+1+2i,$$

for the function i remains the same for projections from simple points of

a curve. Hence

$$\rho \geqslant n-2p+1+2i.$$

And indeed the same line of argument can be used in the case of a scroll with a single double curve; and we obtain the same result except that the correction ϵ does not arise. It is necessary to add that the above result is effectively the same as Segre's result. For when the plane section is non-special, the two results are identical. On the other hand, when the plane section is special, the above result gives no useful information; for when a curve is special we have by Clifford's theorem

$$r = n-p+i \leqslant n/2 \leqslant p-i,$$

and hence

$$n-2p+2i \leqslant 0.$$

22. We can however deduce some useful inferences. In conjunction with Segre's lemma, the above result yields the upper and lower limits for the normal space in terms of the order and the genus. For instance, for a F_n^2 the normal space is either S_{n-2} or S_{n-3}; for a F_n^3 it is S_{n-3}, S_{n-4}, or S_{n-5}. And, in general, when the hyperplane section is not special,

$$n-2p+1 \leqslant \rho \leqslant n-p.$$

Again, given a scroll F_n^p in ordinary space, the system of plane sections is an ∞^3 system of curves of grade n. This system may or may not be complete. If the normal space for the scroll is S_ρ, then the complete system of which the system of plane sections forms part is a system of freedom ρ. On any one particular curve of the complete system, the remaining curves of the system cut out a linear series of sets of n points, and this series is of freedom $\rho-1$. This series is called the characteristic series of the completed system of plane sections; and we may consider the characteristic series as cut out on any one curve of the system, and in particular on one of the plane sections. But on a curve C_n^p normal in S_r the complete series of sets of n points is a series of freedom r. There is thus a deficit in the freedom of the characteristic series given by

$$\delta = r-(\rho-1).$$

If we assume that

$$\rho \geqslant n-2p+1+2i,$$

it follows that

$$\delta \leqslant p-i.$$

Thus when the plane section is non-special, we find that the deficit is not greater than the genus, which is a well known result. On the other hand, when the plane section is special we can find a further limit. We have remarked that, for $i > 0$,

$$n-2p+2i \leqslant 0.$$

Hence $$n-2p+2i+3 \leqslant 3,$$

and since we always have $$\rho \geqslant 3,$$

it follows that $$\rho \geqslant n-2p+2i+3.$$

Therefore $$\delta \leqslant p-i-2.$$

We shall now show how, when

$$0 < p \leqslant n-3,$$

we may construct a scroll F_n^p for which the deficit δ is as low as one. For suppose

$$p = n-3-\epsilon.$$

Consider now a plane curve C_n^p with a $(n-2)$-fold point and $(\epsilon+1)$ double points. Its genus is given by

$$p = \frac{(n-1)(n-2)}{2} - \frac{(n-2)(n-3)}{2} - (\epsilon+1)$$

$$= n-3-\epsilon.$$

We can now make the curve C_n^p the base of a scroll F_n^p with a $(n-2)$-fold directrix and a 2-fold directrix, besides ϵ double generators, in a variety of ways. The equation of the scroll would be of the form

$$x^2(z, t)_{n-2} + xy(z, t)_{n-2} + y^2(z, t)_{n-2} = 0.$$

Let $$x+\lambda_\kappa y = 0, \quad z+\mu_\kappa t = 0 \quad (\kappa = 1 \dots \epsilon)$$

represent the double generators. Consider the transformation

$$\frac{x}{X} = \frac{y}{Y} = \frac{z}{Z} = \frac{t}{T} = \frac{(x+\lambda_\kappa y)\, t/(z+\mu_\kappa t)}{\theta_\kappa} \quad (\kappa = 1 \dots \epsilon);$$

we obtain thereby a scroll F_n^p situated in $S_{\epsilon+3}$ which projects from the θ vertices into the given scroll. Now it is obvious that the plane section of the scroll is non-special; for it is a hyperelliptic curve and no hyperelliptic curve can be special. Hence we have

$$r = n-p = \epsilon+3;$$

Segre's lemma gives $\rho \leqslant r$, and it can be proved (cf. § 25 below) that in the present case $\rho \geqslant r$; hence

$$\rho = \epsilon+3.$$

Thus the deficit is given by

$$\delta = r-(\rho-1) = 1.$$

23. We now continue the work in §§ 18, 19; and we propose to prove a result due to Prof. Baker. The result is as follows. *Given a scroll F_n^p with an α directrix and a β directrix, its normal space is given by*

$$\rho = 1+u+v,$$

where u, v are defined with reference to an arbitrary plane section; u is the freedom of the system of adjoints of order $n-1$ passing through the free intersections of a fixed adjoint of order $n-2$ and having the foot of the α directrix for an α-fold point, v being defined similarly with respect to the foot of the β directrix.

The equation of the scroll F_n^p with two directrices is of the form

$$[(x, y)_\alpha (z, t)_\beta] = 0,$$

where

$$\alpha+\beta = n.$$

Any adjoint of order $n-2$ is of the form

$$[(x, y)_{\alpha-1} (z, t)_{\beta-1}] = 0.$$

Consider now an adjoint of order $n-1$. If $x^p y^q z^r t^s$ is the general term in its equation, we have

$$p+q+r+s = n-1, \quad p+q \geqslant \alpha-1, \quad r+s \geqslant \beta-1,$$

and therefore

$$r+s \leqslant \beta, \quad p+q \leqslant \alpha.$$

Thus $(p+q)$ is either α or $\alpha-1$, and $(r+s)$ is either β or $\beta-1$. Therefore the equation of any adjoint of order $n-1$ is of the form

$$[(x, y)_\alpha (z, t)_{\beta-1}]+[(x, y)_{\alpha-1} (z, t)_\beta] = 0.$$

Indicating the two parts of the expression on the left by A and B, we shall show that A as well as B represents an adjoint surface. We observe first that A and B have no single term common. Keeping the equations of the two directrices unaltered, we can so choose the vertices of the fundamental tetrahedron that any given k-fold generator is represented by

$$x = z = 0.$$

Since $A+B$ is an adjoint, it follows that for every term in $A+B$ we must have

$$p+r \geqslant k-1.$$

Since A and B have no single term common, it follows that for every term in A as well as for every term in B the above relation is satisfied. Thus not only does $A+B$ behave adjointly with respect to the k-fold generator, but each of A and B taken separately behaves adjointly; and

what is true of one multiple generator is equally true of every other multiple generator. Hence A is an adjoint surface, and so also is B.

Further, the α directrix is not an $(\alpha-1)$-fold line but an α-fold line on A. We may therefore say that A is an adjoint surface of order $n-1$, which is super-adjoint with respect to the α directrix. And when we wish to be understood to mean such a surface, we shall speak of it as a Φ_α of order $n-1$. So also we shall speak of a Φ_β of order $n-1$. The surface B is thus a Φ_β of order $n-1$. Thus the most general adjoint Φ_{n-1} is the sum of adjoints Φ_α and Φ_β of order $n-1$.

Again, if $A+B$ is to pass through any assigned generator, it is necessary and sufficient that each of A and B should pass through it. To prove this it is enough to change the axes as above and repeat the argument. Consider now a fixed adjoint Φ_{n-2}, C, cutting out a set of $(n+2p-2)$ generators. Every adjoint Φ_{n-1} through these generators will then split up into the sum of a Φ_α and a Φ_β of order $n-1$, each of which passes through all these generators. Let now u be the freedom of the system of adjoints Φ_α of order $n-1$ passing through the set, and v the freedom of the system of adjoints Φ_β of order $n-1$ passing through the set. Then the freedom of the system of adjoints Φ_{n-1} passing through the fixed set is given by

$$\rho = 1+u+v,$$

and S_ρ is the normal space for the scroll. Suppose now there are no irreducible adjoints Φ_α, Φ_β of order $n-1$. Then we have at least the surfaces

$$C(x+\lambda y) = 0, \quad C(z+\mu t) = 0,$$

satisfying the conditions. Thus we have

$$u \geqslant 1, \quad v \geqslant 1,$$

and hence

$$\rho \geqslant 3.$$

24. We proved in § 19 that ω_{n-2} was zero; and this was found to result from the fact that the Φ_{n-2} was also a scroll with the two directrices for $(\alpha-1)$-fold and $(\beta-1)$-fold directrices, and that on every ϕ_{n-2} we could build up a Φ_{n-2}. Now we observe that A is a scroll with the two directrices for α-fold and $(\beta-1)$-fold directrices; and that the section of A by an arbitrary plane is a ϕ_α of order $n-1$, that is a curve of order $n-1$ which is not only adjoint to the section of the given scroll by the same plane but is super-adjoint with respect to the foot of the α directrix.

It follows* as before that the freedom of the system of surfaces A is precisely the same as the freedom of the system of adjoints ϕ_α of order $n-1$ for an arbitrary plane section of the scroll.

Again, in order that a Φ_α of order $n-1$ should pass through an assigned generator, it is enough to make it pass through any one arbitrary point of the generator; for the surface already passes through $n-1$ points of the generator situated on the two directrices. Thus if the adjoint ϕ_α of order $n-1$ passes through any assigned point of the plane section of the scroll, then the adjoint Φ_α of order $n-1$ built up on it passes entirely through the generator issuing from that point. Therefore the freedom of the system of Φ_α of order $n-1$ passing through the set of generators cut out by C the fixed Φ_{n-2} is precisely the freedom of the system of ϕ_α of order $n-1$ passing through the set of points cut out on the plane section of the scroll by the section of C. Precisely similar remarks apply to v, and we obtain the result

$$\rho = 1+u+v,$$

where u, v are defined with reference to a plane section of the scroll: u is the freedom of the system of ϕ_α of order $n-1$ passing through the free intersections of a fixed ϕ_{n-2}, and v is defined similarly.

25. The above result is a precise result, and it requires for its evaluation only a knowledge of the plane section. But in any particular case its evaluation is a matter of considerable difficulty. We observe that the freedom of the system of ϕ_{n-1} passing through the free intersections of a fixed ϕ_{n-2} is r, where S_r is the normal space for the plane section of the scroll. In order to make any such ϕ_{n-1} super-adjoint at the foot of the α directrix, it is necessary to subject it to $(\alpha-\epsilon_1)$ further conditions where

$$0 \leqslant \epsilon_1 \leqslant \alpha.$$

We thus have $\quad u = r-(\alpha-\epsilon_1), \quad v = r-(\beta-\epsilon_2) \quad (0 \leqslant \epsilon_2 \leqslant \beta).$

Hence

$$\begin{aligned}\rho &= 1+u+v \\ &= n-2p+1+2i+\epsilon_1+\epsilon_2.\end{aligned}$$

This is practically the result obtained in § 21, except that a new significance is here attached to the correction.

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* The point of the argument is simply this: given a plane curve C_n^p with a n_1-fold point and a n_2-fold point, where $n_1+n_2 = n$ and two skew lines passing through these two singular points, there is one and only one way of building up a scroll F_n^p with the given curve as base, and having the given lines as directrices. Compare §§ 19, 26.

But we may proceed in a different way. The general system of ϕ_{n-1} passing through the free intersections of a fixed ϕ_{n-2} cuts out on the curve a g_n^r where S_r is the normal space for C_n^p the plane section of the scroll. This linear series is complete and is imaged by the hyperplane sections of the normal curve γ_n^p situated in S_r. Again, the system of ϕ_α of order $n-1$ passing through the free intersections of the fixed ϕ_{n-2} cuts out a g_β^u and is a part of the general system of ϕ_{n-1} mentioned above. Thus the g_β^u is a part of the g_n^r; and therefore its image is also a part of the image of the latter. Indicate by P the group of α points distinct or otherwise of γ which project into the foot of the α directrix on the plane section under consideration. It is then easy to see that if S_h is the space in which the points P are situated, we have

$$h \leqslant r-2.$$

In particular, the plane section C_n^p is the projection of a curve on a quadric;* and if it is normal in space higher than S_3, the latter curve is again the projection of a curve on a hyper-cone. To the α-fold point of C_n^p correspond α points on a generator of the quadric, and the corresponding α points on the curve in S_4 are situated in a generator plane or on a line.

The system of hyperplanes of S_r which pass through the space S_h is then the image of the g_β^u. Thus we have

$$u = r-h-1,$$

and in virtue of the above inequality, we have

$$u \geqslant 1.$$

It remains to determine h. In† general α arbitrary points of a curve in S_r determine a $S_{\alpha-1}$ which does not meet the curve in any further points. We may therefore say that

$$h = \alpha-1,$$

and therefore that $u = r-\alpha,$

provided $r-\alpha \geqslant 1.$

* Cf. § 26.

† Cf. Severi, *Geometria Algebrica*, Padua, 1908, p. 171. There is a proviso that $\alpha-1 \leqslant r-2$, which is not of importance.

Similar considerations apply to v. *Hence we have**

$$\rho = 1+[r-\alpha]+[r-\beta],$$

where $[x]$ *stands for* x *when* x *is greater than zero, and equals one for all other values of* x. But whenever we have

$$r-\alpha \leqslant 1, \quad r-\beta \leqslant 1$$

simultaneously, a correction, to be added to ρ, may arise. This correction occurs when there exists an α-fold generator ($\alpha < \beta$) and it equals unity; a simple example is the scroll F_7^3 of the [3, 4] type with a triple generator. Again, the correction may be as high as two; a simple example is the F_8^3 of the type [4, 4] with a fourfold generator. Such a correction occurs in particular when the two directrices are of equal multiplicity.

26. We come now to a different question. Given a twisted curve C_n^p, under what circumstances can we transform it birationally into a plane curve of the same order which can be made the base of a scroll? We shall limit ourselves to the consideration of scrolls with two directrices.

Consider a curve of order n lying on a quadric, and meeting each generator of the one system in α points and each generator of the other system in β points, so that

$$\alpha+\beta = n.$$

We can project the curve from an arbitrary point of the quadric into a plane curve of the same order having an α-fold point and a β-fold point, besides other possible multiple points. Conversely, such a plane curve can always be thought of as the projection of a curve of the same order lying on a quadric and meeting the generators of either system in α and β points. For the equation of the plane curve is of the form

$$[(z, x)_\alpha (z, y)_\beta] = 0.$$

Considered as a locus in space it represents a cone with (0, 0, 0, 1) for vertex and having the two lines

$$z = x = 0, \quad z = y = 0,$$

as α-fold and β-fold generators. The residual intersection of the cone with the quadric

$$zt = xy$$

is a curve of the requisite nature. Thus the question at the beginning of

* I owe this result to Prof. Baker, whose proof, however, was different.

the article is the same as the other question, When can we transform birationally a given twisted curve into a curve of the same order lying on a quadric?

It is difficult to give a complete answer, but some particular classes of curves may be considered. We begin with the following lemma. Any curve C_n which possesses a g^1_α and a g^1_β can always be transformed birationally into a plane curve of order n having an α-fold point and a β-fold point. For let the two linear series be given by loci of the form

$$A-\lambda A'=0, \qquad B-\mu B'=0;$$

so that between λ and μ there is a (β, α)-correspondence. Consider now two networks of lines in a plane having for their centres the origin and the point x_0, y_0, and given by

$$y-\lambda x=0, \qquad (y-y_0)-\mu(x-x_0)=0.$$

If the (β, α) relation between λ, μ be indicated by

$$F(\overset{\beta}{\lambda}, \overset{\alpha}{\mu})=0, \tag{i}$$

the locus of the points of intersection of corresponding lines in the two networks is given by

$$F\left(\overset{\beta}{\frac{y}{x}}, \overset{\alpha}{\frac{y-y_0}{x-x_0}}\right)=0.$$

This equation may be written in the form

$$[(x, y)_\beta (x-x_0, y-y_0)_\alpha]=0, \tag{ii}$$

and is of order n, provided the value y_0/x_0 of λ does not correspond to the same value of μ; that is, provided y_0/x_0 is not one of the roots of the equation

$$F(\overset{\beta}{\theta}, \overset{\alpha}{\theta})=0.$$

The curve represented by an equation of the form (i) can be made the base of a scroll in a variety of ways; for we have only to take two non-intersecting lines through the α point and the β point, and set up between the points of the two lines the correspondence indicated by the relation (i). Our endeavour is thus to find two such linear series if possible on the given curve.

This is easily done when the given twisted curve C^ρ_n is hyperelliptic; for on such a curve we have a g^1_2, and the planes passing through an arbitrary bisecant yield a g^1_{n-2}. We may therefore limit ourselves to the

case of non-hyperelliptic curves. On such a curve, the residue of an arbitrary point with respect to the canonical series is a g_p^1; and on taking two arbitrary points, we obtain two such series. Thus any curve of order greater than or equal to $2p$ permits a transformation of the desired type.

We shall now show that on any curve which is not hyperelliptic there always exists a $g^1_{p-1-\epsilon}$, where ϵ has some particular value positive or zero. Projecting the canonical curve C^p_{2p-2} normally situated in S_{p-1} from points on itself, we obtain a C^p_{p+1} situated in a plane; and it has multiple points equivalent in effect to

$$\frac{p(p-1)}{2} - p = \frac{p(p-3)}{2}$$

double points. Thus* the lines through one of these multiple points yield on the plane C^p_{p+1} a $g^1_{p-1-\epsilon}$ ($\epsilon \geqslant 0$). Thus on the given curve also there exists a $g^1_{p-1-\epsilon}$, where ϵ has some definite value positive or zero.

In particular when the order of the curve is $2p-1$, we have not only a $g^1_{p-1-\epsilon}$ but also a $g^1_{p+\epsilon}$, for, on a non-hyperelliptic curve, we have linear series of any order greater than or equal to p. Consider again the case where the order of the curve is $2p-2$, we have on the one hand a $g^1_{p-1-\epsilon}$, and we require a $g^1_{p-1+\epsilon}$, where ϵ has some definite positive integral value which may be zero. Such a series certainly exists when ϵ is greater than zero; and it exists as certainly when ϵ is zero, for that only means that all the singularities of the plane C^p_{p+1} are double points, and there are certainly at least two double points.

We have thus considered all curves of order

$$n \geqslant 2p-2,$$

besides hyperelliptic curves. Limiting ourselves to the case of twisted curves of order less than nine, the only curves that do not fall under the former class are the

$$C^5_7,\ C^6_7;\quad C^6_8,\ C^7_8,\ C^8_8,\ C^9_8.$$

These curves are all special, and hence none of them is hyperelliptic. It is known† that the twisted C^5_7 lies on a cubic surface, and it can be shown‡ that it posseses a g^1_3 and a g^1_4. Again a C^6_7 is the curve of that order of maximum genus; hence it lies on a quadric. The curve† C^6_8 lies on a

* It is necessary that p should be greater than 3. But this is hardly a restriction.

† Cf. Noether, *Berliner Abhandlungen*, 1882, pp. 92, 96.

‡ Cf. Baker, *Proc. London Math. Soc.*, Ser. 2, Vol. 11, p. 294.

cubic surface and permits two 4-point chords, the planes through which yield two series g_4^1.

Passing to the curve C_8^7, we have a canonical curve C_{12}^7 in S_6. The given curve is normal in S_3, hence* 5 arbitrary points of the canonical curve determine a S_4 which passes through 3 further points. The system of hyperplanes (S_5) through this S_4 yields a g_4^1. Varying the arbitrary points we obtain another g_4^1. Lastly, it is known† that the curves C_8^8, C_8^9 lie on a quadric. Thus a transformation of the type indicated at the beginning of this article is always possible in the case of non-hyperelliptic curves, of curves of order not less than $2p-2$, and finally of all curves of order less than nine.

There are questions raised in this paper which have not been answered. A detailed study of scrolls which in their normal space are situated on one or more quadric loci would be interesting. But this must be reserved for a future occasion.

[*Added November* 1920.—A correspondent, to whom the author is much indebted, sends, as an example of cases for which the theorem of § 7 fails, that of a F_7^3 constructed by placing in correspondence a C_4^3 with a C_5^3, in different planes, with two common self-corresponding points. This surface is normal in S_3, but has not two directrices. It is not shown, in fact, in § 8 (p. 238), that (even if the argument in § 7 be complete) it is possible to place the two curves in correspondence, so as to have $n-2$ common corresponding points. The result of § 10 is therefore unproved. And, in fact, a F_9^3, normal in S_5, may be constructed by placing in correspondence a C_7^3, having a triple point, with a C_2 taken doubly, these curves being in independent planes in S_5.

Further, attention should be called to the facts that the inequality $\frac{1}{2}n \leqslant \mu$, of § 4, evidently assumes the existence of two C_μ, and that the statement (§ 10), that two projections of the same scroll are in *homographic* uniform correspondence is inaccurate.

The failure of § 7 may affect individual statements made elsewhere in the paper; the author will be grateful for other corrections.]

* Cf. Segre, *Annali di Math.*, Vol. 22, p. 126.

† Cf. Noether, *Berlin Abhand.*, 1882, pp. 92, 96.

THE ZEROS OF LOMMEL'S POLYNOMIALS

By G. N. WATSON.

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THE zeros of the polynomial $R_{m,\nu}(z)$, defined by the equation

$$R_{m,\nu}(z) = \sum_{n=0}^{\leqslant \frac{1}{2}m} (-)^n {}_{m-n}C_n \frac{\Gamma(\nu+m-n)(\frac{1}{2}z)^{-m+2n}}{\Gamma(\nu+n)},$$

have been examined by Hurwitz* when the index ν is an unrestricted real number. The ultimate object of his investigation was to obtain information concerning the zeros of Bessel functions by an application of the formula

$$\lim_{m\to\infty} \frac{(\frac{1}{2}z)^{\nu+m} R_{m,\nu+1}(z)}{\Gamma(\nu+m+1)} = J_\nu(z).$$

It appears, however, that the reasoning applied by Hurwitz to $R_{m,\nu}(z)$ is not altogether satisfactory when ν has any assigned negative value (less than -1); and in view of the importance of his results and of the applications of them, it seems desirable to give a rigorous demonstration of them.

Since the power-series which represent the Lommel polynomial $R_{m,\nu}(z)$ and the Bessel function $J_\nu(z)$ proceed in powers of z^2, it is convenient to make a change of variable; and accordingly the functions $f_\nu(z)$ and $g_{m,\nu}(z)$ are defined by the equations†

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* *Math. Ann.*, Vol. 33 (1889), pp. 246–266. The investigation is reproduced by Nielsen, *Handbuch der Cylinderfunktionen* (1904), pp. 163–172.

† The notation $R_{m,\nu}(z)$ is that used by Lommel, *Math. Ann.*, Vol. 4 (1871), pp. 108–111. But the notation used by Hurwitz has been modified. In the first place, Hurwitz uses n to denote the (unrestricted) order of the Bessel function and restricts ν to be an integer; but Lommel's notation in which ν is unrestricted and n is an integer seems greatly preferable. Secondly the alternation of the signs in the series for $f_\nu(z)$ and $g_{m,\nu}(z)$, which takes effect in this paper but not in the memoir by Hurwitz, makes the transition, to Bessel functions and Lommel polynomials from functions of Hurwitz' types, seem more natural.

$$f_\nu(z) = \sum_{n=0}^{\infty} \frac{(-)^n z^n}{n!\,\Gamma(\nu+n+1)},$$

$$g_{m,\nu}(z) = \sum_{n=0}^{\leqslant \frac{1}{2}m} (-)^n {}_{m-n}C_n \frac{\Gamma(\nu+m-n+1)\, z^n}{\Gamma(\nu+n+1)},$$

so that $J_\nu(z) = (\tfrac{1}{2}z)^\nu f_\nu(\tfrac{1}{4}z^2)$, $R_{m,\nu+1}(z) = (\tfrac{1}{2}z)^{-m} g_{m,\nu}(\tfrac{1}{4}z^2)$,

$$\operatorname{Lim}_{m\to\infty} \{g_{m,\nu}(z)/\Gamma(\nu+m+1)\} = f_\nu(z).$$

With this notation, the recurrence formulæ, due to Lommel, assume the forms

(1) $$g_{m+1,\nu}(z) = (\nu+m+1)\, g_{m,\nu}(z) - z g_{m-1,\nu}(z),$$

(2) $$g_{m+1,\nu-1}(z) = \nu g_{m,\nu}(z) - z g_{m-1,\nu+1}(z),$$

(3) $$\frac{1}{z^{\nu-1}} \frac{d}{dz} \{z^\nu g_{m,\nu}(z)\} = z g_{m-1,\nu}(z) + g_{m+1,\nu+1}(z),$$

(4) $$z^{m+2} \frac{d}{dz} \left\{ \frac{g_{m,\nu}(z)}{z^{m+1}} \right\} = g_{m+1,\nu-1}(z) - g_{m+1,\nu}(z).$$

These are, of course, easily established from the definition of $g_{m,\nu}(z)$.

Again, Crelier's three-term relation,* which in Lommel's notation is written

$$R_{n,\nu}(z)\, R_{p-q-1,\nu+q+1} + R_{p,\nu}(z)\, R_{q-n-1,\nu+n+1}(z) + R_{q,\nu}(z)\, R_{n-p-1,\nu+p+1}(z) = 0,$$

assumes the form

$$z^q g_{n,\nu}(z)\, g_{p-q-1,\nu+q+1}(z) + z^n g_{p,\nu}(z)\, g_{q-n-1,\nu+n+1}(z) + z^p g_{q,\nu}(z)\, g_{n-p-1,\nu+p+1}(z) = 0.$$

We shall require the special case of this obtained by giving n, p, q the values m, 0, $m+2$, that is to say

$$z^{m+2} g_{m,\nu}(z)\, g_{-m-3,\nu+1}(z) + z^m g_{0,\nu}(z)\, g_{1,\nu+m+1}(z) + g_{m+2,\nu}(z)\, g_{m-1,\nu-1}(z) = 0.$$

Since functions of negative parameter are expressible in terms of functions of positive parameter by the relation (due to Graf)

$$z^{m+2} g_{-m-3,\nu+1}(z) = -\, g_{m+1,\nu+1}(z),$$

we have

(5) $$g_{m,\nu}(z)\, g_{m+1,\nu+1}(z) - g_{m+2,\nu}(z)\, g_{m-1,\nu+1}(z) = z^m g_{0,\nu}(z)\, g_{1,\nu+m+1}(z)$$
$$= (\nu+m+2)\, z^m.$$

* *Ann. di Mat.*, Ser. 2, Vol. 24 (1896), p. 143.

If, for brevity, we omit the suffix denoting the order ν, the result of eliminating alternate functions from the system (1) is

$$(6)\quad \begin{cases} (\nu+2)\,g_4(z) = c_2(z)\,g_2(z) - (\nu+4)\,z^2 g_0(z), \\ (\nu+4)\,g_6(z) = c_4(z)\,g_4(z) - (\nu+6)\,z^2 g_2(z), \\ \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ (\nu+2s)\,g_{2s+2}(z) = c_{2s}(z) - (\nu+2s+2)\,z^2 g_{2s-2}(z), \\ \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ (\nu+2m-2)\,g_{2m}(z) = c_{2m-2}(z)\,g_{2m-2}(z) - (\nu+2m)\,z^2 g_{2m-4}(z), \end{cases}$$

where $c_m(z)$ denotes $(\nu+m+1)\{(\nu+m)(\nu+m+2)-2z\}$. We have now enumerated the fundamental properties of the functions $g_{m,\nu}(z)$ which will be subsequently required.

2. Hurwitz shows that the system of functions $g_{2m}(z)$, $g_{2m-2}(z)$, ..., $g_0(z)$ (with certain modifications of sign when ν is negative), form a set of Sturm's functions possessing the property that the real zeros of contiguous functions alternate: to prove this, it is sufficient to prove that, for sufficiently large values of m, the real zeros of $g_{2m}(z)$ and $g_{2m-2}(z)$ alternate, and then the existence of the system (6) shows that the functions form a set of Sturm's functions.

Hurwitz claims to prove that the real zeros of $g_{2m}(z)$ and $g_{2m-2}(z)$ alternate by proving that the quotient $g_{2m}(z)/g_{2m-2}(z)$ is monotonic except at the zeros of the denominator, where the quotient obviously has an infinite discontinuity. The alternation of the zeros is then evident from a consideration of the graph of the quotient. But Hurwitz' proof, which is substantially as follows, seems defective when $\nu+2$ is negative.

If
$$g_{p,\nu}(z)\,g'_{q,\nu}(z) - g_{q,\nu}(z)\,g'_{p,\nu}(z) = \Delta_{p,q},$$

we have
$$g^2_{2m-2}(z)\,\frac{d}{dz}\left\{\frac{g_{2m}(z)}{g_{2m-2}(z)}\right\} = -\Delta_{2m,2m-2},$$

while it follows without difficulty from (1) that

$$\begin{cases} \Delta_{2m,2m-2} = g^2_{2m-2}(z) + (\nu+2m)\,\Delta_{2m-1,2m-2}, \\ \Delta_{2m-1,2m-2} = z^2\Delta_{2m-3,2m-4} + (\nu+2m-2)\,g^2_{2m-3}(z), \end{cases}$$

and so $\Delta_{2m,2m-2} = g^2_{2m-2}(z) + (\nu+2m)\sum_{r=1}^{m-1}(\nu+2r)\,z^{2m-2r-2}\,g^2_{2r-1}(z)$.

Hence, if $m \geqslant 1$, $\Delta_{2m,2m-2}$ is a sum of positive terms *provided that*

$\nu+2$ *is positive*, and the monotonic property of the quotient is then manifest.

When ν has an assigned value* less than -2, the terms of low rank in the summation are negative; an approximation to the general term of high rank is

$$(\nu+2m)(\nu+2r)\, z^{2m-2r+2} \{\Gamma(\nu+2r)\}^2 f_\nu^2(z),$$

and, by taking m sufficiently large, enough terms of high rank are obtained† to neutralise the terms of low rank, *except when* z *lies in the neighbourhood of a zero of* $f_\nu(z)$.

After this point, the demonstration of Hurwitz seems to me to be defective; in the first place he does not consider the neighbourhood of the zeros of $f_\nu(z)$, but only the zeros themselves; this would be adequate if only a bounded domain of values of z were being considered, for, if any point near a zero of $f_\nu(z)$ were taken in such a domain, it would be possible to choose m so large that $|\{g_{m,\nu}(z)/\Gamma(\nu+m+1)\}|$ was greater than some positive number [say $|\frac{1}{2}f_\nu(z)|$] there; but the larger m is taken, the greater the number of neighbourhoods to be considered, and the argument seems to fail. Again, if we only consider the actual zeros (as Hurwitz does), his argument seems to fail for much the same reason. For he uses the approximation

$$\Delta_{2m-1,\,2m-2} = \left(\frac{\pi}{\sin\nu\pi}\right)^4 z^{2m-2}+\ldots;$$

this is obtained by expressing $\Delta_{2m-1,\,2m-2}$ in terms of functions of the type $g_{m,\nu}(z)$, using the relation

$$g_{m,\nu}(z) = -\frac{\pi}{\sin\nu\pi}\left[z^{m+1}f_{\nu+m+1}(z)f_{-\nu}(z)-(-)^{m+1}f_\nu(z)f_{-\nu-m-1}(z)\right],$$

and replacing $f_\nu(z)$ by zero after the differentiations have been performed.

The approximation given is the lowest term in the expansion of $\Delta_{2m-1,\,2m-2}$ in ascending powers of z when this simplification has been made; subsequent terms in the expansion are $O(m^{-1})$, $O(m^{-2})$, ... when m is large. Now, if $|z|$ were bounded, it would follow that, by taking m sufficiently large, we could make $\Delta_{2m-1,\,2m-2}$ positive; but since the largest zero of $g_{m,\nu}(z)$ is as large as $O(m)$, the argument seems to fail.

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* Since the circumstances of the problem demand that $m\to\infty$, we are evidently at liberty to take m so large that $\nu+2m$ is positive.

† Hurwitz, *loc. cit.*, p. 256, states that "So ist für jeden reellen Werth von z [$\Delta_{2m,\,2m-2}$] positiv, sobald [m] eine gewisse Zahl überschreitet."

3. We now proceed to the construction of a more rigorous investigation. We first observe that, by the formulæ stated in § 1,

$$z^{-\nu-2m+2}\, g^2_{2m-2,\,\nu}(z)\, \frac{d}{dz}\left\{\frac{z^\nu g_{2m,\,\nu}(z)}{z^{-2m+1} g_{2m-2,\,\nu}(z)}\right\}$$

$$= g_{2m-2,\,\nu}(z)\{z g_{2m-1,\,\nu}(z) + g_{2m+1,\,\nu-1}(z)\} - g_{2m,\,\nu}(z)\{g_{2m-1,\,\nu-1}(z) - g_{2m-1,\,\nu}(z)\}$$

$$= (\nu+2m)\, g^2_{2m-1,\,\nu}(z) + \{g_{2m-2,\,\nu}(z)\, g_{2m+1,\,\nu-1}(z) - g_{2m-1,\,\nu-1}(z)\, g_{2m,\,\nu}(z)\}$$

$$= (\nu+2m)\{g^2_{2m-1}(z) - z^{2m-1}\},$$

and the quotient considered is the only one which supplies after differentiation two terms one of which is a square, and the other a power of z.

Now when z is *negative* and m is taken so large that $\nu+2m$ is positive, it is clear from this analysis that

$$(-z)^{\nu+2m-1}\, g_{2m,\,\nu}(z)/g_{2m-2,\,\nu}(z)$$

is a decreasing function of z, and so the negative zeros of $g_{2m,\,\nu}(z)$ and $g_{2m-2,\,\nu}(z)$ must alternate.

We now give m any value so large that $\nu+2m$ is positive, and we choose s to be the integer such that

$$-2s > \nu > -2s-2.$$

And then by (6) the set of functions

$$g_{2m}(z),\ g_{2m-2}(z),\ \ldots,\ g_{2s+2}(z),\ g_{2s}(z),\ -g_{2s-2}(z),$$
$$+g_{2s-4}(z),\ \ldots,\ (-)^{s-1} g_2(z),\ (-)^s g_0(z),$$

form a set of Sturm's functions when z is negative.

The signs of the set of functions when z is $-\infty$ are

$$+,\ +,\ \ldots,\ +,\ +,\ -,\ +,\ \ldots,\ (-)^{s-1},\ (-)^s.$$

The signs of the set of functions when z is zero are

$$\pm,\ \pm,\ \ldots,\ \pm,\ +,\ -,\ +,\ \ldots,\ (-)^{s-1},\ (-)^s,$$

the upper signs being taken when ν lies between $-2s$ and $-2s-1$, and the lower signs being taken when ν lies between $-2s-1$ and $-2s-2$. The number of changes of sign that are gained* is 0 or 1 according to the value of ν; and so there is no negative zero or one negative zero in the respective cases.

* The reason why changes of sign are *gained* is that the coefficients of z^m in $g_{2m}(z)$ and z^{m-1} in $g_{2m-2}(z)$ have opposite signs. In the usual version of Sturm's theorem, they have the same sign.

Hence, when $\nu+2m$ is positive, $g_{2m}(z)$ has no negative zero when $-2s>\nu>-2s-1$, and it has one negative zero when

$$-2s-1>\nu>-2s-2.$$

This is Hurwitz' result concerning negative zeros of $g_{2m}(z)$.

4. The method just used is not applicable for positive values of z because $g^2_{2m-1}(z)-z^{2m-1}$ is not obviously one-signed. But we can determine the number of positive zeros of $g_{2m}(z)$ by the recurrence formulæ given at the end of § 1, combined with an application of Descartes' rule of signs. The result is:

When $-2s>\nu>-2s-1$, $g_{2m}(z)$ *has* $m-2s$ *positive zeros; when* $-2s-1>\nu>-2s-2$, $g_{2m}(z)$ *has* $m-2s-1$ *positive zeros, provided that, in each case* m *is so large that* $m+\nu$ *is positive.*

From Descartes' rule of signs it follows that the numbers just stated are the maximum numbers of positive zeros of the functions. For when ν lies between $-2s$ and $-2s-1$, the signs of the coefficients of $1, z, z^2, \ldots, z^{2s}, z^{2s+1}, z^{2s+2}, z^{2s+3}, \ldots, z^m$ in $g_{2m}(z)$ are

$$+, +, +, \ldots, +, -, +, -, \ldots, (-)^m,$$

and there are but $m-2s$ alternations of sign.

When ν lies between $-2s-1$ and $-2s-2$, the signs of the coefficients of $1, z, z^2, \ldots, z^{2s+1}, z^{2s+2}, z^{2s+3}, z^{2s+4}, \ldots, z^m$ in $g_{2m}(z)$ are

$$-, -, -, \ldots, -, +, -, +, \ldots, (-)^m,$$

and there are but $m-2s-1$ changes of sign.

Next we prove by induction that there are as many as $m-2s$ (or $m-2s-1$) positive zeros.

When ν lies between $-2s$ and $-2s-1$, the coefficients in $g_{4s}(z)$ have no alternations of sign and so $g_{4s}(z)$ has no positive zeros. Next

$$g_{4s+2}(0)=+, \quad g_{4s+2}(\infty)=-,$$

so g_{4s+2} has one positive zero $\alpha_{1,1}$.

Again, $g_{4s+4}(0)=+$, $g_{4s+4}(\infty)=+$, and from the recurrence formulæ $g_{4s+4}(\alpha_{1,1})=-$, since $g_{4s}(\alpha_{1,1})=+$. Hence $g_{4s+4}(z)$ has two positive zeros $\alpha_{1,2}$ and $\alpha_{2,2}$, where $\alpha_{1,2}<\alpha_{1,1}<\alpha_{2,2}$, and so

$$g_{4s+2}(\alpha_{1,2})=+, \quad g_{4s+2}(\alpha_{2,2})=-.$$

The mode of procedure is now obvious, and the proof when

$$-2s-1>\nu>-2s-2$$

is left to the reader.

It follows that, when $-2s > \nu > -2s-2$ and $m > s$, then $g_{2m,\nu}(z)$ has $2s$ complex zeros.

5. To investigate the complex zeros of $f_\nu(z)$, we follow Hurwitz by writing

$$\phi_m(x, y) = \frac{g_{2m+1}(z)\, g_{2m}(z') - g_{2m+1}(z')\, g_{2m}(z)}{z'-z},$$

where $z = x+iy$, $z' = x-iy$, and then the complex zeros of $g_{2m}(z)$ lie on the curve

$$\phi_m(x, y) = 0.$$

The terms of highest degree in $\phi_m(x, y)$ are

$$\tfrac{1}{6}m(m+1)(\nu+m)(\nu+m+1)\{(\nu+m)(2m+1)+m-1\}(x^2+y^2)^{m-1},$$

and, as Hurwitz shows,

$$\phi_{m+1}(x, y) = (\nu+2m+2)\, g_{2m+2}(z)\, g_{2m+2}(z') + (x^2+y^2)\, \phi_m(x, y),$$

and hence, for sufficiently large values of m, the curve $\phi_m(x, y) = 0$ lies wholly in the finite part of the plane, and $\phi_{m+1}(x, y) = 0$ lies wholly inside one or more of the portions of $\phi_m(x, y) = 0$. It follows that, as $m \to \infty$, the complex zeros of $g_{2m}(z)$ lie in a bounded region of the z-plane.

Again, since

$$\left| f_\nu(z) - \frac{g_{2m}(z)}{\Gamma(\nu+2m+1)} \right|$$

can be made as small as we please in any bounded domain of the z plane, by taking m sufficiently large, it follows that for sufficiently large values of m the number of zeros of $f_\nu(z)$ in any small area is equal to the number of zeros of $g_{2m}(z)$ in that area; and so $f_\nu(z)$ has $2s$ complex zeros. None of these zeros are real,* for if a zero were real it would be a limit point of two conjugate complex zeros of $g_{2m}(z)$ and so it would count as a double zero of $f_\nu(z)$; and $f_\nu(z)$ has no double zeros.

Again, from the series for $f_\nu(z)$ it is seen that, for the appropriate values of ν, it has one negative zero, and it cannot have more than one negative zero for the reasons just stated.

Hence $f_\nu(z)$ has $2s$ complex zeros, one negative zero or none, and it is well known to have an infinity of positive zeros. And these are Hurwitz' results concerning the zeros of Bessel functions.

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* Hurwitz deduces this from a consideration of his approximation for $\Delta_{2m-1\ 2m-2}$.

ON FOURIER'S COEFFICIENTS OF BOUNDED FUNCTIONS*

By H. STEINHAUS.

(*Communicated by* G. H. HARDY.)

[Received December 8th, 1919.—Read December 11th, 1919.]

IT is well known that, $f(t)$ and its square $f^2(t)$ being integrable functions, the series

$$2\pi a_0^2+\pi \sum_{n=0} (a_n^2+b_n^2) = \int_0^{2\pi} f^2(t)\,dt \tag{1}$$

is convergent if a_n, b_n are the Fourier's coefficients of $f(t)$, *i.e.* if

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(t)\,dt, \quad a_n = \frac{1}{\pi}\int_0^{2\pi} f(t)\cos nt\,dt, \quad b_n = \frac{1}{\pi}\int_0^{2\pi} f(t)\sin nt\,dt$$

$$(n = 1, 2, \ldots).$$

For bounded functions, that is to say for such functions $f(t)$ that

$$|f(t)| \leqslant M \quad (0 \leqslant t \leqslant 2\pi),$$

M being independent of t, the following question presents itself naturally:

Is it possible to diminish the exponent 2 without depriving the series (1) of its convergence, by putting $2-\epsilon$ ($\epsilon > 0$) instead of 2? If no positive ϵ exists, which suffices to render

$$\sum_{n=1}^{\infty} (|a_n|^{2-\epsilon}+|b_n|^{2-\epsilon}) \tag{2}$$

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* [The manuscript of this paper was received by me, after considerable delay in the post, on December 8th, 1919. The author was unacquainted with two recent memoirs bearing on the same point, viz.:—

T. Carleman, "Über die Fourierkoeffizienten einer stetigen Funktion," *Acta Mathematica*, Vol. 41 (1918), pp. 377–384;

E. Landau, "Bemerkungen zu einer Arbeit von Herrn Carleman," *Math. Zeitschrift*, Vol. 5 (1919), pp. 147–153.

In these memoirs the problem put by Mr. Steinhaus is solved in a number of different manners, and it is shown that the same conclusion holds even for *continuous* functions. But the solution given by Mr. Steinhaus is different, and interesting in itself.—G. H. H.]

convergent for all bounded functions, is it at least always possible to find, for every *given* bounded function, an appropriate positive ϵ rendering (2) convergent?

The answer is *negative*, even if we choose the second formulation of our problem. In fact, the following statement is true:

There exists a bounded integrable function $f(t)$, independent of ϵ, which renders the series (2) *divergent for every positive ϵ, however small.*

The object of this paper is to establish this proposition. For the sake of brevity, we write, for any integrable function $f(t)$,

$$\frac{1}{\pi}\int_0^{2\pi} f(t)\cos ntdt = f_{2n-1}, \qquad \frac{1}{\pi}\int_0^{2\pi} f(t)\sin ntdt = f_{2n} \quad (n=1, 2, \ldots),$$

and we call $\{f_n\}$ shortly the "Fourier sequence of f". In a former publication* the following theorem was proved:

The summability ($C1$) *of*

$$\sum_{k=1}^{\infty} h_k x_k, \tag{3}$$

for all Fourier's sequences $\{x_k\}$ of bounded functions $x(t)$, is a necessary and sufficient condition that $\{h_k\}$ should be the Fourier's sequence of an integrable function $h(t)$.

We draw the immediate conclusion that, if a given sequence $\{h'_k\}$ is *not* a Fourier's sequence, there exists another sequence $\{x'_k\}$ which is the Fourier's sequence of a bounded function and which renders

$$\sum_{k=1}^{\infty} h'_k x'_k \tag{3'}$$

not summable ($C1$), and, *a fortiori*, divergent.

Messrs. Hardy and Littlewood have shown in their important and beautiful papers on Diophantine Approximation,† that

$$\sum_{\nu=1}^{\infty} \frac{\cos \nu^2 t}{\nu^{\frac{1}{2}}} \tag{4}$$

is not a Fourier's series.

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* H. Steinhaus, "Additive und stetige Funktionaloperationen," *Mathematische Zeitschrift*, Vol. 5 (1919), pp. 186–221, Parallelsatz 3, Umkehrung des Parallelsatzes 3.

† G. H. Hardy and J. E. Littlewood, "Some Problems of Diophantine Approximation," *Acta Mathematica*, Vol. 38 (1914), Part 2, p. 237.

The sequence $\{\bar{h}_k\}$ defined by

$$\bar{h}_{2n} = 0 \quad (n = 1, 2, \ldots), \tag{5}$$

$$\bar{h}_{2n-1} = \frac{1}{\nu^{\frac{1}{2}}} \quad (n = 1, 4, \ldots, \nu^2, \ldots),$$

$$\bar{h}_{2n-1} = 0 \quad (n \neq \nu^2),$$

is therefore not a Fourier's sequence; nevertheless

$$\sum_{k=1}^{\infty} |\bar{h}|^{2+\delta} \tag{6}$$

is convergent for every positive δ, as results from (5). The conclusion we had drawn from our theorem quoted above permits us to affirm the existence of a bounded integrable function $\bar{x}(t)$, whose Fourier's sequence $\{\bar{x}_k\}$ renders

$$\sum_{k=1} \bar{h}_k \bar{x}_k \tag{7}$$

divergent. We can now assert the divergence of

$$\sum_{k=1}^{\infty} |\bar{x}_k|^{2-\epsilon} \tag{8}$$

for $1 > \epsilon > 0$ (which implies the divergence of the same series for $\epsilon \geqslant 1$, and consequently for all $\epsilon > 0$). In fact, the convergence of (8), *i.e.* of $\sum_{k=1}^{\infty} |\bar{x}_k|^{1+(1-\epsilon)}$, and the convergence of (6), which holds for every positive δ, and therefore for

$$\delta = \frac{\epsilon}{1-\epsilon},$$

implies the absolute convergence of (7),* contrary to what has been shown; and $\bar{x}(t)$ is therefore the bounded integrable function, the existence of which we proposed to demonstrate.

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* If $\sum_{k=1}^{\infty} |x_k|^{1+j}$ and $\sum_{k=1}^{\infty} |h_k|^{1+1/j}$ are convergent, then $\sum_{k=1}^{\infty} |h_k x_k|$ is convergent, for $j > 0$. Cf. F. Riesz, "Les systèmes d'équations linéaires à une infinité d'inconnues" (Paris, Gauthier-Villars, 1913), Chap. 3, §33, pp. 43-45. In our case $j = 1-\epsilon$. Riesz puts $1+j = \rho > 1$.

ON THE DIOPHANTINE EQUATION $ay^2+by+c = dx^n$

By EDMUND LANDAU *and* ALEXANDER OSTROWSKI.

(*Communicated by* G. H. HARDY.)

[Received December 19th, 1919.—Read February 12th, 1920.]

1. We owe to Mr. A. Thue* the important theorem:

If $F(u, v)$ is a homogeneous form with rational integral coefficients, and is not a power of a linear or quadratic form, then the equation

$$F(u, v) = f$$

has, for every $f \neq 0$, at most a finite number of rational integral solutions.

It was shown by Landau,† in response to a question‡ in *l'Intermédiaire des Mathématiciens*, and by use of the particular case of Thue's theorem in which F is a cubic form, that the equation $y^2-2 = x^3$ has only a finite number of solutions. The method may be extended, by the aid of considerations drawn from the theory of ideals, so as to lead to the following theorem:

The equation

$$ay^2+by+c = dx^n, \tag{1}$$

where $n \geqslant 3$, $a \neq 0$, $b^2-4ac \neq 0$, $d \neq 0$, and all the letters denote rational integers, has at most a finite number of solutions.

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* "Om en general i store hele tal ulösbar ligning," *Skrifter udgivne af Videnskabs-Selskabet i Christiania*, 1908 I, Mathematisk-Naturvidenskabelig Klasse (1909), No. 7, 15 S.; "Über Annäherungswerte algebraischer Zahlen," *Journal für die reine und angewandte Mathematik*, Bd. 135 (1909), S. 284–305. In the second memoir the theorem is proved only for an irreducible $F(u, v)$; but the extension to the general case (treated in the first memoir) is immediate.

† *L'Intermédiaire des Mathématiciens*, t. 8 (1901), pp. 145–147, and t. 20 (1913), p. 154.

‡ 1360.

This theorem, the proof of which is the object of the present note, is new only when $n>3$, for when $n=3$ it results immediately from a combination of Thue's theorem with the results obtained by Mr. Mordell, from the theory of numbers and the theory of invariants, in his memoir "Indeterminate equations of the third and fourth degrees."*

In the special case when $a=1$, $b=0$, $d=1$, $n\geqslant 3$, our theorem shows that:

If all squares and all n-th powers $\geqslant 0$ are arranged together in order of magnitude, numbers which are both squares and n-th powers occurring only once, in a series

$$z_1(=0),\ z_2(=1),\ z_3(=4),\ \ldots,\ z_m,\ z_{m+1},\ \ldots,$$

then $z_{m+1}-z_m$ tends to infinity with m.

This special case is trivial† when n is even; it may be expressed, when n is odd, in the form:

If $\vartheta(t)$ denotes the distance of t from the nearest rational integer, and x runs through all rational integers $\geqslant 0$ which are not squares, then

$$x^{\frac{1}{2}n}\,\vartheta(x^{\frac{1}{2}n})\to\infty.$$

[For, if the sign is chosen appropriately,

$$\left(x^{\frac{1}{2}n}\pm\vartheta(x^{\frac{1}{2}n})\right)^2=x^n\pm 2x^{\frac{1}{2}n}\,\vartheta(x^{\frac{1}{2}n})+\left(\vartheta(x^{\frac{1}{2}n})\right)^2$$

is a square, whose distance from x^n is

$$\left|\pm 2x^{\frac{1}{2}n}\,\vartheta(x^{\frac{1}{2}n})+\left(\vartheta(x^{\frac{1}{2}n})\right)^2\right|\leqslant 2x^{\frac{1}{2}n}\,\vartheta(x^{\frac{1}{2}n})+\tfrac{1}{4},$$

and tends to infinity.]

2. *Proof of the theorem.*—The equation (1) may be written

$$(2ay+b)^2-(b^2-4ac)=4adx^n.$$

Hence, if (1) has an infinity of solutions, so has an equation

$$y^2-k=lx^n\quad(k\neq 0,\ l\neq 0). \tag{2}$$

We need therefore only consider the equation (2). There are two cases.

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* *Quarterly Journal of Pure and Applied Mathematics*, Vol. 45 (1914), pp. 170–186. See also the note "A statement by Fermat," *Proc. London Math. Soc.* (*Records &c.*), Ser. 2, Vol. 18 (1919), pp. v–vi.

† Since then every z_m is a square.

I. Let k be a square m^2. Then

$$(y+m)(y-m) = lx^n.$$

We ignore the trivial solutions $x = 0$, $y = \pm m$. Any prime factor of $y+m$, which is not a factor of $2m$ or of l, occurs in $y+m$ with a multiplicity divisible by n, since it does not divide $y-m$, and divides lx^n exactly as often as x^n. Hence

$$y+m = \pm p_1^{a_1} \dots p_j^{a_j} z^n,$$

where $p_1, p_2, \dots, p_j$ are the different prime factors of $2ml$, the exponents $a_1, \dots, a_j$ are positive or zero, and z is a rational integer. If every a_i is reduced to modulus n, we obtain

$$y+m = qu^n, \tag{3}$$

where q can assume only a finite system of values, exclusive of zero. In exactly the same way

$$y-m = rv^n, \tag{4}$$

where r can assume only a finite system of values, exclusive of zero.

For each pair q, r, occurring in (3) and (4), the equation

$$2m = qu^n - rv^n$$

has, by Thue's theorem, at most a finite number of solutions; for the n roots of $q\tau^n - r = 0$ are all different. Thus at most a finite number of values of y are possible.

II. Suppose that k is not a square. Then in (2) $x \neq 0$, and so the ideal equation

$$[y+\sqrt{k}][y-\sqrt{k}] = [l][x]^n$$

holds in the quadratic corpus $P(\sqrt{k})$.* Every prime ideal which divides $[y+\sqrt{k}]$, but neither $[2\sqrt{k}]$ nor $[l]$, occurs in $[y+\sqrt{k}]$ with an exponent divisible by n; for it does not divide $[2\sqrt{k}]$, and therefore not $[y-\sqrt{k}]$. Thus

$$[y+\sqrt{k}] = \mathfrak{p}_1^{a_1} \mathfrak{p}_2^{a_2} \dots \mathfrak{p}_j^{a_j} \mathfrak{z}^n,$$

where $\mathfrak{p}_1, \dots, \mathfrak{p}_j$ are the different prime ideals which divide $[2\sqrt{k} \,.\, l]$, the exponents $a_1, \dots, a_j$ are positive or zero, and $\mathfrak{z}$ is an ideal. If each a_i is

* $[\alpha]$ denotes the principal ideal (Hauptideal) of the integers of the corpus divisible by α.

reduced to modulus n, we obtain

$$[y+\sqrt{k}] = \mathfrak{q}\mathfrak{u}^n, \tag{5}$$

where $\mathfrak{q}$ belongs to a finite system of ideals, and $\mathfrak{u}$ is an ideal.

It is enough to show that, for a fixed $\mathfrak{q}$, (5) can be satisfied by an ideal $\mathfrak{u}$ for at most a finite number of values of y. If this were not so, there would be a class of ideals $\mathfrak{K}$ such that (5) could be satisfied by an ideal $\mathfrak{u}$ of $\mathfrak{K}$ for an infinity of values of y. If $\mathfrak{w}$ is a fixed representative of the class inverse to $\mathfrak{K}$, so that* $\mathfrak{u}\mathfrak{w} \sim [1]$, then it follows from (5) that

$$\mathfrak{q}\mathfrak{u}^n \sim [1],$$

$$\mathfrak{q} \sim \mathfrak{q}(\mathfrak{u}\mathfrak{w})^n \sim \mathfrak{w}^n.$$

There are therefore two integers s and γ, of which s may be supposed rational, and both are independent of $\mathfrak{u}$, in the corpus $P(\sqrt{k})$, such that

$$[s]\mathfrak{q} = [\gamma]\mathfrak{w}^n.$$

It now follows from (5) that

$$[s][y+\sqrt{k}] = [s]\mathfrak{q}\mathfrak{u}^n = [\gamma]\mathfrak{w}^n\mathfrak{u}^n = [\gamma](\mathfrak{w}\mathfrak{u})^n = [\gamma][\xi]^n,$$

where ξ is an integer of the corpus. Therefore

$$s(y+\sqrt{k}) = \epsilon\gamma\xi^n,$$

where ϵ is a unity.

If $k<0$ the number of unities is finite, and if $k>0$ all unities are expressible in terms of a fundamental unity η in the form $\pm\eta^t$, where t is a rational integer. Thus every unity is the product of (a) a unity chosen from a finite system, and (b) the n-th power of a unity. Accordingly

$$s(y+\sqrt{k}) = \beta\xi^n, \tag{6}$$

where β belongs to a finite system of integers of the corpus, excluding zero, and ξ is an integer of the corpus. It is therefore enough to show that, for a fixed positive or negative s, and a fixed non-zero β, at most a finite number of values of y can occur in (6).

We choose a base 1, ω of the integers of the corpus. Then

$$\xi = u+v\omega,$$

where u, v are rational integers. Denoting generally by μ' the number

* The symbol $\sim$ denotes equivalence.

conjugate to μ, we have,* from (6),

$$\frac{2s\sqrt{k}}{\omega-\omega'} = \frac{\beta(u+v\omega)^n-\beta'(u+v\omega')^n}{\omega-\omega'}. \tag{7}$$

The right-hand side is a binary form $F(u, v)$ with rational integral coefficients, since for every integer $\mu = u_0+v_0\omega$ the number $\frac{\mu-\mu'}{\omega-\omega'} = v_0$ is rational and integral. If δ is a root of $\delta^n = \beta'/\beta$, and ρ runs through the n-th roots of unity, then

$$F(u, v) = \frac{\beta}{\omega-\omega'} \Pi_\rho \{u+v\omega-\rho\delta(u+v\omega')\}.$$

No two of the linear factors are the same, or differ only by a constant factor; for, if $\rho_1 \neq \rho_2$,

$$\begin{vmatrix} 1-\rho_1\delta, & \omega-\rho_1\delta\omega' \\ 1-\rho_2\delta, & \omega-\rho_2\delta\omega' \end{vmatrix} = \begin{vmatrix} 1, & -\rho_1\delta \\ 1, & -\rho_2\delta \end{vmatrix} \begin{vmatrix} 1, & \omega \\ 1, & \omega' \end{vmatrix} = \delta(\rho_1-\rho_2)(\omega'-\omega) \neq 0.$$

Thus Thue's theorem may be applied to (7); at most a finite number of values of u and v, of ξ, and of y, can occur; and our theorem is proved.

3. From our theorem it is very easy to deduce that of Pólya:† *the greatest prime factor of* ay^2+by+c ($a \neq 0$, $b^2-4ac \neq 0$) *tends to infinity with* $|y|$. For if, for an infinity of values of y, ay^2+by+c were composed only of a finite system of primes $p_1, \ldots, p_j$, then for every fixed n greater than 2, and for at least one number d, formed by powers of these primes, the equation (1) would have an infinity of solutions.

Göttingen, December 14th, 1919.

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* Compare the similar argument used by Pólya, "Zur arithmetischen Untersuchung der Polynome," *Math. Zeitschrift*, Bd. 1 (1918), S. 143–148 (S. 147).

† *L.c.*, S. 144.

A MULTIPLE INTEGRAL OF IMPORTANCE IN THE THEORY OF STATISTICS

By G. F. S. HILLS.

[Read December 11th, 1919.]

MULTIPLE integrals of the type

$$V(h) = \iint \dots x_1^{a_1} x_2^{a_2} \dots x_n^{a_n} e^{-\frac{1}{2}\chi}\, dx_1 dx_2 \dots dx_n,$$

where χ is an essentially positive quadratic form in the variables $x_1, x_2, \dots, x_n$, where the indices $a_1, a_2, \dots, a_n$ are positive integers or zeros, and the integration is taken over the finite hyper-space within $\chi = h$, are of importance in the theory of statistics.

In this paper we shall obtain an expression for the multiple integral

$$W_k(h) = \iint \dots \chi'^k e^{-\frac{1}{2}\chi}\, dx_1 dx_2 \dots dx_n,$$

where

$$\chi = a_{11}x_1^2 + \dots + 2a_{12}x_1x_2 + \dots + a_{nn}x_n^2,$$

$$\chi' = a_{11}x_1^2 + \dots + 2a'_{12}x_1x_2 + \dots + a'_{nn}x_n^2,$$

χ being essentially positive and k a positive integer, the integration being taken over the hyper-space within $\chi = h$. Expressions for the multiple integrals of type V are then deducible. For brevity we shall write

$$W_k(\infty) = W_k.$$

Integrals of the type $V(\infty)$ have been discussed by Dr. Isserlis* and Sverder Bergstrom,† and particular cases by Prof. Wicksell.‡ Dr. Isserlis, working from the point of view of the theory of statistics and utilising the

* *Biometrika*, Vol. 12, p. 134.
† *Biometrika*, Vol. 12 (misc.), p. 177.
‡ *Phil. Mag.*, Vol. 37 (1919), p. 446.

known properties of regression planes, obtains an expression for $V(\infty)$ where the indices $a_1 a_2 \ldots a_n$ are each unity, and then shows how an expression for any particular case can be written down when the indices are integers other than unity. A particular case of W_k for $k = 1$ has been discussed by Mr. Arthur Black.*

Dr. Isserlis and Sverder Bergstrom both employ the method of mathematical induction. The method in the present paper is direct.

The first step is to identify the integral W_k, barring a numerical coefficient, with the coefficient of ϕ^k in the expansion of $M^{-\frac{1}{2}}$, where M is the discriminant of $\chi+\phi\chi'$, *i.e.*

$$M \equiv \Delta+\phi\Theta_1+\phi^2\Theta_2+\ldots+\phi^n\Delta'.$$

Consider the multiple integral

$$W = \iint_{-} \ldots e^{-\frac{1}{2}(\chi+\phi\chi')}\, dx_1\, dx_2 \ldots dx_n,$$

evaluating in the usual way,† we have, if $\chi+\phi\chi'$ is essentially positive,

$$W = (2\pi)^{n/2}\, M^{-\frac{1}{2}}.$$

As χ' is not essentially positive, in order to ensure that $\chi+\phi\chi'$ is always positive, ϕ must lie between the positive and negative roots of $M(\phi)$ nearest zero; or, if there are no positive roots, ϕ must be greater than the negative root nearest zero.

It is clear that for values of ϕ within $\pm\phi_1$, where ϕ_1 is the numerically smallest root of M, both expressions for W can be expanded in powers of ϕ. The first expression gives

$$W = \iint \ldots \left[1-\frac{\chi'}{2}\phi+\frac{\chi'^2}{2^2}\frac{\phi^2}{2!}-\ldots\right] e^{-\frac{1}{2}\chi}\, dx_1 dx_2 \ldots dx_n$$

$$= W_0-\frac{W_1}{2}\phi+\frac{W_2}{2^2}\frac{\phi^2}{2!}-\ldots(-)^k\frac{W_k}{2^k}\frac{\phi^k}{k!}\ldots,$$

and the second gives

$$W = (2\pi)^{n/2}[R_0-R_1\phi+R_2\phi^2 \ldots (-)^k R_k\phi^k \ldots],$$

where $(-)^k R_k$ is the coefficient of ϕ^k in $(\Delta+\phi\Theta_1+\phi^2\Theta_2+\ldots+\phi^n\Delta')^{-\frac{1}{2}}$.

Hence $$W_k = (2\pi)^{n/2}\, 2^k k!\, R_k.$$

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* *Camb. Phil. Trans.*, Vol. 16 (1898), p. 219.

† The method of the last example in this paper may be employed.

A Linear Relation between $W_0, W_1, \ldots, W_k$.

If k is small $M^{-\frac{1}{2}}$ can be expanded and the coefficient of ϕ^k picked out, so that

$$R_0 = \Delta^{-\frac{1}{2}}, \quad R_1 = +\tfrac{1}{2}\frac{\Theta_1}{\Delta}, \text{ &c., &c.}$$

To obtain a linear relation between $W_0, W_1, \ldots, W_k$, we differentiate for ϕ the equality

$$W_0 - \frac{W_1}{2}\phi + \frac{W_2}{2^2}\frac{\phi^2}{2!} \ldots (-)^k \frac{W_k}{2^k}\frac{\phi^k}{k!} \ldots = \frac{(2\pi)^{n/2}}{(\Delta+\phi\Theta_1+\phi^2\Theta_2+\ldots+\phi^n\Theta_n)^{\frac{1}{2}}},$$

where we write $\Delta' = \Theta_n, \quad \Delta = \Theta_0,$

whence

$$\left[W_1 - \frac{W_2}{2}\phi + \ldots(-)^k \frac{W_{k+1}\phi^k}{2^k k!} \ldots\right][\Theta_0+\phi\Theta_1+\ldots+\phi^n\Theta_n]$$

$$= \left[W_0 - \frac{W_1}{2}\phi + \ldots(-)^k \frac{W_k\phi^k}{2^k k!} \ldots\right][\Theta_1+2\phi\Theta_2+\ldots+n\phi^{n-1}\Theta_n].$$

Equating the coefficients of ϕ^k on both sides and making slight reductions, we have

$$\frac{W_{k+1}}{k!}\Theta_0 = (2k+1)\frac{W_k}{k!}\Theta_1 - 2.2k\frac{W_{k-1}}{(k-1)!}\Theta_2 + \ldots$$

$$\ldots(-)^r 2^r(2k-r+1)\frac{W_{k-r}}{(k-r)!}\Theta_{r+1}\ldots(-)^{n-1}2^{n-1}\frac{(2k-n+2)}{}\frac{W_{k-n+1}}{(k-n+1)!}\Theta_n;$$

if $k < n$ the last term on the right is $(-2)^k(k+1)W_0\Theta_{k+1}$.

Hence we obtain, replacing Θ_0 by Δ,

$$W_1 = \frac{W_0\Theta_1}{\Delta},$$

$$W_2 = \frac{W_0}{\Delta^2}(3\Theta_1^2 - 4\Delta\Theta_2),$$

$$W_3 = \frac{W_0}{\Delta^3}(15\Theta_1^3 - 36\Delta\Theta_1\Theta_2 + 24\Delta^2\Theta_3),$$

$$W_4 = \frac{W_0}{\Delta^4}(105\Theta_1^4 - 360\Delta\Theta_1^2\Theta_2 + 144\Delta^2\Theta_2^2 + 288\Delta^2\Theta_1\Theta_3 - 192\Delta^3\Theta_4),$$

&c., &c.

Particular Forms of W_k.

As $\Theta_1\Theta_2 \dots \Theta_n$ are polynomials of weights $1, 2, \dots, n$ in the coefficients $a'_{11}a'_{12}\dots$ of χ', it is clear that W_k is a polynomial of weight k in those coefficients, as of course it must be from its other expression as an integral. We can therefore compare coefficients of products of the coefficients $a'_{11}a'_{12}\dots$ in the two expressions for W_k and thus obtain integrals of type V.

Before considering suitable forms of W_k to facilitate such comparison it might be noted that in the case $n = 2$ a general expression for W_k is obtainable by writing

$$M = \Delta(1-2\mu h+h^2),$$

where $$h = \left(\frac{\Delta'}{\Delta}\right)^{\frac{1}{2}}\phi \quad \text{and} \quad \mu = -\tfrac{1}{2}\frac{\Theta}{(\Delta\Delta')^{\frac{1}{2}}},$$

then $$W_k = 2^k k!\left(\frac{2\pi}{\Delta^{\frac{1}{2}}}\right)\left(\frac{\Delta'}{\Delta}\right)^{\frac{1}{2}k} P_k(\mu),$$

where $P_k(\mu)$ is the Legendre polynomial.

Returning to the general case it will be seen by considering the determinantal form of M that if χ' is a perfect square, say

$$\chi' = (\xi_1 x_1+\xi_2 x_2+\dots+\xi_n x_n)^2,$$

then all the invariants $\Theta_2 = \Theta_3 = \dots = \Theta_n = 0$, and thus M reduces to the simple form

$$M = \Delta+\phi\Theta_1,$$

where $$\Theta_1 = A_{11}\xi_1^2+\dots+2A_{12}\xi_1\xi_2+\dots+A_{nn}\xi_n^2,$$

and A_{pq} is the co-factor of the element a_{pq} in Δ.

Hence

$$U_k = \int\int_{-\infty}^{+\infty} \dots (\xi_1 x_1+\xi_2 x_2+\dots+\xi_n x_n)^{2k} e^{-\frac{1}{2}\chi}\, dx_1 dx_2 \dots dx_n$$

$$= \text{coefficient of } (-\phi)^k \text{ in } \frac{2^k.k!\,(2\pi)^{n/2}}{(\Delta+\phi\Theta_1)}$$

$$= 1.3.5 \dots (2k-1)\frac{(2\pi)^{n/2}}{\Delta^{\frac{1}{2}}}\left(\frac{\Theta_1}{\Delta}\right)^k$$

$$= 1.3.5 \dots (2k-1)\frac{(2\pi)^{n/2}}{\Delta^{\frac{1}{2}+k}}(A_{11}\xi_1^2+\dots+2A_{12}\xi_1\xi_2+\dots)^k.$$

We can now readily compare coefficients. Taking out the coefficient of $\xi_1^{\alpha_1}\xi_2^{\alpha_2}\dots\xi_n^{\alpha_n}$ in both expressions for U_k, where

$$\alpha_1+\alpha_2+\dots+\alpha_n = 2k,$$

we have

$$\frac{(2k)!}{\alpha_1!\,\alpha_2!\dots\alpha_n!}\iint_{-\infty}^{+\infty}\dots x_1^{\alpha_1}x_2^{\alpha_2}\dots x_n^{\alpha_n}e^{-\frac{1}{2}\chi}\,dx_1dx_2\dots dx_n$$

$$= 1.3.5\dots(2k-1)\,\frac{(2\pi)^{n/2}}{\Delta^{k+\frac{1}{2}}}\,\Sigma\,\frac{k!}{\lambda_{11}!\,\lambda_{12}!\dots\lambda_{nn}!}A_{11}^{\lambda_{11}}\dots(2A_{12})^{\lambda_{12}}\dots A_{nn}^{\lambda_{nn}},$$

where the summation extends to all sets of positive integral values (including zeros) of the indices $\lambda_{11}\dots\lambda_{12}\dots\lambda_{nn}$ connected by the n relations

$$2\lambda_{11}+\lambda_{12}+\dots+\lambda_{1n} = \alpha_1,$$

$$\lambda_{12}+2\lambda_{22}+\dots+\lambda_{2n} = \alpha_2,$$

$$\dots\quad\dots\quad\dots\quad\dots$$

$$\lambda_{1n}+\lambda_{2n}+\dots+2\lambda_{nn} = \alpha_n$$

Hence

$$V = \iint_{-\infty}^{+\infty}\dots x_1^{\alpha_1}x_2^{\alpha_2}\dots x_n^{\alpha_n}e^{-\frac{1}{2}\chi}\,dx_1dx_2\dots dx_n$$

$$= \frac{\alpha_1!\,\alpha_2!\dots\alpha_n!}{2^k}\,\frac{(2\pi)^{n/2}}{\Delta^{k+\frac{1}{2}}}\,\Sigma\,\frac{A_{11}^{\lambda_{11}}}{\lambda_{11}!}\dots\frac{(2A_{12})^{\lambda_1}}{\lambda_{12}!}\dots\frac{A_{nn}^{\lambda_{nn}}}{\lambda_{nn}!}.$$

where $$\alpha_1+\alpha_2+\dots+\alpha_n = 2k.$$

Some Special Cases.

The following special cases may be noted, the integrals in all cases extending to infinity,

(I) $$\iint\dots x_1^{2k}e^{-\frac{1}{2}\chi}\,dx_1dx_2\dots dx_n = 1.3\dots(2k-1)\,\frac{(2\pi)^{n/2}}{\Delta^{\frac{1}{2}}}\left(\frac{A_{11}}{\Delta}\right)^k,$$

(II) $$\iint\dots x_1x_2\dots x_ne^{-\frac{1}{2}\chi}\,dx_1dx_2\dots dx_n \quad (n \text{ even}).$$

Here $\lambda_{pp} = 0$, and if $\lambda_{pq} = 1$ $(p \neq q)$, then all other indices λ having

either p or q as a suffix are zero. Thus in each set there will be $n/2$ indices of type λ_{pq} where no suffix p or q is repeated in the set.

The integral is therefore equal to

$$\frac{(2\pi)^{n/2}}{\Delta^{(n+1)/2}} \Sigma A_{12} A_{34} \dots A_{n-1,\,n},$$

where the summation consists of $1.3 \dots (n-1)$ terms, each term being the product of $n/2$ co-factors A_{pq}, no suffix p or q being repeated in a term. It is readily obtained direct from U_k by taking out the coefficient of $\xi_1 \xi_2 \dots \xi_n$ in

$$(A_{11}\xi_1^2 + \dots + 2A_{12}\xi_1\xi_2 + \dots + A_{nn}\xi_n^2)^{n/2}.$$

It is this integral which Dr. Isserlis obtains first. In his paper he gives a handy condensation rule for writing down from this result the value of the integral for a smaller number of variables but where the indices $a_1 a_2 \dots a_n$ are not unity.

(III) $$\iint x_1^\alpha x_2^\beta e^{-\frac{1}{2}\chi} dx_1 dx_2,$$

where $$\chi = a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2$$

and $\alpha+\beta$ is even.

Case (i). $$\alpha = 2p, \quad \beta = 2q,$$

here $$2\lambda_{11} + \lambda_{12} = 2p,$$

$$\lambda_{12} + 2\lambda_{22} = 2q.$$

Hence $$\lambda_{11} = p-\mu, \quad \lambda_{12} = 2\mu, \quad \lambda_{22} = q-\mu,$$

where μ takes integral values from 0 to q, including 0 and q, if $q \not> p$. The integral reduces to

$$\frac{2p!\,2q!}{2^{p+q}} \frac{(2\pi)}{\Delta^{p+q+\frac{1}{2}}} \Sigma 2^{2\mu} \frac{A_{11}^{p-\mu}}{(p-\mu)!} \frac{A_{22}^{q-\mu}}{(q-\mu)!} \frac{A_{12}^{2\mu}}{(2\mu)!}.$$

Case (ii). $$\alpha = 2p+1, \quad \beta = 2q+1,$$

here $$\lambda_{11} = p-\mu, \quad \lambda_{22} = q-\mu, \quad \lambda_{12} = 2\mu+1.$$

The integral becomes

$$\frac{(2p+1)!\,(2q+1)!}{2^{p+q+1}}\,\frac{(2\pi)}{\Delta^{p+q+\frac{3}{2}}}\,\Sigma\,2^{2\mu+1}\,\frac{A_{11}^{p-\mu}}{(p-\mu)!}\,\frac{A_{22}^{q-\mu}}{(q-\mu)!}\,\frac{A_{12}^{2\mu+1}}{(2\mu+1)!},$$

the summation for values of μ as above.

(IV) $$G=\iint\ldots X_1^{a_1}X_2^{a_2}\ldots X_n^{x_n}e^{-\frac{1}{2}\chi}\,dx_1dx_2,\ldots dx_n,$$

where $$X_p=L_{p1}x_1+L_{p2}x_2+\ldots+L_{pn}x_n,$$

and $$\chi=a_{11}x_1^2+\ldots+2a_{12}x_1x_2+\ldots,$$

as before. Transform the variables to $y_1, y_2, \ldots, y_n$, where

$$y_p=X_p,$$

and suppose $$\chi=b_{11}y_1^2+\ldots+2b_{12}y_1y_2+\ldots,$$

write $$J=\frac{\partial(y_1, y_2, \ldots, y_n)}{\partial(x_1, x_2, \ldots, x_n)}.$$

Denoting this generalised form of V by G, we see that

$$G=\frac{1}{J}\iint\ldots y_1^{a_1}y_2^{a_2}\ldots y_n^{a_n}e^{-\frac{1}{2}\chi}\,dy_1dy_2\ldots dy_n=\frac{1}{J}\,V' \text{ say,}$$

where χ is expressed in terms of the coefficients $b_{11}b_{12}\ldots$, and the variables $y_1y_2\ldots$. Write Δ_b for the discriminant of χ in this form, and let B_{pq} be the co-factor of the element b_{pq} in Δ_b.

Referring to the expression previously obtained for V', we see the only quantities contained in it affected by the change of variables are the quantities $\Delta_b^{\frac{1}{2}}$ and the ratios $\frac{B_{pp}}{\Delta_b}, \frac{B_{pq}}{\Delta_b}, \ldots$.

Now $J^2\Delta_b=\Delta$, and the values of the ratios referred to in terms of the original coefficients $a_{11}a_{12}\ldots$ of χ and $L_{p1}L_{p2}\ldots$ of $X_1X_2\ldots$ are obtained by considering the invariant Θ_1 of two quadratic forms χ and χ', *i.e.*

$$\Theta_1=J^2[b'_{11}B_{11}+\ldots+2b'_{12}B_{12}+\ldots]$$
$$=a'_{11}A_{11}+\ldots+2a'_{12}A_{12}+\ldots.$$

where χ' is supposed to take the two forms

$$a'_{11}x_1^2+\ldots+2a'_{12}x_1x_2\ldots$$

and $$b'_{11}y_1^2+\ldots+2b'_{12}y_1y_2\ldots,$$

according as the variables are $x_1, x_2, \ldots$ or $y_1, y_2, \ldots$.

Take χ' in the special form $b'_{pp}y_p^2$ consisting of one term only when expressed in the variables y, so that

$$\chi' = b'_{pp}y_p^2 = b'_{pp}(L_{p1}x_1+L_{p2}x_2+\ldots)^2,$$

$$\chi = b_{11}y_1^2+\ldots+2b_{12}y_1y_2+\ldots = a_{11}x_1^2+\ldots+2a_{12}x_1x_2+\ldots.$$

When the variables are $y_1, y_2, \ldots,$

$$\Theta_1 = J^2b'_{pp}B_{pp}.$$

When the variables are $x_1, x_2, \ldots,$

$$\Theta_1 = b'_{pp}[A_{11}L_{p1}^2+\ldots+A_{qq}L_{pq}^2+\ldots+2A_{rs}L_{pr}L_{ps}+\ldots].$$

Hence

$$\frac{B_{pp}}{\Delta_b} = \frac{1}{\Delta}[A_{11}L_{p1}^2+A_{22}L_{p2}^2+\ldots+2A_{rs}L_{pr}L_{ps}+\ldots],$$

and similarly, by considering χ' in the special form $2b'_{pq}y_py_q$, we get

$$\frac{B_{pq}}{\Delta_b} = \frac{1}{\Delta}[A_{11}L_{p1}L_{q1}+\ldots+(L_{p1}L_{q2}+L_{p2}L_{q1})A_{12}+\ldots].$$

Substituting these values and the value of Δ_b in the expression for V', we obtain the value of the integral G in terms of its original coefficients. It will be noted that J does not appear in the final result.

The Integral $W_k(h)$ where the Limits of Integration are Finite.

In the preceding cases the integration extends to infinity in all directions. By means however of Dirichlet's integrals we can obtain expressions for the above integrals within the hyper-space bounded by $\chi = h$, where h is a constant, χ being as before essentially positive. We have

$$W_k(h) = \iint\ldots\chi'^k e^{-\frac{1}{2}\chi}\,dx_1dx_2\ldots dx_n$$

within $\chi = h.$

Transform to normal coordinates $u_1u_2\ldots u_n$ by a linear transformation, so that

$$\chi = u_1^2+u_2^2+\ldots+u_n^2,$$

and $$\chi' = c_1u_1^2+c_2u_2^2+\ldots+c_nu_n^2.$$

Let $$J = \frac{\partial(u_1, u_2, \ldots, u_n)}{\partial(x_1, x_2, \ldots, x_n)}$$

Then $$W_k(h) = \frac{1}{J} \Sigma Q \iint \ldots u_1^{2a_1} u_2^{2a_2} \ldots u_n^{2a_n} F(u_1^2+u_2^2+\ldots+u_n^2)\, du_1 du_2 \ldots du_n,$$

where $$Q = \frac{k!\, c_1^{a_1} c_2^{a_2} \ldots c_n^{a_n}}{a_1!\, a_2! \ldots a_n!},$$

and the summation is for all positive integral or zero values of the indices $a_1, a_2, \ldots$ subject to the condition

$$a_1+a_2+\ldots+a_n = k,$$

the integration being taken over the space within

$$\chi = u_1^2+u_2^2+\ldots+u_n^2 = h.$$

In the integral we have for the time written $F(\chi)$ in place of $e^{-\frac{1}{2}\chi}$. This brings out more clearly the Dirichlet character, and moreover shows that the result is not special to the exponential form.

Quoting the result given in Williamson, *Int. Calc.* (7th ed.), p. 319, we have within the above limits

$$\iint \ldots u_1^{2a_1} u_2^{2a_2} \ldots u_n^{2a_n} F(u_1^2+u_2^2+\ldots+u_n^2)\, du_1 du_2 \ldots du_n$$
$$= \Gamma(a_1+\tfrac{1}{2})\, \Gamma(a_2+\tfrac{1}{2}) \ldots \Gamma(a_n+\tfrac{1}{2}) \int_0^h \frac{F(u)\, u^{k+n/2-1}}{\Gamma(k+n/2)}\, du,$$

remembering that there are 2^n quadrants and that the integral is the same for each. It will be noted that the integral on the right does not involve the separate indices $a_1, a_2, \ldots$, but only their sum k. It is therefore common to all terms of the summation in $W_k(h)$, so that

$$W_k(h) = \frac{1}{J} K \int_0^h \frac{F(u)\, u^{k+n/2-1}}{\Gamma(k+n/2)}\, du.$$

where $$K = \Sigma Q\, \frac{1}{2^{2k}}\, \frac{(2a_1)!\, (2a_2)! \ldots (2a_n)!}{a_1!\ \ a_2!\ \ \ldots\ a_n!}\, (\pi)^{n/2},$$

as $$\Gamma(a+\tfrac{1}{2}) = \frac{(2a)!}{2^a a!}\, \pi^{\frac{1}{2}}.$$

The quantity K is clearly the coefficient of $(-\phi)^k$ in the product

$$k!\, (\pi)^{n/2}\, P_1 P_2 \ldots P_n,$$

where $$P \equiv 1-\beta_1 c_s\phi+\beta_2 c_s^2\phi^2-\ldots(-)^r\beta_r c_s^{a_r}\phi^{a_r}\ldots,$$

and $$\beta_r = \frac{(2a_r)!}{2^{2a_r}a_r!\,a_r!}.$$

Hence K is the coefficient of $(-\phi)^k$ in

$$k!\,(\pi)^{n/2}\left[(1+c_1\phi)(1+c_2\phi)\ldots(1+c_n\phi)\right]^{-\frac{1}{2}},$$

provided ϕ lies between $\pm 1/c_1$, where c_1 is the numerically greatest of the quantities $c_1, c_2, \ldots, c_n$.

The quantity within the square brackets is M/J^2, where M is the discriminant of $\chi+\phi\chi'$ expressed in the original variables $x_1, x_2, \ldots, x_n$.

Hence $$W_k(h) = k!\,(\pi)^{n/2}\,R_k\int_0^h \frac{F(u)\,u^{k+n/2-1}}{\Gamma(k+n/2)}\,du,$$

where R_k is the coefficient of $(-\phi)^k$ in $M^{-\frac{1}{2}}$. Writing

$$F(u) = e^{-\frac{1}{2}u} = e^{-v},$$

we obtain, finally,

$$W_k(h) = (2\pi)^{n/2}\,k!\,2^k.R_k\int_0^{h/2}\frac{e^{-v}v^{k+n/2-1}}{\Gamma(k+n/2)}\,dv,$$

which gives the result previously obtained when $h=\infty$.

As an example of this method consider the case of $U_k(h)$, a particular case of $W_k(h)$, where

$$\chi' = (\xi_1x_1+\xi_2x_2+\ldots+\xi_nx_n)^2.$$

Changing to coordinates $u_1, u_2, \ldots, u_n$,

χ becomes $u_1^2+u_2^2+\ldots+u_n^2$,

χ' ,, $c_1u_1^2$ (one term only).

Hence, inserting the value of the Dirichlet integral,

$$U_k(h) = \frac{c_1^k}{J}\,\Gamma(k+\tfrac{1}{2})\,\Gamma(\tfrac{1}{2})\ldots\Gamma(\tfrac{1}{2})\int_0^h\frac{e^{-\frac{1}{2}u}u^{k+n/2-1}}{\Gamma(k+n/2)}\,du.$$

Now $J=\Delta^{\frac{1}{2}}$ and $$\frac{c_1}{1}=\frac{\Theta_1}{\Delta}=\frac{A_{11}\xi_1^2+\ldots+2A_{12}\xi_1\xi_2+\ldots}{\Delta},$$

and after easy reduction

$$U_k(h) = 1.3.5\ldots(2k-1)\left(\frac{\Theta_1}{\Delta}\right)^k\frac{(2\pi)^{n/2}}{\Delta^{\frac{1}{2}}}\int_0^{h/2}\frac{e^{-v}v^{k+n/2-1}}{\Gamma(k+n/2)}dv,$$

which agrees with the previous result when $h=\infty$.

It is clear that the ratio of the incomplete integrals of form $W_k(h)$ or the derived form $V(h)$ bounded by $\chi = h$ to the corresponding complete integrals bounded by $\chi = \infty$, is the quantity $1-P$, where

$$P = \int_{h/2}^{\infty} \frac{e^{-v} v^{k+n/2-1}}{\Gamma(k+n/2)}\, dv$$

$$= \frac{\int_{\sqrt{h}}^{\infty} e^{-\frac{1}{2}x^2} x^{2k+n-1}\, dx}{\int_{0}^{\infty} e^{-\frac{1}{2}x^2} x^{2k+n-1}\, dx},$$

where $$v = \tfrac{1}{2}x^2,$$

thus identifying the quantity P with that tabulated by Karl Pearson for a number of values of h, and the index $2k+n-1$, in his paper on the deviations from the probable in the *Phil. Mag.*, July 1900, p. 157.

A PROPERTY OF POLYNOMIALS WHOSE ROOTS ARE REAL

By G. S. Le Beau.

[Read March 11th, 1920.]

1. Let $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$, be the roots, supposed real, unequal, and in ascending order of magnitude, of a polynomial

$$f(x) \equiv x^n - a_1 x^{n-1} + a_2 x^{n-2} - \ldots,$$

and let $\beta_1, \beta_2, \ldots, \beta_{n-1}$, be the roots, also real and in ascending order of magnitude, of

$$\phi(x) \equiv x^{n-1} - b_1 x^{n-2} + \ldots;$$

further, let the roots of $\phi(x)$ separate those of $f(x)$, so that

$$\alpha_1 < \beta_1 \leqslant \alpha_2 \leqslant \beta_2 \leqslant \ldots \leqslant \beta_{n-1} < \alpha_n.$$

Let $f(x)$ be divided by $\phi(x)$ and let the quotient be $x - \gamma$ and the remainder $-A\psi(x)$, where

$$\psi(x) \equiv x^{n-2} - c_1 x^{n-3} + \ldots.$$

Then denoting the roots of $\psi(x)$ by $\delta_1, \delta_2, \ldots, \delta_{n-2}$, it is immediately seen that the δ's are all real and that they separate the β's, and further that γ and the δ's separate the α's. We can thus take γ and the δ's together as a new set of β's; that is, we take, as a new $\phi(x)$, the polynomial

$$(x-\gamma)(x-\delta_1)(x-\delta_2)\ldots(x-\delta_{n-2})$$

and divide $f(x)$ by it as before, obtaining a new γ and new δ's, which separate the α's as before, and so on, continuing in this way indefinitely. We shall show that $\delta_1, \delta_2, \ldots, \delta_{n-2}$ tend to the limits $\alpha_2, \alpha_3, \ldots, \alpha_{n-1}$.

2. Considering any stage of the process, let the greatest value of

$$|\beta_s - \tfrac{1}{2}(\alpha_1 + \alpha_n)| \quad (s = 1, 2, 3, \ldots, n-1)$$

be denoted by g. Then since

$$\beta_1 \leqslant \delta_1 \leqslant \beta_2 \leqslant \delta_2 \leqslant \ldots \leqslant \delta_{n-2} \leqslant \beta_{n-1}, \tag{1}$$

it follows that $$|\delta_s - \tfrac{1}{2}(a_1+a_n)| \leqslant g \quad (s = 1, 2, 3, \ldots, n-2). \tag{2}$$

Also $$\gamma = \sum_{s=1}^{n} a_s - \sum_{s=1}^{n-1} \beta_s,$$

and since $$a_1 \leqslant \beta_1 \leqslant a_2 \leqslant \beta_2 \leqslant \ldots \leqslant \beta_{n-1} \leqslant a_n, \tag{3}$$

we get $$\begin{aligned} g &\geqslant \tfrac{1}{2}(a_1+a_n) - \beta_1 \geqslant \gamma - \tfrac{1}{2}(a_1+a_n) \\ &\geqslant \tfrac{1}{2}(a_1+a_n) - \beta_{n-1} \\ &\geqslant -g. \end{aligned}$$

Hence $$|\gamma - \tfrac{1}{2}(a_1+a_n)| \leqslant g. \tag{4}$$

From (2) and (4) we get that if g' is the greatest value of $|\beta_s - \tfrac{1}{2}(a_1+a_n)|$ for the next stage of the process,

$$g' \leqslant g.$$

Thus at no stage does g increase, and, since it is essentially positive, it tends to a definite limit.

3. Again, from the equation

$$f(x) = (x-\gamma)\phi(x) - A\psi(x),$$

we get, on equating coefficients of x^{n-2},

$$\begin{aligned} A &= \Sigma\beta_1\beta_2 - \Sigma a_1 a_2 + \gamma\Sigma\beta_1 \\ &= \Sigma\beta_1\beta_2 - \Sigma a_1 a_2 + (\Sigma a_1 - \Sigma\beta_1)\Sigma\beta_1 \\ &= (a_n - \beta_{n-1}) \sum_{s=1}^{n-1} (\beta_s - a_s) + (a_{n-1} - \beta_{n-2}) \sum_{s=1}^{n-2} (\beta_s - a_s) \\ &\quad + \ldots + (a_2 - \beta_1)(\beta_1 - a_1), \end{aligned} \tag{5}$$

so that it follows from (3) that A cannot be negative.

If we denote the value of A at the next stage by A', we have

$$A' = (\Sigma\delta_1\delta_2 + \gamma\Sigma\delta_1) - \Sigma a_1 a_2 + (\gamma + \Sigma\delta_1)(\Sigma a_1 - \gamma - \Sigma\delta_1),$$

since γ and the δ's constitute the new β's.

Thus $$A' = \Sigma\delta_1\delta_2+\gamma\Sigma\delta_1-\Sigma\alpha_1\alpha_2+(\gamma+\Sigma\delta_1)(\Sigma\beta_1-\Sigma\delta_1)$$
$$= \Sigma\delta_1\delta_2-\Sigma\alpha_1\alpha_2+\gamma\Sigma\beta_1+(\Sigma\beta_1-\Sigma\delta_1)\Sigma\delta_1. \quad (6)$$

From (5) and (6), we get

$$A'-A = \Sigma\delta_1\delta_2-\Sigma\beta_1\beta_2+(\Sigma\beta_1-\Sigma\delta_1)\Sigma\delta_1$$
$$= (\beta_{n-1}-\delta_{n-2})\sum_{s=1}^{n-2}(\delta_s-\beta_s)+(\beta_{n-2}-\delta_{n-3})\sum_{s=1}^{n-3}(\delta_s-\beta_s)$$
$$+\ldots+(\beta_2-\delta_1)(\delta_1-\beta_1), \quad (7)$$

so that, from (1), it follows that

$$A' \geqslant A.$$

Since all the β's lie between assignable limits, it is obvious that A cannot exceed an assignable value. Hence A tends to a definite limit.

4. It follows from this result that, ϵ being a given arbitrarily small positive quantity, from and after a certain stage the expression on the right-hand side of (7) will be less than ϵ^2. Since all the products of differences of which this expression is composed are positive, and the expression contains the product $(\beta_{s+1}-\delta_s)(\delta_s-\beta_s)$, we shall then have

$$(\beta_{s+1}-\delta_s)(\delta_s-\beta_s) < \epsilon^2 \quad (s = 1, 2, 3, \ldots, n-2),$$

and hence either $\beta_{s+1}-\delta_s < \epsilon$ or $\delta_s-\beta_s < \epsilon$. (8)

Writing $x = \delta_s$ in the identity

$$f(x) = (x-\gamma)\,\phi(x)-A\psi(x),$$

we get $$f(\delta_s) = (\delta_s-\gamma)\,\phi(\delta_s),$$

and it follows from (8) that

$$|f(\delta_s)| < \epsilon\,(\alpha_n-\alpha_1)^{n-1}.$$

Hence from and after a certain stage, δ_s differs from some one of the α's by less than an arbitrarily small ϵ_1. If ϵ_1 is sufficiently small, this one of the α's cannot be either α_1 or α_n. For, if the initial values of $\beta_1-\alpha_1$ and $\alpha_n-\beta_{n-1}$ both exceed h, say, it follows from the result of § 2 that, at every stage of the process, the difference between any δ and either α_1 or α_n also exceeds h.

We thus arrive at the result that, ϵ being given and arbitrarily small, from and after a certain stage in the process, every δ will differ from some one of $a_2, a_3, \ldots, a_{n-1}$, by a quantity less than ϵ.

5. We will next prove that if the process is carried sufficiently far, and if ϵ is sufficiently small, it is impossible that two δ's should differ from the same a by quantities less than ϵ.

For suppose that 2ϵ is less than the difference between any two consecutive a's, and that δ and δ' both differ from a_r by less than ϵ. Since the interval between two consecutive a's cannot contain more than one δ, it follows that δ and δ' are consecutive δ's. Let β, β', β'', be the consecutive β's which they separate. Then we shall have either

$$\text{(i)} \quad \beta < a_{r-1} < \delta < \beta' < a_r < \delta' < \beta'',$$

or

$$\text{(ii)} \quad \beta < \delta < a_r < \beta' < \delta' < a_{r+1} < \beta''.$$

We can suppose the process carried so far that the expression on the right-hand side of (7) is less than either $\frac{1}{2}\epsilon(a_r - a_{r-1})$ or $\frac{1}{2}\epsilon(a_{r+1} - a_r)$. Hence $(\beta'' - \delta')(\delta - \beta)$ is less than either of these quantities.

Now, in case (i),

$$\delta - \beta > \tfrac{1}{2}(a_r - a_{r-1}), \quad \beta'' - \delta' < \epsilon, \quad \beta'' - a_r < 2\epsilon,$$

and in case (ii),

$$\beta'' - \delta' > \tfrac{1}{2}(a_{r+1} - a_r), \quad \delta - \beta < \epsilon, \quad a_r - \beta < 2\epsilon.$$

Thus in either case we shall have two of the δ's and two of the β's all differing from a_r by less than 2ϵ.

Hence from the equation

$$f(x) = (x - \gamma)\phi(x) - A\psi(x),$$

we get, observing that the right-hand side must be divisible by $x - a_r$,

$$(x - a_1)(x - a_2) \ldots (x - a_n) = (x - a_r)^2 F(x) + \epsilon(x - a_r) G(x),$$

where $|G(a_r)|$ cannot exceed an assignable fixed value, independent of ϵ.

Dividing by $x - a_r$ and putting $x = a_r$, we get

$$f'(a_r) = \epsilon G(a_r),$$

which is impossible if ϵ is sufficiently small. This proves the result required.

6. If, then, ϵ_1 is sufficiently small, and we choose any ϵ less than ϵ_1, we shall have, from and after a certain stage, every δ differing from one of $a_2, a_3, \ldots, a_{n-1}$ by less than ϵ, and no two δ's differing from the same a by less than ϵ. Thus we shall have

$$|\delta_1-a_2|<\epsilon, \quad |\delta_2-a_3|<\epsilon, \quad \ldots, \quad |\delta_{n-2}-a_{n-1}|<\epsilon.$$

Thus $\delta_1, \delta_2, \ldots, \delta_{n-2}$ tend to the limits $a_2, a_3, \ldots, a_{n-1}$.

If γ and γ' are the values of γ obtained in two consecutive stages of the operation, it immediately follows from this result, combined with the equation

$$\gamma = \sum_{s=1}^{n} a_s - \sum_{s=1}^{n-1} \beta_s,$$

that $\gamma+\gamma'$ tends to the limit a_1+a_n. Hence the alternate values of γ tend to limits l, l', such that $l+l' = a_1+a_n$, and we evidently have that if L is the limiting value of A,

$$(x-a_1)(x-a_n) = (x-l)(x-l')-L,$$

so that

$$L = ll'-a_1 a_n.$$

The values of l, l', L, depend on the initial choice of $\beta_1, \beta_2, \ldots, \beta_{n-}$

7. We have

$$L = (l-a_1)(a_n-l) = (l-a_1)(l'-a_1) = (a_n-l)(a_n-l').$$

It may be observed that

$$|(a_r-l)(a_r-l')| \leqslant (l-a_1)(l'-a_1) \quad (r = 2, 3, \ldots, n-1).$$

For suppose, if possible, that

$$\left|\frac{(a_r-l)(a_r-l')}{(a_1-l)(a_1-l')}\right| = 1+\lambda,$$

where λ is positive.

From the equation

$$(x-a_1)(x-a_2)\ldots(x-a_n)$$
$$= (x-\gamma')(x-\gamma)(x-\delta_1)\ldots(x-\delta_{n-2}) - A(x-\delta_1')(x-\delta_2')\ldots(x-\delta_{n-2}'),$$

we get, putting $x = a_r$,

$$\frac{a_r-\delta_{r-1}'}{a_r-\delta_{r-1}} = \frac{(a_r-\gamma')(a_r-\gamma)}{A} \prod_s \left(\frac{a_r-\delta_s}{a_r-\delta_s'}\right),$$

where the product is taken for every value of s from 1 to $n-2$ inclusive, except $r-1$. Since $\delta_1, \delta_2, \ldots$ tend to limits $\alpha_2, \alpha_3, \ldots$, it follows that this product tends to the limit unity. Also

$$\text{Lim} \left| \frac{(\alpha_r-\gamma')(\alpha_r-\gamma)}{A} \right| = 1+\lambda.$$

Hence from and after a certain stage, we shall have

$$\left| \frac{(\alpha_r-\gamma')(\alpha_r-\gamma)}{A} \right| > 1+\tfrac{1}{2}\lambda, \quad \text{and} \quad \left| \prod_s \left(\frac{\alpha_r-\delta_s}{\alpha_r-\delta_s'}\right) \right| > \frac{1}{1+\frac{1}{2}\lambda},$$

so that from and after this stage

$$\left| \frac{\alpha_r-\delta_{r-1}'}{\alpha_r-\delta_{r-1}} \right| > 1,$$

which is impossible, since $\text{Lim}\ \delta_{r-1} = \alpha_r$.

8. It has been supposed that the roots of $f(x)$ are unequal, but we may evidently dispense with this condition, for if $f(x)$ contains a factor $(x-\alpha)^r$, we must suppose the original $\phi(x)$ to contain a factor $(x-\alpha)^{r-1}$, so that $\psi(x)$ also contains a factor $(x-\alpha)^{r-1}$, and so on. In fact, the polynomial $\psi(x)$ obtained at any stage is $(x-\alpha)^{r-1}\psi_1(x)$, where $\psi_1(x)$ is the polynomial obtained at the corresponding stage of the process of repeated division of $f(x)/(x-\alpha)^{r-1}$. The theorem thus holds in all cases when the roots of $f(x)$ are real.

If we divide $f(x)$ by the successive polynomials $\psi(x)$, we get a succession of quadratics, whose roots tend to the limits α_1 and α_n. Taking a polynomial $\psi(x)$, whose roots are sufficiently close to $\alpha_2, \alpha_3, \ldots, \alpha_{n-1}$, we can apply the same process to it, and in this way we can obtain a quadratic whose roots are as close as we please to α_2 and α_{n-1}, and so on. We can thus obtain a set of equations, which are all quadratic if n be even, or one linear and the rest quadratic if n be odd, whose roots approximate as closely as we please to α_1 and α_n, α_2 and α_{n-1}, α_3 and α_{n-2}, &c.

In particular, we can take, for the initial $\phi(x)$, the first derived $f'(x)$ of $f(x)$. In this case, the coefficients in the quadratic equations are rational functions of the coefficients of $f(x)$. The process can thus be applied to an algebraic function y defined by an equation $f(y, x) = 0$, and we get the following result :—If for all real values of x in the range $a \leqslant x \leqslant b$, the values of y given by the equation $f(y, x) = 0$ are real, these values

may be represented, to an arbitrarily close approximation, as the values of rational functions of x, or the roots of quadratic equations whose coefficients are rational functions of x; these equations may be obtained by the process described above, the initial divisor being $f_y'(y, x)$. As a trivial example, if $|x| \leqslant 1$, successive approximations to one value of y given by the equation

$$y^3-3y^2+4x^2 = 0$$

are $$y = 2x^2, \quad y = (2x^4-4x^2)/(2x^4-2x^2-1),$$

and so on.

A POINT IN THE DYNAMICAL THEORY OF THE TIDES

By E. G. C. POOLE.

[Read March 11th, 1920.]

1. *Statement of the Problem.*

It is well known* that, in the case of free tides symmetrical about the axis of a rotating globe, in an ocean of uniform depth, the elevation ζ satisfies the equation

$$\frac{d}{d\mu}\left(\frac{1-\mu^2}{f^2-\mu^2}\frac{d\zeta}{d\mu}\right)+\beta\zeta=0, \qquad \text{(A)}$$

where β is a known positive constant, μ is the cosine of the polar distance, and f is half the frequency per day of the tide. If the ocean covers the whole globe, it is physically necessary that ζ and $d\zeta/d\mu$ remain finite at the poles $\mu=\pm1$, and these boundary conditions require that f^2 should satisfy certain conditions. If, following Darwin, we expand ζ in a power series of *even* powers of μ only, then we find tides which are symmetrical about the equator, and for which f^2 must satisfy

$$1-\frac{\beta f^2}{2.3}+\frac{\frac{\beta}{4.5}}{\left(1-\frac{\beta f^2}{4.5}\right)}+\frac{\frac{\beta}{6.7}}{\left(1-\frac{\beta f^2}{6.7}\right)}+\ldots=0, \qquad (1)$$

while for the unsymmetrical tides, where ζ is a series of odd powers, we must have

$$1-\frac{\beta f^2}{1.2}+\frac{\frac{\beta}{3.4}}{\left(1-\frac{\beta f^2}{3.4}\right)}+\frac{\frac{\beta}{5.6}}{\left(1-\frac{\beta f^2}{5.6}\right)}+\ldots=0. \qquad (2)$$

These formulæ and the method by which they are obtained suggest that a

* Lamb, *Hydrodynamics*, 4th edition, pp. 322–325 and 335–337. The mutual attraction of the particles of water is neglected.

series of values of f^2 are given approximately by $\beta f^2 = n(n+1)$, where n is an integer.

If again, following Hough, we expand in zonal harmonics, we have the equations

$$L_2 - \cfrac{1}{\cfrac{5.7^2.9}{L_4 -} \; \cfrac{1}{\cfrac{9.11^2.13}{L_5 -} \cdots}} = 0 \qquad (1a)$$

and

$$L_1 - \cfrac{1}{\cfrac{3.5^2.7}{L_3 -} \; \cfrac{1}{\cfrac{7.9^2.11}{L_5 -} \cdots}} = 0, \qquad (2a)$$

where

$$L_n \equiv \frac{f^2-1}{n(n+1)} + \frac{2}{(2n-1)(2n+3)} - \frac{1}{\beta}. \qquad (3)$$

Hough shows how to calculate with any required degree of accuracy a set of values of f^2, to which the first approximations are given by putting $L_n = 0$, or

$$\beta(f^2-1) = n(n+1) - \frac{2\beta\, n(n+1)}{(2n-1)(2n+3)}.$$

In the present paper, we propose to supplement this discussion by a direct proof, based on the methods of Sturm and Liouville, that these are in fact the only roots of the equation for f^2, so that the sets of roots suggested by the two methods are in fact identical and complete.

2. *Prof. Love's Transformation.*

The treatment of the equation is greatly simplified by the use of a new variable suggested to the writer by Prof. Love. We put

$$\frac{1-\mu^2}{f^2-\mu^2}\,\frac{d\xi}{d\mu} = -\beta\xi,$$

so that

$$\frac{d\xi}{d\mu} = \zeta. \qquad (4)$$

In the tidal theory, ξ is a factor in the expressions for the horizontal components of velocity. Then

$$(1-\mu^2)\frac{d^2\xi}{d\mu^2} + \beta(f^2-\mu^2)\,\xi = 0. \qquad (B)$$

This equation is of the same form as that discussed by M. Abraham* in connection with the electrical oscillations on a rod. It also bears a simple relation to the equation of wave motions symmetrical about an axis. For, if we have

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Pi}{\partial\rho}\right)+\frac{\partial^2\Pi}{\partial z^2}+k^2\Pi=0, \tag{5}$$

and if we put $Q=\rho\dfrac{\partial\Pi}{\partial\rho}$, so that

$$\rho\frac{\partial}{\partial\rho}\left(\frac{1}{\rho}\frac{\partial Q}{\partial\rho}\right)+\frac{\partial^2 Q}{\partial z^2}+k^2Q=0, \tag{6}$$

then on introducing elliptic coordinates

$$z=c\lambda\mu, \quad \rho=c\sqrt{[(\lambda^2-1)(1-\mu^2)]},$$

we find
$$(\lambda^2-1)\frac{\partial^2 Q}{\partial\lambda^2}+(1-\mu^2)\frac{\partial^2 Q}{\partial\mu^2}+k^2c^2(\lambda^2-\mu^2)Q=0, \tag{7}$$

and if Q is of the form $E(\lambda)F(\mu)$, then $E(\lambda)$ and $F(\mu)$ each satisfy equations of the form (B), where $\beta=k^2c^2$ and the value of f^2 is the same in each case.

3. *Demonstration.*

Corresponding to the boundary conditions that ξ, $d\xi/d\mu$ should be finite at $\mu=\pm1$, it is sufficient to postulate that $\bar{\xi}=0$ at $\mu=\pm1$. We will now consider a particular solution $\bar{\xi}(\mu)$ which vanishes when $\mu=-1$, and we will examine the conditions that it should vanish also at $\mu=+1$ and at a certain number of points in the interval.

By a direct application of the methods, of which a sketch is given by Lord Rayleigh (*Theory of Sound*, Vol. 1, pp. 217–223), we can establish the following theorems:—

I. *No critical value of f^2 can be imaginary.*

For suppose $f^2=p+iq$, and $\bar{\xi}=u+iv$. Then equating real and imaginary parts in the equation (B), and remembering that μ is real and

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* M. Abraham: (1) "Die elektrischen Schwingungen in einem stabförmigen Leiter," *Wiedemann's Ann.*, Vol. 66 (1898), p. 435; (2) "Über einige bei Schwingsproblemen auftretenden Differentialgleichungen," *Math. Ann.*, Vol. 51, p. 81.

β real and positive, we have

$$\left.\begin{aligned}(1-\mu^2)\frac{d^2u}{d\mu^2}+\beta(p-\mu^2)u-\beta qv=0,\\ (1-\mu^2)\frac{d^2v}{d\mu^2}+\beta(p-\mu^2)v+\beta qu=0.\end{aligned}\right\} \tag{8}$$

Hence $$\left(v\frac{d^2u}{d\mu^2}-u\frac{d^2v}{d\mu^2}\right)=\beta q\left(\frac{u^2+v^2}{1-\mu^2}\right).$$

Integrating from -1 to $+1$, we have $u=0$, $v=0$ at both limits. Hence

$$\beta q\int_{-1}^{+1}\frac{(u^2+v^2)}{(1-\mu^2)}\,d\mu=0. \tag{9}$$

But the integrand is essentially positive. Hence we must have $q=0$, and f^2 is real.

II. *No critical value of f^2 can be negative.*

For suppose $f^2=-a^2$. Then

$$\frac{d^2\bar{\xi}}{d\mu^2}=\beta\frac{(a^2+\mu^2)}{(1-\mu^2)}\bar{\xi}. \tag{10}$$

Hence $\bar{\xi}$, $d^2\bar{\xi}/d\mu^2$ have the same sign throughout the interval $-1\leqslant\mu\leqslant+1$. Now $\bar{\xi}$ vanishes at $\mu=-1$. Hence, as μ increases from -1, $\bar{\xi}$ will take the sign of $d\bar{\xi}/d\mu$, which we may take positive.

Now, as μ increases from -1, $\bar{\xi}$ will increase steadily until $d\bar{\xi}/d\mu=0$; but $d\bar{\xi}/d\mu$ also increases steadily until $d^2\bar{\xi}/d\mu^2=0$, because the latter is positive so long as $\bar{\xi}$ is positive. Hence both the quantities $\bar{\xi}$, $d\bar{\xi}/d\mu$ must increase steadily throughout the interval and cannot vanish at any other point within or on the boundary of the interval, except at $\mu=-1$. It follows that no negative value of f^2 can give a solution which vanishes at both limits.

III. *The abscissa of each zero of $\bar{\xi}(\mu)$ is a monotonic decreasing function of f^2.*

For let $\bar{\xi}$, $\bar{\xi}'$ be the solutions which vanish at $\mu=-1$ and which correspond to parameters f^2 and f'^2, where $f^2<f'^2$. Then between every pair of zeros of $\bar{\xi}$, there is at least one zero of $\bar{\xi}'$. For we have

$$\left(\bar{\xi}'\frac{d^2\bar{\xi}}{d\mu^2}-\bar{\xi}\frac{d^2\bar{\xi}'}{d\mu^2}\right)=\beta\frac{(f'^2-f^2)\,\bar{\xi}\,\bar{\xi}'}{(1-\mu^2)}. \tag{11}$$

Integrating between two values of μ which make $\bar{\xi}$ vanish, it is easily shown that, if $\bar{\xi}'$ retains the same sign throughout the interval, then the two sides of the equation

$$\left(\bar{\xi}'\frac{d\bar{\xi}}{d\mu}-\bar{\xi}\frac{d\bar{\xi}'}{d\mu}\right)_{\mu_1}^{\mu_2}=\beta(f'^2-f^2)\int_{\mu_1}^{\mu_2}\frac{\bar{\xi}\,\bar{\xi}'}{1-\mu^2}\,d\mu \tag{12}$$

must have opposite signs. Hence it is impossible for $\bar{\xi}'$ to keep the same sign throughout the interval, and hence the roots of $\bar{\xi}$ are separated by roots of $\bar{\xi}'$.

We conclude from this that, as f^2 is increased, the respective roots of $\bar{\xi}(\mu)$ move steadily towards the lower limit, so that no root can ever be lost. It follows that, if for two values of f^2 the difference in the number of zeros lying within the interval $-1\leqslant\mu\leqslant 1$ (including its extremities), is K, then there are exactly K critical values of f^2 satisfying the conditions $f^2\leqslant f_n^2\leqslant f'^2$.

IV. *If* $2(n+1)>\beta$, *and if* $n(n+1)+\beta\leqslant\beta f^2\leqslant(n+1)(n+2)$, *where* n *is an integer, then* f^2 *cannot have a critical value between these limits, and there are exactly* n *zeros of* $\bar{\xi}(\mu)$ *between* $\mu=-1$ *and* $\mu=+1$, *not counting the lower limit.*

Consider the two equations

$$(1-\mu^2)\frac{d^2y}{d\mu^2}+n(n+1)\,y=0, \tag{13}$$

$$(1-\mu^2)\frac{d^2Y}{d\mu^2}+(n+1)(n+2)\,Y=0. \tag{14}$$

Particular solutions of these equations, which vanish at *both* limits $\mu=\pm 1$ and at $n-1$ and n intermediate points respectively, are clearly

$$y=(1-\mu^2)\frac{dP_n}{d\mu},\qquad Y=(1-\mu^2)\frac{dP_{n+1}}{d\mu}, \tag{15}$$

where P_n, P_{n+1} are the Legendre polynomials of order n and $n+1$.

Now we have identically

$$\left(y\frac{d^2\bar{\xi}}{d\mu^2}-\bar{\xi}\frac{d^2y}{d\mu^2}\right)=\{n(n+1)-\beta f^2+\beta\mu^2\}\,\bar{\xi}\frac{dP_n}{d\mu}, \tag{16}$$

$$\left(Y\frac{d^2\bar{\xi}}{d\mu^2}-\bar{\xi}\frac{d^2Y}{d\mu^2}\right)=\{(n+1)(n+2)-\beta f^2+\beta\mu^2\}\,\bar{\xi}\frac{dP_{n+1}}{d\mu}. \tag{17}$$

Integrating (16) between two zeros of y, we can show that the two sides cannot have the same sign unless there is at least one zero of $\bar{\xi}$ between every two successive zeros of y. But similarly, from (17), we can see that there is at least one zero of Y between two successive ones of $\bar{\xi}$. Hence, we conclude, firstly, that because y has $n-1$ internal zeros and one at each extremity, hence $\bar{\xi}$ has at *least* n internal zeros, besides that at $\mu = -1$. But if $\bar{\xi}$ had $n+1$ or more zeros, or if there were a zero at $\mu = +1$, then Y would have to possess at least $n+1$ internal zeros in the interval $-1 < \mu = 1$, and this is not the case.

Hence $\bar{\xi}$ has exactly n zeros between -1 and $+1$, besides the zero at $\mu = -1$, for the given range of values of f^2.

It follows that the critical value of f^2, which can be approximately calculated by Hough's method, and which lies between $n(n+1)/\beta$ and $1+n(n+1)/\beta$ is that for which $\bar{\xi}$ has exactly $n-1$ internal roots in the interval $-1 < \mu < 1$, besides the roots $\mu = \pm 1$. Further, there are *no other critical values of* f^2 besides those whose existence is revealed by Hough's method.

THE DIVISORS OF NUMBERS

By P. A. MacMahon.

[Read January 15th, 1920.]

1. The excess of the number of divisors of a number n which have the form $4m+1$ over the number of divisors which have the form $4m+3$ is a quantity of arithmetical importance which was studied by Jacobi in the *Fundamenta Nova*, and later by Glaisher,* who denoted it by $E(n)$. The present paper is a study of other numerical quantities of a similar nature.

If $\sigma_s(n)$ denote the sum of the s-th powers of the divisors of n, we have the well known identities

$$\sum_1^\infty \frac{mq^m}{1-q^m} = \sum_1^\infty \sigma_s(n)q^n, \quad \sum_1^\infty \frac{q^m}{(1-q^m)^2} = \sum_1^\infty \sigma_1(n)q^n,$$

and thence the well known double identity

$$\sum_1^\infty \frac{mq^m}{1-q^m} = \sum_1^\infty \frac{q^m}{(1-q^m)^2} = \sum_1^\infty \sigma_1(n)q^n,$$

which means when interpreted that the number of the divisors of n is equal to the number of the conjugates of divisors of n.

In general a double identity can always be obtained, a fact which will be in constant evidence in what follows.

If d be a divisor of n and a an arbitrary quantity, it is easy to establish the relations

$$(1) \qquad \sum_1^\infty \frac{aq^m}{1-aq^m} = \sum_1^\infty \frac{a^m q^m}{1-q^m} = \sum_1^\infty (\Sigma a^d)q^n,$$

by the method of expansion in row and summation by column introduced by Lambert.†

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* *Proc. London Math. Soc.*, Ser. 1, Vol. 15.

† See also *Combinatory Analysis*.

We immediately deduce the relations

$$(2) \qquad \left(a\frac{d}{da}\right)^s \sum_1^\infty \frac{aq^m}{1-aq^m} = \left(a\frac{d}{da}\right)^s \sum_1^\infty \frac{a^m q^m}{1-q^m} = \sum_1 (\Sigma d^s a^d) q^n,$$

$$(3) \qquad \left(q\frac{d}{dq}\right)^u \left(a\frac{d}{da}\right)^s \sum_1^\infty \frac{aq^m}{1-aq^m} = \left(q\frac{d}{dq}\right)^u \left(a\frac{d}{da}\right) \sum_1^\infty \frac{a^m q^m}{1-q^m}$$

$$= \sum_1^\infty (\Sigma n^u d^s a^d) q^n,$$

which will be dealt with later in the paper.

At present I proceed in another manner from the relation (1).

Putting $a = e^{ix}$,

where $i = \sqrt{(-1)}$,

we find that

$$\frac{e^{ix}q^m}{1-e^{ix}q^m} = \frac{q^m \cos x - q^{2m}}{1-2q^m \cos x + q^{2m}} + i\,\frac{q^m \sin x}{1-2q^m \cos x + q^{2m}},$$

and since

$$\int \frac{e^{ix}q^m}{1-e^{ix}q^m}\,dx = \tan^{-1}\frac{q^m \sin x}{1-q^m \cos x} + \tfrac{1}{2}i \log (1-2q^m \cos x + q^{2m}),$$

the relations

$$(4) \qquad \sum_1^\infty \frac{q^m \cos x - q^{2m}}{1-2q^m \cos x + q^{2m}} = \sum_1^\infty \frac{q^m \cos mx}{1-q^m} = \sum_1^\infty (\Sigma \cos dx)\, q^n,$$

$$(5) \qquad \Sigma \frac{q^m \sin x}{1-2q^m \cos x + q^{2m}} = \sum_1^\infty \frac{q^m \sin mx}{1-q^m} = \sum_1^\infty (\Sigma \sin dx)\, q^n,$$

lead by integration to the relations

$$(6) \quad \sum_1^\infty \tan^{-1}\frac{q^m \sin x}{1-q^m \cos x} = \sum_1^\infty \frac{1}{m}\frac{q^m \sin mx}{1-q^m} = \sum_1^\infty \left(\Sigma \frac{1}{d} \sin dx\right) q^n,$$

$$(7) \quad \sum_1^\infty \log \frac{1}{1-2q^m \cos x + q^{2m}} = 2\sum_1^\infty \frac{1}{m}\frac{q^m \cos mx}{1-q^m} = 2\sum_1^\infty \left(\Sigma \frac{1}{d} \cos dx\right) q^n.$$

The left-hand side of relation (7) may be written

$$\log \frac{1}{\prod_1^\infty (1-2q^m \cos x + q^{2m})},$$

and transforming by a well known formula in the *Fundamenta Nova*, we obtain

$$\log \frac{\sin \frac{1}{2}x \prod_1^\infty (1-q^m)}{\sin \frac{1}{2}x - q \sin \frac{3}{2}x + q^3 \sin \frac{5}{2}x - \dots} = 2 \sum_1^\infty \frac{1}{m} \frac{q^m \cos mx}{1-q^m}$$

$$= 2 \sum_1^\infty \left(\Sigma \frac{1}{d} \cos dx \right) q^n,$$

where the exponents of q in the denominator series are the triangular numbers.

Differentiating with regard to x and changing sign throughout, we obtain the formula

$$(8) \quad \frac{\cos \frac{1}{2}x - 3q \cos \frac{3}{2}x + 5q^3 \cos \frac{5}{2}x - 7q^6 \cos \frac{7}{2}x + \dots}{\sin \frac{1}{2}x - q \sin \frac{3}{2}x + q^3 \sin \frac{5}{2}x - q^6 \sin \frac{7}{2}x - \dots}$$

$$= \cot \tfrac{1}{2}x + 4 \sum_1^\infty \frac{q^m \sin mx}{1-q^m} = \cot \tfrac{1}{2}x + 4\Sigma(\Sigma \sin dx) q^n,$$

which is fundamental for this research.

If in $\sin mx$ we give x the special value π/p where p is an integer, and m the successive values 1, 2, 3, ..., the values of $\sin mx$ recur with a period $2p$. Thus, to take the simplest case possible, $p = 2$, $\sin mx$ has the values 1, 0, −1, 0, ... in a period of 4.

In other words, if m have the forms $4m+1$, $4m+3$, the values are 1, −1 respectively, while the value for the forms $4m$, $4m+2$ is zero.

Hence we see that $\Sigma \sin dx$,

where d denotes a divisor of n, represents the excess of the number of divisors which have the form $4m+1$ over the number of divisors which have the form $4m+3$. In fact

$$\Sigma(\Sigma \sin dx) q^n = \Sigma E(n) q^n$$

in Glaisher's notation, when x has the value $\frac{1}{2}\pi$.

Before proceeding to the consideration of the identity (8) which as has been seen is derived directly from the *Fundamenta Nova*, an important simplification can be made, because by simple trigonometry it can be thrown into the form

$$(9) \quad \frac{2q \sin x - (q+3q^3) \sin 2x + (2q^3+4q^6) \sin 3x - (3q^6+5q^{10}) \sin 4x + \dots}{1-(1+q)\cos x + (q+q^3)\cos 2x - (q^3+q^6)\cos 3x + (q^6+q^{10})\cos 4x - \dots}$$

$$= 2 \sum_1^\infty (\Sigma \sin dx) q^n.$$

In the fraction the general terms of numerator and denominator are respectively

$$(-)^{m+1}\left\{(m-1)\,q^{\frac{1}{2}m(m-1)}+(m+1)\,q^{\frac{1}{2}m(m+1)}\right\}\sin mx,$$

$$(-)^{m}\left\{q^{\frac{1}{2}m(m-1)}+q^{\frac{1}{2}m(m+1)}\right\}\cos mx.$$

Writing these for brevity

$$N_m \sin mx, \qquad D_m \cos mx,$$

we have

$$\frac{N_1\sin x+N_2\sin 2x+N_3\sin 3x+\ldots}{1+D_1\cos x+D_2\cos 2x+D_3\cos 3x+\ldots}=2\sum_1^\infty(\Sigma\sin dx)\,q^n.$$

The case $x=\frac{\pi}{2}$ gives

$$(10)\qquad \frac{q-q^3-2q^6+2q^{10}+3q^{15}-3q^{21}-4q^{28}+4q^{36}+\ldots}{1-q-q^3+q^6+q^{10}-q^{15}-q^{21}+q^{28}+q^{36}-\ldots}=\sum_1^\infty E^{(2)}(n)\,q^n,$$

equivalent to Glaisher's formula (*loc. cit.*, p. 4).

I have in the above denoted by $E^{(2)}(n)$ the quantity for which Glaisher's symbol is $E(n)$, because it is convenient to denote by $E^{(p)}(n)$ the arithmetical quantity obtained by putting $x=\frac{\pi}{p}$ in the formula.

Before proceeding to the general case I work out a few of the elementary cases.

The case $x=\frac{\pi}{3}$.

$\sin mx$ has the series of values of period 6,

$$\frac{\sqrt{3}}{2},\ \frac{\sqrt{3}}{2},\ 0,\ -\frac{\sqrt{3}}{2},\ -\frac{\sqrt{3}}{2},\ 0,$$

and $\cos mx$ the series of values

$$\tfrac{1}{2},\ -\tfrac{1}{2},\ -1,\ -\tfrac{1}{2},\ \tfrac{1}{2},\ 1,$$

and we observe that if $E^{(3)}(n)$ represents the excess of the number of divisors of n which have the forms $6m+1$, $6m+2$ over the number of divisors which have the forms $6m+4$, $6m+5$,

$$\Sigma\left(\Sigma\sin\frac{d\pi}{3}\right)q^n=\frac{\sqrt{3}}{2}\Sigma\,E^{(3)}(x)q^n.$$

Inserting the values of the sines and cosines and throwing out the factors 2 and $\frac{\sqrt{3}}{3}$ we reach the relation

$$(11)\quad \frac{q-3q^3+3q^6+q^{10}-6q^{15}+6q^{21}+q^{28}-9q^{36}+9q^{45}-\cdots}{1-2q+q^3+q^6-2q^{10}+q^{15}+q^{21}-2q^{28}+q^{36}+\ldots} = \sum_1^\infty E^{(3)}(n)q^n.$$

In both numerator and denominator of this fraction the signs *recur* in the order $+, -, +$.

In the numerator

when the q exponent is of form $\frac{1}{2}(3m+1)(3m+2)$ the coefficient is unity,

" " $\frac{1}{2}(3m+2)(3m+3)$ " $-3(m+1)$,

" " $\frac{1}{2}(3m+3)(3m+4)$ " $+3(m+1)$.

In the denominator the coefficients recur in the order $1, -2, 1$. Thence the recurring formula

(12) $E^{(3)}(n)-2E^{(3)}(n-1)+E^{(3)}(n-3)+E^{(3)}(n-6)-\ldots = 0$ (or the number above specified according as n is not or is a triangular number).

The case $n = \frac{\pi}{4}$.

We have here a period of eight in the values of the sines and cosines.

$\sin\frac{m\pi}{4}$ has the values $\frac{1}{\sqrt{2}}, 1, \frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}, -1, -\frac{1}{\sqrt{2}}, 0,$

$\cos\frac{m\pi}{4}$ " $\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}, -1, -\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, 1.$

Whence we gather that if $E^{(4)}(n)$ represents the excess of the number of divisors of n which have the forms $8m+1$, $8m+3$ over the number of divisors which have the forms $8m+5$, $8m+7$; and if $E^{(44)}(n)$ represents the excess of the number of divisors of n which have the form $8m+2$ over the number of divisors which have the form $8m+6$,

$$\Sigma\left(\Sigma\frac{d\pi}{4}\right)q^n = \frac{1}{\sqrt{2}}E^{(4)}(n)\,q^n+\Sigma E^{(44)}(n)\,q^n.$$

We find

$$\frac{\sqrt{2}(N_2-N_6+N_{10}-N_{14}+\ldots)+N_1+N_3-N_5-N_7+\ldots}{\sqrt{2}(1-D_4+D_7-D_{12}+D_{15}-\ldots)+D_1-D_3-D_5+D_7+\ldots}$$
$$= \sqrt{2}\,\Sigma E^{(4)}(n)\,q^n+\Sigma E^{(44)}(n)\,q^n,$$

simplifying to $$\frac{\sqrt{2}\,a+b}{\sqrt{2}\,c+d} = \sqrt{2}\,A+B,$$

where
$$\left.\begin{aligned} a &= -q-3q^3+5q^{15}+7q^{21} \\ b &= 2q+2q^3+4q^6-4q^{10}-6q^{15} \end{aligned}\right\} \quad A = \Sigma E^{(4)}(n)\,q^n,$$

$$\left.\begin{aligned} c &= 1-q^6-q^{10}-q^{21}-q^{28}+\ldots \\ d &= -1-q+q^3+q^6+q^{10}+q^{15}-\ldots \end{aligned}\right\} \quad B = \Sigma E^{(44)}(n)\,q^n.$$

Thence the relations
$$dA+cB = a,$$
$$2cA+dB = b,$$

and we are able to express A, B separately in terms of a, b, c, d; but if we did so the laws of the q series involved would not be so clear, so instead of writing

$$A = \frac{ad-bc}{d^2-2c^2}, \qquad B = \frac{bd-2ac}{d^2-2c^2},$$

I prefer to retain the above simultaneous relations which are at length

(13) $$(-1-q+q^3+q^6+q^{10}+q^{15}-\ldots)\,\Sigma E^{(4)}(n)\,q^n$$
$$+(1-q^6-q^{10}-q^{21}-q^{28}+\ldots)\,\Sigma E^{(44)}(n)\,q^n$$
$$= -q-3q^3+5q^{15}+7q^{21}-\ldots,$$

(14) $$2(1-q^6-q^{10}-q^{21}-q^{28}+\ldots)\,\Sigma E^{(4)}(n)\,q^n$$
$$+(-1-q+q^3+q^6+q^{10}+q^{15}-\ldots)\,\Sigma E^{(44)}(n)\,q^n,$$
$$= 2\,(q+q^3+2q^6-2q^{10}-3q^{15}-\ldots),$$

leading to the simultaneous formulæ

(15) $$-E^{(4)}(n)-E^{(4)}(n-1)+E^{(4)}(n-3)+E^{(4)}(n-6)+E^{(4)}(n-10)$$
$$+E^{(4)}(n-15)-\ldots+E^{(44)}(n)-E^{(44)}(n-6)-E^{(44)}(n-10)$$
$$-E^{(44)}(n-21)-E^{(44)}(n-28)+\ldots$$
$$= \text{coefficient of } q^n \text{ in } -q-3q^3+5q^{15}+7q^{21}-\ldots,$$

(16) $2E^{(4)}(n)-2E^{(4)}(n-6)-2E^{(4)}(n-10)-2E^{(4)}(n-21)-2E^{(4)}(n-28)+\ldots$

$$-E^{44}(n)-E^{44}(n-1)+E^{(44)}(n-3)+E^{(44)}(n-6)+E^{(44)}(n-10)+\ldots$$

$$= \text{coefficient of } q^n \text{ in } 2(q+q^3+2q^6-2q^{10}-3q^{15}+\ldots).$$

The actual values of the trigonometrical functions are not necessary for the investigation. This will now appear.

The case $x = \frac{\pi}{5}$.

The values of the sines and cosines in a period of ten are

$$\sin\frac{\pi}{5},\ \sin\frac{2\pi}{5},\ \sin\frac{2\pi}{5},\ \sin\frac{\pi}{5},\ 0,\ -\sin\frac{\pi}{5},\ -\sin\frac{2\pi}{5},\ -\sin\frac{2\pi}{5},$$

$$-\sin\frac{\pi}{5},\ 0;$$

$$\cos\frac{\pi}{5},\ \cos\frac{2\pi}{5},\ -\cos\frac{2\pi}{5},\ -\cos\frac{\pi}{5},\ -1,\ -\cos\frac{\pi}{5},\ -\cos\frac{2\pi}{5},$$

$$\cos\frac{2\pi}{5},\ \cos\frac{\pi}{5},\ 1.$$

Whence, if $E^{(5)}(n)$ represents the excess of the number of divisors of n which have the forms $10m+1$, $10m+4$ over the number of divisors which have the forms $10m+6$, $10m+9$; and if $E^{(55)}(n)$ represents the excess of the number of divisors of n which have the forms $10m+2$, $10m+3$ over the number of divisors which have the forms $10m+7$, $10m+8$,

$$\Sigma\left(\Sigma\frac{d\pi}{5}\right)q^n = \sin\frac{\pi}{5}\Sigma E^{(5)}(n)q^n + \sin\frac{2\pi}{5}\Sigma E^{(55)}(n)q^n.$$

The left-hand side of the identity becomes as regards the numerator

$$\sin\frac{\pi}{5}(N_1+N_4-N_6-N_9+\ldots)+\sin\frac{2\pi}{5}(N_2+N_3-N_7-N_8+\ldots),$$

where the N subscripts are of forms $5m\pm1$, $5m\pm2$, respectively.

This is

$$\sin\frac{\pi}{5}(2q-3q^6-5q^{10}+5q^{15}+7q^{21}-8q^{36}-10q^{45}+\ldots)$$

$$+\sin\frac{2\pi}{5}(-q-q^3+4q^6-6q^{21}-8q^{28}+9q^{36}+\ldots)$$

$$=a_1\sin\frac{\pi}{5}+a_2\sin\frac{2\pi}{5}\text{ suppose,}$$

and as regards the denominator

$$1+\cos\frac{\pi}{5}(D_1-D_4-D_6+D_9+\ldots)+\cos\frac{2\pi}{5}(D_2-D_3-D_7+D_8+\ldots)$$

$$-D_5+D_{10}-D_{15}+\ldots.$$

This is

$$1+q^{10}+q^{15}+q^{45}+q^{55}+q^{105}+q^{120}+\ldots$$

$$+\cos\frac{\pi}{5}(-1-q-q^6-q^{10}-q^{15}-q^{21}-q^{36}-q^{45}-\ldots)$$

$$+\cos\frac{2\pi}{5}(q+2q^3+q^6+q^{21}+2q^{28}+q^{36}+\ldots)$$

$$=b_0+b_1\cos\frac{\pi}{5}+b_2\cos\frac{2\pi}{5}\text{ suppose.}$$

So that writing the right-hand side of the identity

$$2A_1\sin\frac{\pi}{5}+2A_2\sin\frac{2\pi}{5},$$

where $A_1=\Sigma E^{(5)}(n)\,q^n, \quad A_2=\Sigma E^{(55)}(n)\,q^n,$

$$\frac{a_1\sin\frac{\pi}{5}+a_2\sin\frac{2\pi}{5}}{b_0+b_1\cos\frac{\pi}{5}+b_2\cos\frac{2\pi}{5}}=2A_1\sin\frac{\pi}{5}+2A_2\sin\frac{2\pi}{5},$$

whence

$$\{(2b_0-b_2)A_1+(b_1+b_2)A_2\}\sin\frac{\pi}{5}+\{(b_1+b_2)A_1+(2b_0+b_1)A_2\}\sin\frac{2\pi}{5}$$

$$=a_1\sin\frac{\pi}{5}+a_2\sin\frac{2\pi}{5},$$

a relation which is of the form

$$a_1\sin\frac{\pi}{5}+a_2\sin\frac{2\pi}{5}=0,$$

and which can only be satisfied when

$$a_1=a_2=0.$$

Hence

$$(17)\qquad \begin{cases}(2b_0-b_2)A_1+(b_1+b_2)A_2=a_1,\\ (b_1+b_2)A_1+(2b_0+b_1)A_2=a_2,\end{cases}$$

where

$$2b_0-b_2=2-q-2q^3-q^6+2q^{10}+2q^{15}-q^{21}-2q^{28}-q^{36}+2q^{45}+2q^{55}+\ldots,$$

$$b_1+b_2=-1+2q^3-q^{10}-q^{15}+2q^{28}-q^{45}+\ldots,$$

$$2b_0+b_1=1-q-q^6+q^{10}+q^{15}-q^{21}-q^{36}+q^{45}+\ldots.$$

In the first of these series when the q exponent is

$\frac{1}{2}5m(5m+1)$	the coefficient is	$+2$,
$\frac{1}{2}(5m+1)(5m+2)$	,,	-1,
$\frac{1}{2}(5m+2)(5m+3)$	,,	-2,
$\frac{1}{2}(5m+3)(5m+4)$	,,	-1,
$\frac{1}{2}(5m+4)(5m+5)$	,,	$+2$,

while in a_1 and a_2 the law is evident.

We are led to the simultaneous formulæ

$$(18)\quad 2E^{(5)}(n)-E^{(5)}(n-1)-2E^{(5)}(n-3)-E^{(5)}(n-6)+2E^{(5)}(n-10)+\ldots$$

$$-E^{(55)}(n)+2E^{(55)}(n-3)-E^{(55)}(n-10)-E^{(55)}(n-15)+\ldots$$

$$=\text{coefficient of } q^n \text{ in } 2q-3q^6-5q^{10}+5q^{15}+7q^{21}-\ldots,$$

$$(19)\quad -E^{(5)}(n)+2E^{(5)}(n-3)-E^{(5)}(n-10)-E^{(5)}(n-15)+\ldots$$

$$+E^{(55)}(n)-E^{(55)}(n-1)-E^{(55)}(n-6)+E^{(55)}(n-10)+E^{(55)}(n-15)-\ldots$$

$$= \text{coefficient of } q^n \text{ in } -q-q^3+4q^6-6q^{21}-8q^{28}+9q^{36}+\ldots .$$

From these formulæ we calculate in succession

$$E^{(5)}(1)=1,\quad E^{(5)}(2)=1,\quad E^{(5)}(3)=1,\quad E^{(5)}(4)=2,\quad E^{(5)}(5)=1,$$

$$E^{(5)}(6)=0,\quad \&c.;$$

$$E^{(55)}(1)=0,\quad E^{(55)}(2)=1,\quad E^{(55)}(3)=1,\quad E^{(55)}(4)=1,\quad E^{(55)}(5)=0,$$

$$E^{(55)}(6)=2,\quad \&c.,$$

as a verification.

For the moment I omit the case $x=\frac{\pi}{6}$ because generally for the case $x=\frac{\pi}{6p}$ there is an exception to the general rule, due to the fact that in the period of $12p$ values of $\sin\frac{m\pi}{6p}$ there are two which are positive rational and not zero, viz. when $m=p$ and when $m=3p$.

The case $x=\frac{\pi}{7}$ is very important, because it points clearly to a general law.

The sines and cosines in a period of fourteen are

$$\sin\frac{\pi}{7},\ \sin\frac{2\pi}{7},\ \sin\frac{3\pi}{7},\ \sin\frac{3\pi}{7},\ \sin\frac{2\pi}{7},\ \sin\frac{\pi}{7},\ 0,\ -\sin\frac{\pi}{7},\ -\sin\frac{2\pi}{7},$$

$$-\sin\frac{3\pi}{7},\ -\sin\frac{3\pi}{7},\ -\sin\frac{2\pi}{7},\ -\sin\frac{\pi}{7},\ 0;$$

$$\cos\frac{\pi}{7},\ \cos\frac{2\pi}{7},\ \cos\frac{3\pi}{7},\ -\cos\frac{3\pi}{7},\ -\cos\frac{2\pi}{7},\ -\cos\frac{\pi}{7},\ -1,\ -\cos\frac{\pi}{7},$$

$$-\cos\frac{2\pi}{7},\ -\cos\frac{3\pi}{7},\ \cos\frac{3\pi}{7},\ \cos\frac{2\pi}{7},\ \cos\frac{\pi}{7},\ 1.$$

Whence, if $E^{(7)}(n)$ represents the excess of the number of divisors of n of the forms $14m+1$, $14m+6$ over the number of divisors of the form $14m+8$, $14m+13$; and $E^{(77)}(n)$ represents the excess of the number of divisors of n of the forms $14m+2$, $14m+5$ over the number of divisors

of the forms $14m+9$, $14m+12$; and $E^{(777)}(n)$ represents the excess of the number of divisors of the forms $14m+3$, $14m+4$ over the number of divisors of the forms $14m+10$, $14m+11$,

$$\sum_1^\infty \left(\Sigma \sin \frac{d\pi}{7}\right) q^n$$

$$= \sin\frac{\pi}{7}\,\Sigma E^{(7)}(n)\,q^n + \sin\frac{2\pi}{7}\,\Sigma E^{(77)}(n)\,q^n + \sin\frac{3\pi}{7}\,\Sigma E^{(777)}(n)\,q^n$$

$$= A_1 \sin\frac{\pi}{7} + A_2 \sin\frac{2\pi}{7} + A_3 \sin\frac{3\pi}{7} \text{ suppose.}$$

The numerator of the left-hand side of the identity is

$$\sin\frac{\pi}{7}\,(N_1 - N_6 + N_8 - N_{13} + \ldots)$$

$$+\sin\frac{2\pi}{7}\,(-N_2 + N_5 - N_9 + N_{12} - \ldots)$$

$$+\sin\frac{3\pi}{7}\,(N_3 - N_4 + N_{10} - N_{11} + \ldots),$$

and the denominator

$$1 + D_7 + D_{14} + D_{21} + D_{28} + \ldots$$

$$+\cos\frac{\pi}{7}\,(-D_1 - D_6 - D_8 - D_{13} - \ldots)$$

$$+\cos\frac{2\pi}{7}\,(D_2 + D_5 + D_9 + D_{12} + \ldots)$$

$$+\cos\frac{3\pi}{7}\,(-D_3 - D_4 - D_{10} - D_{11} - \ldots),$$

where in the coefficients of

$\sin\frac{\pi}{7}$, $\cos\frac{\pi}{7}$ the subscripts are of the form $7m \pm 1$,

$\sin\frac{2\pi}{7}$, $\cos\frac{2\pi}{7}$ „ „ $7m \pm 2$,

$\sin\frac{3\pi}{7}$, $\cos\frac{3\pi}{7}$ „ „ $7m \pm 3$.

Writing the identity in the abbreviated form

$$\frac{a_1 \sin\frac{\pi}{7} + a_2 \sin\frac{2\pi}{7} + a_3 \sin\frac{3\pi}{7}}{b_0 + b_1 \cos\frac{\pi}{7} + b_2 \cos\frac{2\pi}{7} + b_3 \cos\frac{3\pi}{7}} = 2A_1 \sin\frac{\pi}{7} + 2A_2 \sin\frac{2\pi}{7} + 2A_3 \sin\frac{3\pi}{7},$$

we find without difficulty the relation

$$\{(2b_0 - b_2)A_1 + (b_1 - b_3)A_2 + (b_2 + b_3)A_3 - a_1\} \sin\frac{\pi}{7}$$

$$+ \{(b_1 - b_3)A_1 + (2b_0 + b_3)A_2 + (b_1 + b_2)A_3 - a_2\} \sin\frac{2\pi}{7}$$

$$+ \{(b_2 + b_3)A_1 + (b_1 + b_2)A_2 + (2b_0 + b_1)A_3 - a_3\} \sin\frac{3\pi}{7} = 0.$$

This can be converted into a quartic equation in $\sin\frac{\pi}{7}$; but it is known that $\sin\frac{\pi}{7}$ satisfies a sextic equation, and it follows that the relation cannot exist unless the trigonometrical functions have zero coefficients.

Hence

$$(20) \qquad \begin{cases} (2b_0 - b_1)A_1 + (b_1 - b_3)A_2 + (b_2 + b_3)A_3 = a_1, \\ (b_1 - b_3)A_1 + (2b_0 + b_3)A_2 + (b_1 + b_2)A_3 = a_2, \\ (b_2 + b_3)A_1 + (b_1 + b_2)A_2 + (2b_0 + b_1)A_3 = a_3, \end{cases}$$

three simultaneous linear equations in

$$A_1 = \sum_1^\infty E^{(7)}(n) q^n,$$

$$A_2 = \sum_1^\infty E^{(77)}(n) q^n,$$

$$A_3 = \sum_1^\infty E^{(777)}(n) q^n,$$

leading to three simultaneous recurrent formulæ in the arithmetical quantities

$$E^{(7)}(n), \quad E^{(77)}(n), \quad E^{(777)}(n).$$

We can proceed to the case

$$x = \frac{\pi}{2p+1},$$

and are led to a relation

$$\frac{a_1 \sin\frac{\pi}{2p+1} + a_2 \sin\frac{2\pi}{2p+1} + a_3 \sin\frac{3\pi}{2p+1} + \ldots + a_p \sin\frac{p\pi}{2p+1}}{b_0 + b_1 \sin\frac{\pi}{2p+1} + b_2 \sin\frac{2\pi}{2p+1} + b_3 \sin\frac{3\pi}{2p+1} + \ldots + b_p \sin\frac{p\pi}{2p+1}}$$

$$= 2A_1 \sin\frac{\pi}{2p+1} + 2A_2 \sin\frac{2\pi}{2p+1} + \ldots + 2A_p \sin\frac{p\pi}{2p+1}.$$

and thence to the relation, for p of unlimited magnitude,

$$\begin{aligned}
&\{(2b_0-b_2)A_1 + (b_1-b_3)A_2 + (b_2-b_4)A_3 + (b_3-b_5)A_4 + (b_4-b_6)A_5 + \ldots - a_1\} \sin\frac{\pi}{2p+1} \\
&\{(b_1-b_3)A_1 + (2b_0-b_4)A_2 + (b_1-b_5)A_3 + (b_2-b_6)A_4 + (b_3-b_7)A_5 + \ldots - a_2\} \sin\frac{2\pi}{2p+1} \\
&\{(b_2-b_4)A_1 + (b_1-b_5)A_2 + (2b_0-b_6)A_3 + (b_1-b_7)A_4 + (b_2-b_8)A_5 + \ldots - a_3\} \sin\frac{3\pi}{2p+1} \\
&\{(b_3-b_5)A_1 + (b_2-b_6)A_2 + (b_1-b_7)A_3 + (2b_0-b_8)A_4 + (b_1-b_9)A_5 + \ldots - a_4\} \sin\frac{4\pi}{2p+1} \\
&\{(b_4-b_6)A_1 + (b_3-b_7)A_2 + (b_2-b_8)A_3 + (b_1-b_9)A_4 + (2b_0-b_{10})A_5 + \ldots - a_5\} \sin\frac{5\pi}{2p+1} \\
&\quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots
\end{aligned}$$

0.

The symmetry of the A matrix and the law of the elements will be observed.

To write down the matrix for a given value of p we notice first that

$$\cos\frac{(p+k+1)\pi}{2p+1} = -\cos\frac{(p-k)\pi}{2p+1},$$

and thence that $$b_{p+k+1} = -b_{p-k}.$$

Making this substitution, when k is zero or any positive number not ex-

ceeding $p-1$, we obtain the A matrix appertaining to

$$x = \frac{\pi}{2p+1}.$$

Thus the A matrix for $x = \frac{\pi}{11}$ is (retaining only the b terms)

(22)
$$\begin{array}{ccccc}
(2b_0-b_2) & (b_1-b_3) & (b_2-b_4) & (b_3-b_5) & (b_4+b_5) \\
(b_1-b_3) & (2b_0-b_4) & (b_1-b_5) & (b_2+b_5) & (b_3+b_4) \\
(b_2-b_4) & (b_1-b_5) & (2b_0+b_5) & (b_1+b_4) & (b_2+b_3) \\
(b_3-b_5) & (b_2+b_5) & (b_1+b_4) & (2b_0+b_3) & (b_1+b_2) \\
(b_4+b_5) & (b_3+b_4) & (b_2+b_3) & (b_1+b_2) & (2b_0+b_1)
\end{array}$$

The exact form of the relation for any value of p, which may be written

$$a_1 \sin\frac{\pi}{2p+1} + a_2 \sin\frac{2\pi}{2p+1} + \ldots + a_p \sin\frac{p\pi}{2p+1} = 0,$$

is therefore manifest.

This relation when expressed in powers of $\sin\frac{\pi}{2p+1}$ is of degree $2p-2$, and it is known that $\sin\frac{\pi}{2p+1}$ satisfies an equation of degree $2p$.

Hence the relation is impossible unless

$$a_1 = a_2 = \ldots = a_p = 0.$$

Hence we have the p relations obtained by giving m the values 1, 2, 3, ..., p in the relation

$$\begin{aligned}
&(b_{m-1}-b_{m+1})A_1 + (b_{m-2}-b_{m+2})A_2 + \ldots + (b_1-b_{2m-1})A_{m-1} \\
&+(2b_0-b_{2m})A_m \\
&+(b_1-b_{2m+1})A_{m+1} + (b_2-b_{2m+2})A_{m+2} + \ldots + (b_{p-m}-b_{p+m})A_p = a_m,
\end{aligned}$$

in which the first row of terms is to contain $m-1$ terms, and we are to write $-b_{p-k}$ for b_{p+k+1} throughout.

The arithmetical q series

$$A_1, A_2, \ldots, A_p,$$

which appertain to this case, may be defined. If we write down the $4p+2$ recurring values of $\sin \frac{m\pi}{2p+1}$, obtained by giving m the values 1, 2, 3, ..., the particular value $\sin \frac{k\pi}{2p+1}$, where $k \leqslant p$, occurs in the k-th and $(2p-k+1)$-th places and $-\sin \frac{k\pi}{2p+1}$ in the $(2p+k+1)$-th and $(4p-k+2)$-th places.

Hence $$A_k = \Sigma E^{(2p+1)^k}(n) q^n$$

is interpreted by the statement that $E^{(2p+1)^k}(n)$ represents the excess of the number of divisors of n which have the forms

$$(4p+2)m+k, \quad (4p+2)m+2p-k+1$$

over the number of divisors which have the forms

$$(4p+2)m+2p+k+1, \quad (4p+2)m+4p-k+2.$$

The matrix has row and column symmetry. Looking to the case of $x = \frac{\pi}{11}$ it will be observed that, if every element of the matrix which involves a b with a subscript > 3 be deleted, the remaining elements consolidated in a rectilinear manner constitute the matrix for the case $x = \frac{\pi}{7}$. The deleted elements occur in four sinister diagonal lines. By deletion of the elements which involve a b with a subscript > 4 we would similarly obtain the matrix for the case $x = \frac{\pi}{9}$. Generally we can in this way proceed from the matrix for the case $x = \frac{\pi}{2p+1}$ to the matrices corresponding to the division of π into an uneven number of parts and conversely proceed to higher matrices by a law that is readily ascertained.

Before taking the general case $x = \frac{\pi}{2p}$, p not a multiple of 3, we will consider the case $x = \frac{\pi}{8}$.

We are led to the relation

$$\sin\frac{\pi}{8}(N_1+N_7-N_9-N_{15}+\ldots)+\sin\frac{2\pi}{8}(N_2+N_6-N_{10}-N_{14}+\ldots)$$
$$+\sin\frac{3\pi}{8}(N_3+N_5-N_{11}-N_{13}+\ldots)+N_4-N_{12}+N_{20}-N_{28}+..$$

$$\overline{1+\cos\frac{\pi}{8}(D_1-D_7-D_9+D_{15}+\ldots)+\cos\frac{2\pi}{8}(D_2-D_6-D_{10}+D_{14}+\ldots)}$$
$$+\cos\frac{3\pi}{8}(D_3-D_5-D_{11}+D_{13}+\ldots)$$

$$=2A_1\sin\frac{\pi}{8}+2A_2\sin\frac{2\pi}{8}+2A_3\sin\frac{3\pi}{8}+2A_4,$$

and thence writing the left-hand side

$$\frac{a_1\sin\frac{\pi}{8}+a_2\sin\frac{2\pi}{8}+a_3\sin\frac{3\pi}{8}+a_4}{1+b_1\cos\frac{\pi}{8}+b_2\cos\frac{2\pi}{8}+b_3\cos\frac{3\pi}{8}}$$

to
$$\{(2-b_2)A_1+(b_1-b_3)A_2+\quad b_2A_3+2b_3A_4\}\sin\frac{\pi}{8}$$
$$+\{(b_1-b_3)A_1+\quad 2A_2+(b_1+b_3)A_3+2b_2A_4\}\sin\frac{2\pi}{8}$$
$$+\{\quad b_2A_1+(b_1+b_3)A_2+(2+b_2)A_3+2b_1A_4\}\sin\frac{3\pi}{8}$$
$$+\quad b_3A_1+\quad b_2A_2+\quad b_1A_3+\quad 2A_4$$
$$=a_1\sin\frac{\pi}{8}+a_2\sin\frac{2\pi}{8}+a_3\sin\frac{3\pi}{8}+a_4,$$

which is of the form

$$a_1\sin\frac{\pi}{8}+a_2\sin\frac{2\pi}{8}+a_3\sin\frac{3\pi}{8}+a_4=0.$$

This may be arranged as a sextic equation in $\sin\frac{\pi}{8}$, and it is seen to be incompatible with the equation that $\sin\frac{\pi}{8}$ is known to satisfy, viz.

$$16y^6-24y^4+10y^2-1=0,$$

unless
$$a_1=a_2=a_3=a_4=0.$$

Hence we obtain the relations

$$(23)\quad \begin{cases} (2-b_2)A_1+(b_1-b_3)A_2+\quad\ldots\quad b_2A_3+2b_3A_4 = a_1, \\ (b_1-b_3)A_1+\quad 2A_2+(b_1+b_3)A_3+2b_2A_4 = a_2, \\ b_2A_1+(b_1+b_3)A_2+(2+b_2)A_3+2b_1A_4 = a_3, \\ b_3A_1+\quad b_2A_2+\quad b_1A_3+\quad 2A_4 = a_4. \end{cases}$$

If now p be unrestricted in magnitude I find the relation

$$\{(2-b_2)A_1+(b_1-b_3)A_2+(b_2-b_1)A_3+(b_3-b_5)A_4+(b_4-b_6)A_5+\ldots\}\sin\frac{\pi}{2p}$$

$$+\{(b_1-b_3)A_1+(2-b_4)A_2+(b_1-b_5)A_3+(b_2-b_6)A_4+(b_3-b_7)A_5+\ldots\}\sin\frac{2\pi}{2p}$$

$$+\{(b_2-b_4)A_1+(b_1-b_5)A_2+(2-b_6)A_3+(b_1-b_7)A_4+(b_2-b_8)A_5+\ldots\}\sin\frac{3\pi}{2p}$$

$$+\{(b_3-b_5)A_1+(b_2-b_6)A_2+(b_1-b_7)A_3+(2-b_8)A_4+(b_1-b_9)A_5+\ldots\}\sin\frac{4\pi}{2p}$$

$$+\{(b_4-b_6)A_1+(b_3-b_7)A_2+(b_2-b_8)A_3+(b_1-b_9)A_4+(2-b_{10})A_5+\ldots\}\sin\frac{5\pi}{2p}$$

$$+\ldots$$

$$=a_1\sin\frac{\pi}{2p}+a_2\sin\frac{2\pi}{2p}+a_3\sin\frac{3\pi}{2p}+a_4\sin\frac{4\pi}{2p}+a_5\sin\frac{5\pi}{2p}+\ldots,$$

in which the law of formation of the matrix is manifest.

For a definite value of p we have herein to put

$$b_p=0,\qquad b_s=-b_{2p-s},$$

but in the last row $b_s=0$ if $s>p$.

We thus verify the cases $p=2$, $p=4$, while for $p=10$ we find

$$\{(2-b_2)A_1+(b_1-b_3)A_2+(b_2-b_4)A_3+\quad b_3A_4+2b_4A_5\}\sin\frac{\pi}{10}$$

$$\{(b_1-b_3)A_1+(2-b_4)A_2+\quad b_1A_3+(b_2+b_4)A_4+2b_3A_5\}\sin\frac{2\pi}{10}$$

$$\{(b_2-b_4)A_1+\quad b_1A_2+(2+b_4)A_3+(b_1+b_3)A_4+2b_2A_5\}\sin\frac{3\pi}{10}$$

$$\{\quad b_3A_1+(b_2+b_4)A_2+(b_1+b_3)A_3+(2+b_2)A_4+2b_1A_5\}\sin\frac{4\pi}{10}$$

$$\{\quad b_4A_1+\quad b_3A_2+\quad b_2A_3+\quad b_1A_4+\quad 2A_5\}$$

$$=a_1\sin\frac{\pi}{10}+a_2\sin\frac{2\pi}{10}+a_3\sin\frac{3\pi}{10}+a_4\sin\frac{4\pi}{10}+a_5.$$

We gather that generally we have a relation

$$a_1 \sin \frac{\pi}{2p} + a_2 \sin \frac{2\pi}{2p} + \ldots + a_{p-1} \sin \frac{(p-1)\pi}{2p} + a_p = 0,$$

and it is not difficult to prove that this necessitates the relations

$$a_1 = a_2 = a_3 = \ldots = a_{p-1} = a_p = 0,$$

giving the solutions of the problem of expressing the arithmetical functions $A_1, A_2, \ldots, A_n$ in terms of the q series

$$a_1, a_2, a_3, \ldots, a_p, \qquad b_1, b_2, b_3, \ldots, b_{p-1},$$

$$A_k = \Sigma E^{(2p)^k}(n) q^m,$$

where $E^{(2p)^k}(n)$ represents the excess of the number of divisors of n which have the forms $4mp+k$, $4mp+2p-k$ over the number of divisors which have the forms $4mp+2p+k$, $4mp+4p-k$.

For the remaining cases in which $x = \frac{\pi}{n}$, where n is a multiple of 6, it will suffice to show the nature of the results by considering in detail the case $x = \frac{\pi}{6}$.

From the sine and cosine values

$$\frac{1}{2}, \frac{\sqrt{3}}{2}, 1, \frac{\sqrt{3}}{2}, \frac{1}{2}, 0, -\frac{1}{2}, -\frac{\sqrt{3}}{2}, -1, -\frac{\sqrt{3}}{2}, -\frac{1}{2}, 0,$$

$$\frac{\sqrt{3}}{2}, \frac{1}{2}, 0, -\frac{1}{2}, -\frac{\sqrt{3}}{2}, -1, -\frac{\sqrt{3}}{2}, -\frac{1}{2}, 0, \frac{1}{2}, \frac{\sqrt{3}}{2}, 1,$$

we derive a relation of the form

$$\frac{a+b\sqrt{3}}{c+d\sqrt{3}} = A + B\sqrt{3},$$

where

$$a = N_1 + N_5 - N_7 - N_{11} + \ldots + 2(N_3 - N_9 + \ldots),$$

$$b = N_2 + N_4 - N_8 - N_{10} + \ldots,$$

$$c = D_2 - D_4 - D_8 + D_{10} + \ldots + 2(1 - D_6 + D_{12} - \ldots),$$

$$d = D_1 - D_5 - D_7 + D_{11} + \ldots,$$

$$A = \Sigma \left(d_{\substack{12m+1+5 \\ -7-11}}(n) + 2 d_{\substack{12m+3 \\ -9}}(n) \right) q^n,$$

$$B = \Sigma \left(d_{\substack{12m+2+4 \\ -8-10}}(n) \right) q^n,$$

where $d_{\substack{12m+\alpha \\ -\gamma}}(n)$ denotes the excess of the sum of the divisors of n which are of form $12m+\alpha$ over the sum of those of form $12m+\gamma$, and $d_{\substack{12m+\alpha+\beta \\ -\gamma-\delta}}(n)$ denotes the excess of the sum of the divisors of n which are of forms $12m+\alpha$, $12m+\beta$ over the sum of those which have the forms $12m+\gamma$, $12m+\delta$.

We thence derive

$$A = \frac{ac-3bd}{c^2-3d^2}, \qquad B = \frac{bc-ad}{c^2-3d^2},$$

also $$Ac+3Bd = a, \qquad Bc+Ad = b$$

relations which, upon development, yield two simultaneous recurring formulæ.

2. I consider a theory similar to that of § 1, which is concerned only with those divisors of n which possess uneven conjugates. We will use the symbol δ to denote such a divisor.

We have

$$\sum_1^\infty \frac{aq^{2m-1}}{1-aq^{2m-1}} = \sum_1^\infty \frac{a^m q^m}{1-q^{2m}} = \sum_1^\infty (\Sigma a^\delta)\, q^n$$

and

$$\int \frac{e^{ix}q^{2m-1}}{1-e^{ix}q^{2m-1}}\,dx = \tan^{-1}\frac{q^{2m-1}\sin x}{1-q^{2m-1}\cos x} + \tfrac{1}{2}i\log(1-2q^{2m-1}\cos x+q^{4m-2}),$$

but $$\frac{e^{ix}q^{2m-1}}{1-e^{ix}q^{2m-1}} = \frac{q^{2m-1}\cos x-q^{4m-2}}{1-2q^{2m-1}\cos x+q^{4m-2}} + i\,\frac{q^{2m-1}\sin x}{1-2q^{2m-1}\cos x+q^{4m-2}},$$

so that $$\int \frac{q^{2m-1}\cos x-q^{4m-2}}{1-2q^{2m-1}\cos x+q^{4m-2}}\,dx = \tan^{-1}\frac{q^{2m-1}\sin x}{1-q^{2m-1}\cos x},$$

$$\int \frac{q^{2m-1}\sin x}{1-q^{2m-1}\cos x+q^{4m-2}}\,dx = \tfrac{1}{2}\log(1-2q^{2m-1}\cos x+q^{4m-2}),$$

and $$\sum_1^\infty \tfrac{1}{2}\log(1-2q^{2m-1}\cos x+q^{4m-2}) = \sum_1^\infty \left(\Sigma-\frac{1}{\delta}\cos\delta x\right) q^n$$

or $$\log\frac{1}{\prod_1^\infty (1-2q^{2m-1}\cos x+q^{4m-2})} = 2\sum_1^\infty \left(\Sigma\,\frac{1}{\delta}\cos\delta x\right) q^n.$$

Transforming by a well known formula of the *Fundamenta Nova*,

$$\log \frac{\prod_1^\infty (1-q^{2m})}{1-2q\cos x+2q^4\cos 2x-2q^9\cos 3x+\ldots} = 2\sum_1^\infty \left(\Sigma \frac{1}{\delta}\cos \delta x\right) q^n.$$

Differentiation with regard to x gives

$$(24)\quad \frac{q\sin x-2q^4\sin 2x+3q^9\cos 3x-4q^{16}\cos 4x+\ldots}{1-2q\cos x+2q^4\cos 2x-2q^9\cos 3x+2q^{16}\cos 4x-\ldots} = \sum_1^\infty (\Sigma \sin \delta x) q^n,$$

the fundamental formula which presents itself for examination.

The case $x = \frac{\pi}{2}$.

The formula becomes

$$\frac{q-3q^9+5q^{25}-7q^{49}+\ldots}{1-2q^4+2q^{16}-2q^{36}+\ldots} = \Sigma \varepsilon^{(2)}(n) q^n,$$

where $\varepsilon^{(2)}(n)$ represents the excess of the number of divisors of n which have the form $4m+1$ over the number of divisors which have the form $4m+3$.

Thence the formula

$$\varepsilon^{(2)}(n)-2\varepsilon^{(2)}(n-4)+2\varepsilon^{(2)}(n-16)-2\varepsilon(n-36)+\ldots = 0$$

or $(-)^m(2m+1)$ according as n is not or is of the form $(2m+1)^2$.

If we compare the two formulæ

(25)

$$\begin{cases} \dfrac{2q\sin x-(q+3q^3)\sin 2x+(2q^3+4q^6)\sin 3x-\ldots}{1-(1+q)\cos x+(q+q^3)\cos 2x-(q^3+q^6)\cos 3x+\ldots} = 2\sum_1^\infty (\Sigma \sin d x)\, q^n, \\ \dfrac{q\sin x-2q^4\sin 2x+3q^9\sin 3x-\ldots}{1-2q\cos x+2q^4\cos 2x-2q^9\cos 3x+\ldots} = \sum_1^\infty (\Sigma \sin \delta x) q^n, \end{cases}$$

it will appear that the theories of the divisors d and δ from the point of view of this paper are absolutely parallel.

The trigonometrical functions are similarly placed and (bringing the factor 2 on the right-hand side of the first formula over to the left) they only differ in the coefficients of the sines and cosines. We have merely in the formulæ of the previous section to take for

$$N_1,\ N_2,\ N_3,\ \ldots,$$
$$D_1,\ D_2,\ D_3,\ \ldots,$$

the values

$$2q,\ -4q^4,\ 6q^9,\ \ldots,$$

$$-2q,\ +2q^4,\ -2q^9,\ \ldots,$$

and all the formulæ will be suitably transformed to the divisor δ. We can therefore at once proceed to some further developments affecting the divisor δ.

The δ fundamental formula is, in the notation of the *Fundamenta Nova*,

$$\frac{K}{2\pi} Z\left(\frac{Kx}{\pi}\right) = \Sigma(\Sigma \sin \delta x)\, q^n,$$

and squaring each side we find, by p. 136 of that work,

(26)
$$\left(\frac{K}{2\pi}\right)^2 \left\{ Z\left(\frac{Kx}{\pi}\right) \right\}^2$$

$$= \tfrac{1}{2}\Sigma \frac{q^{2m}}{(1-q^{2m})^2} - \tfrac{1}{2}\Sigma \frac{mq^m \cos mx}{1-q^{2m}} + \tfrac{1}{2}\Sigma \frac{q^m(1+q^{2m}) \cos mx}{(1-q^{2m})^2}$$

$$= \{\Sigma(\Sigma \sin \delta x)\, q^n\}^2,$$

a remarkable theorem concerning the divisor δ.

I notice the relations

$$\Sigma \frac{q^{2m}}{(1-q^{2m})^2} = \Sigma \frac{mq^{2m}}{1-q^{2m}},$$

$$\Sigma \frac{mq^m \cos mx}{1-q^{2m}} = \Sigma \frac{(q^{2m-1}+q^{6m-3}) \cos x - 2q^{4m-2}}{(1-2q^{2m-1} \cos x + q^{4m-2})^2}.$$

Differentiating the fundamental relation we get

$$-\tfrac{1}{2} \frac{Q_0 - 2Q_1 \cos x + 2Q_2 \cos 2x - 2Q_3 \cos 3x + \ldots}{(1-2q \cos x + 2q^4 \cos 2x - 2q^9 \cos 3x + \ldots)^2} = \Sigma(\Sigma \delta \cos \delta x) q^n,$$

where

$$Q_0 = 2^2 q^{1^2+1^2} + 4^2 q^{2^2+2^2} + 6^2 q^{3^2+3^2} + \ldots,$$

$$Q_1 = 1^2 q + 3^2 q^{1^2+2^2} + 5^2 q^{2^2+3^2} + 7^2 q^{3^2+4^2} + \ldots,$$

$$Q_2 = 2^2 q^{2^2} + 4^2 q^{1^2+3^2} + 6^2 q^{2^2+4^2} + 8^2 q^{3^2+5^2} + \ldots,$$

$$Q_3 = 3^2 q^{3^2} + 5^2 q^{1^2+4^2} + 7^2 q^{2^2+5^2} + 9^2 q^{3^2+6^2} + \ldots,$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

It will be noticed that in the series Q, the q exponents involve every par-

tition of all numbers into one or two squares,

$$Q_m = \sum_{s=0}^{\infty} (2s+m)^2 q^{s^2+(s+m)^2} \text{ if } m > 0.$$

Moreover

$$(1-2q\cos x+2q^4\cos 2x-2q^9\cos 3x+\ldots)^2$$

$$= R_0-2R_1\cos x+2R_2\cos 2x-2R_3\cos 3x+2R_4\cos 4x-\ldots,$$

where

$$R_0 = 1+2\sum_{1}^{\infty} q^{s^2+s^2},$$

$$R_1 = \quad 2\sum_{0}^{\infty} q^{s^2+(s+1)^2},$$

$$R_2 = q^{1^2+1^2}+2\sum_{1}^{\infty} q^{s^2+(s+2)^2},$$

$$R_3 = \quad 2\sum_{-1}^{\infty} q^{s^2+(s+3)^2},$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots$$

$$R_{2m} = q^{m^2+m^2}+2\sum_{-(m-1)}^{\infty} q^{s^2+(s+m)^2},$$

$$R_{2m+1} = \quad 2\sum_{-1}^{\infty} q^{s^2+(s+2m+1)^2}.$$

We have therefore

$$-\tfrac{1}{2}\frac{Q_0-2Q_1\cos x+2Q_2\cos 2x-\ldots}{R_0-2R_1\cos x+2R_2\cos 2x-\ldots} = \sum_{1}^{\infty}(\Sigma\delta\cos\delta x)q^n,$$

R_{2m} enumerates the compositions of numbers into two squares, the numbers squared differing in magnitude by $2m$ and being drawn from the series

$$-(m-1),\ -(m-2),\ \ldots,\ +\infty,$$

where it must be noted that

$$a^2+b^2, \qquad b^2+a^2$$

are different compositions, except when $a = \pm b$.

Similarly R_{2m+1} enumerates the compositions of numbers into two squares, the numbers squared differing in magnitude by $2m+1$ and being drawn from the series

$$-m,\ -(m-1),\ \ldots,\ +\infty.$$

Putting $x = \frac{1}{2}\pi$,

$$-\tfrac{1}{2}\frac{Q_0-2Q_2+2Q_8-2Q_6+\cdots}{R_0-2R_2+2R_4-2R_6+\ldots} = \sum_1^\infty (\Sigma\,\delta\cos\tfrac{1}{2}\delta\pi)q^n,$$

and the right-hand side denotes the excess of the sum of the divisors δ of n which have the form $4m+4$ over the sum of the divisors which have the form $4m+2$. Calling this $F_1(n)$, we find that

$$(27)\quad F_1(n)-4F_1(n-4)+4F_1(n-8)+4F_1(n-16)-8F_1(n-20)$$
$$+4F_1(n-32)-4F_1(n-36)+8F_1(n-40)+\ldots$$
$$= \text{coefficients of } q^n \text{ in } -\tfrac{1}{2}(Q_0-2Q_2+2Q_4-2Q_6+\ldots).$$

The q function $\qquad R_0-2R_2+2R_4-2R_6+\ldots$

is equal to $\dfrac{2k'K}{\pi}$ when q^4 is written for q. Its elliptic function expression is therefore

$$\frac{1}{\pi}\,\frac{\sqrt{2}}{\pi}\sqrt{[\sqrt{k'}\,(1+k')]}\,K,$$

while Jacobi gives for it the expression

$$1-4\Sigma\psi(n)q^{(4m-1)^2 4n}+4\Sigma\psi(n)q^{2^{l+1}(4m-1)^2 4n},$$

in which n is an uneven number whose prime factors are all of the form $\equiv 1 \bmod 4$; $\psi(n)$ is the number of divisors of n, and l, m assume all integer values from 0 to $+\infty$.*

It is also expressed in the forms

$$1-4\frac{q^4}{1+q^4}+4\frac{q^{12}}{1+q^{12}}-4\frac{q^{20}}{1+q^{20}}+\ldots,$$

$$1-4\frac{q^4}{1+q^8}+4\frac{q^8}{1+q^{16}}-4\frac{q^{12}}{1+q^{24}}+\ldots,$$

$$\left(\frac{(1-q^4)(1-q^8)(1-q^{12})\ldots}{(1+q^4)(1+q^8)(1+q^{12})\ldots}\right)^2.$$

* *Fundamenta Nova*, p. 105.

3. We now take up the relation

$$\left(a\frac{d}{da}\right)^s \sum_1^\infty \frac{aq^m}{1-aq^m} = \sum_1^\infty (\Sigma d^s a^d) q^n = \sum_1^\infty \frac{m^s a^m q^m}{1-q^m}.$$

The left-hand side is

$$\sum_1^\infty \frac{\sum_1^s c_{s,t}(aq^m)^t}{(1-aq^m)^{s+1}},$$

where $\sum_1^\infty c_{s,t}(aq^m)^t$, for the successive values of s, is

$$aq^m,$$
$$aq^m+a^2q^{2m},$$
$$aq^m+4a^2q^{2m}+a^3q^{3m},$$
$$aq^m+11a^2q^{2m}+11a^3q^{3m}+a^4q^{4m},$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

where the numerical coefficients, in tabular form, are

1					
1	1				
1	4	1			
1	11	11	1		
1	26	66	26	1	
1	57	302	302	57	1
…	…	…	…	…	

the number in the s-th row and t-th column being $c_{s,t}$.

We have $$c_{s,t} = tc_{s-1,t}+(s-t+1)c_{s-1,t-1}$$

and $$c_{s,t} = c_{s,s-t+1},$$

$$\sum_1^s c_{s,t} = s!.$$

The coefficients enjoy the further property

$$n^s = c_{s,1}\binom{n}{s}+c_{s,2}\binom{n+1}{s}+c_{s,3}\binom{n+2}{s}+\ldots+c_{s,s}\binom{n+s-1}{s},$$

and we readily obtain the evaluations

$$c_{s,1} = 1^s,$$

$$c_{s,2} = 2^s - \binom{s+1}{1} 1^s,$$

$$c_{s,3} = 3^s - \binom{s+1}{1} 2^s + \binom{s+1}{2} 1^s,$$

$$\dots \quad \dots \quad \dots \quad \dots \quad \dots$$

$$c_{s,t} = t^s - \binom{s+1}{1}(t-1)^s + \binom{s+1}{2}(t-2)^s - \dots (-)^{t+1}\binom{s+1}{t-1} 1^s.$$

Putting $a = 1$, gives

$$\sum_1^\infty \frac{\sum_1^{\cdot} c_{st} q^{mt}}{(1-q^m)^{s+1}} = \sum_1^\infty \sigma_s(n)\, q^n = \sum_1^\infty \frac{m^s q^m}{1-q^m}, \tag{28}$$

and $a = -1$ yields the analogous result connected with the excess of the sum of the s-th powers of the even divisors over the sum of the s-th powers of the uneven divisors.

Putting $a = \sqrt{(-1)} = i$,

$$\Sigma(\Sigma d^s i^d)\, q^n = i\Sigma E_s(n) q^n + \Sigma F_s(n) q^n,$$

where $E_s(n)$ is the excess of the sum of the s-th powers of the $4m+1$ divisors over the sum of the s-th powers of the $4m+3$ divisors, and $F_s(n)$ is the excess of the sum of the s-th powers of the $4m+4$ divisors over the sum of the s-th powers of the $4m+2$ divisors.

If we take $s = 2$,

$$\sum_1^\infty E_2(n) q^n = \frac{q}{1-q} - 3^2 \frac{q^3}{1-q^3} + 5^2 \frac{q^5}{1-q^5} - \dots = \tfrac{1}{4} - \tfrac{1}{4} k'^2 \left(\frac{2K}{\pi}\right)^3,$$

and for $s = 3$,

$$\sum_1^\infty F_3(n) q^n = -2^3 \frac{q^2}{1-q^2} + 4^3 \frac{q^4}{1-q^4} - 6^3 \frac{q^6}{1-q^6} + \dots = -\tfrac{1}{2} + \tfrac{1}{2} k'^2 \left(\frac{2K}{\pi}\right)^4,$$

and we derive

$$\frac{1 + 2\sum_1^\infty F_3(n)\, q^n}{1 - 4\sum_1^\infty E_2(n) q^n} = \frac{2K}{\pi} = 1 + 4q + 4q^2 + 4q^4 + 8q^5 + \dots,$$

a well known series.

Assuming conventionally

$$E_2(0) = -\tfrac{1}{4}, \qquad F_3(0) = \tfrac{1}{2},$$

$$\frac{\sum_0^\infty F_3(n)q^n}{-2\sum_0^\infty E_2(n)q^n} = 1+4q+4q^2+4q^4+8q^5+\ldots,$$

leading to the formula

$$(29)\quad 2E_2(n)+8E_2(n-1)+8E_2(n-2)+8E_2(n-4)+16E_2(n-5)+\ldots = -F_3(n),$$

verified for $n = 4$, because

$$E_2(4) = 1, \quad E_2(3) = 1^2-3^2 = -8, \quad E_2(2) = 1, \quad E_2(0) = -\tfrac{1}{4},$$

$$F_3(4) = 4^3-2^3 = 56,$$

$$2.1+8(-8)+8.1+8(-\tfrac{1}{4}) = -56.$$

As another illustration we derive from well known series for

$$\frac{2kK}{\pi} \text{ and } \frac{4kK^2}{\pi^2},$$

$$(30)\quad F_3(n)+2F_3(n-2)+F_3(n-4)+2F_3(n-6)+2F_3(n-8)+\ldots$$
$$= -2E_2(n)-8E_2(n-1)-12E_2(n-2)-16E_2(n-3)-26E_2(n-4)$$
$$-24E_2(n-5)-28E_2(n-6)-\ldots.$$

The general formulæ which arise from the substitution of i for a are

$$(31)\quad \begin{cases} \sum_1^\infty \dfrac{\sum_1^{s+1}(-)^{t+1}b_{s+1,t}q^{(2t-1)m}}{(1+q^{2m})^{s+1}} = \sum_1^\infty E_s(n)q^n = \sum_1^\infty (-)^{m+1}\dfrac{(2m-1)^s q^{2m-1}}{1-q^{2m-1}}, \\ 2^s\sum_1^\infty \dfrac{\sum_1^{s}(-)^{t+1}c_{s,t}q^{2tm}}{(1+q^{2m})^{s+1}} = \Sigma F_s(n)q^n = 2^s\Sigma(-)^{m+1}\dfrac{m^s q^{2m}}{1-q^{2m}}, \end{cases}$$

where $c_{s,t}$ has the values given above.

As regards $b_{s,t}$ the table of numbers is

1					
1	1				
1	6	1			
1	23	23	1		
1	76	230	76	1	
1	237	1682	1682	237	1
...	...	...	...	...	

the number in the s-th row and t-th column being $b_{s,t}$.

We have $$b_{s,t} = (2t-1)\, b_{s-1,t} + (2s-2t+1)\, b_{s-1,t-1}$$

and $$b_{s,t} = b_{s,\, s-t+1},$$

$$\sum_{1}^{s} b_{s,t} = 2^{s-1}(s-1)!\,.$$

The coefficients enjoy the further property

$$(2n+1)^{s-1} = b_{s,1}\binom{n}{s-1} + b_{s,2}\binom{n+1}{s-1} + b_{s,3}\binom{n+2}{s-1} + \ldots + b_{s,s}\binom{n+s-1}{s-1},$$

and we readily obtain the evaluations

$$b_{s,1} = 1^{s-1},$$

$$b_{s,2} = 3^{s-1} - \binom{s}{1} 1^{s-1},$$

$$b_{s,3} = 5^{s-1} - \binom{s}{1} 3^{s-1} + \binom{s}{2} 1^{s-1},$$

... ...

$$b_{s,t} = (2t-1)^{s-1} - \binom{s}{1}(2t-3)^{s-1} + \binom{s}{2}(2t-5)^{s-1} - \ldots + (-)^{t+1}\binom{s}{t-1} 1^{s-1}.$$

4. The relation

$$\Sigma \frac{m^s a^m q^{m'}}{1-q^{m'}} = \Sigma\,(\Sigma d^s a^d)\, q^n$$

enables us to study the divisors which are restricted in magnitude by upper and lower limits.

For put $a = q^u$, so that

$$\Sigma \frac{m^s q^{m(u+1)}}{1-q^m} = \Sigma (\Sigma d^s)\, q^{u+u^2}.$$

The general term, on the left, under the sign of summation is

$$m^s q^{m(u+1+k)},$$

where m is a divisor of $m(u+1+k)$ which is less than

$$m \frac{u+1+k}{u}$$

for all integer values of u and k.

Hence $$\Sigma \frac{m^s q^{m(u+1)}}{1-q^m} = \Sigma (\Sigma d_u^s)\, q^u,$$

where d_u is a divisor of n which is less than n/u.

Since also $$\Sigma \frac{m^s q^{m(u+v+1)}}{1-q^m} = \Sigma (\Sigma d_{u+v}^s)\, q^n,$$

where d_{u+v} is a divisor of n which is less than $n/(u+v)$, we obtain by subtraction

$$\sum_1^\infty m^s \frac{q^{m(u+1)}(1-q^{mv})}{1-q^m} = \Sigma (\Sigma d_{u,\,u+v}^s)\, q^n, \tag{32}$$

where $d_{u,\,u+v}$ is a divisor of n such that

$$\frac{n}{u} > d_{u,\,u+v} \geqslant \frac{n}{u+v}.$$

In this formula, u, v may be any real positive numerical quantities whatever.

The case $s = 0$ is of particular interest because then the relation becomes

$$\Sigma \frac{q^{m(u+1)}(1-q^{mv})}{1-q^m} = \Sigma \nu_{u,\,u+v}(n)\, q^n, \tag{33}$$

where $\nu_{u,\,u+v}$ denotes the *number* of the divisors $d_{u,\,u+v}$ of n such that

$$\frac{n}{u} > d_{u,\,u+v} \geqslant \frac{n}{u+v}.$$

I proceed to its examination.

Put $v = 1$, and we find that

$$\frac{q^{u+1}}{1-q^{u+1}} = \Sigma \nu_{v,\, u+1}(n)\, q^n,$$

showing, what is otherwise obvious, that $\nu_{u,\, u+1}$ is zero or unity according as n is not or is a multiple of $u+1$.

Put $v = 2$ leading to

$$\frac{\frac{1-q^2}{1-q}\, q^{u+1} - 2q^{2u+3}}{(1-q^{u+1})(1-q^{u+2})} = \Sigma \nu_{u,\, u+2}(n) q^n,$$

and for $v = 3$,

$$\frac{\frac{1-q^3}{1-q}\, q^{u+1} - 2\,\frac{1-q^3}{1-q}\, q^{2u+3} + 3q^{3u+6}}{(1-q^{u+1})(1-q^{u+2})(1-q^{u+3})} = \Sigma \nu_{u,\, u+3}(n)\, q^n,$$

and, finally, the general formula

(34)

$$\frac{1-q^v}{1-q}\, q^{u+1} - 2\,\frac{(1-q^v)(1-q^{v-1})}{(1-q)(1-q^2)}\, q^{2u+3} + 3\,\frac{(1-q^v)(1-q^{v-1})(1-q^{v-2})}{(1-q)(1-q^2)(1-q^3)}\, q^{3u+6} - \ldots$$
$$+ (-)^{v+1}\, v\, q^{vu+\binom{v+1}{2}}$$
$$\overline{(1-q^{u+1})(1-q^{u+2})(1-q^{u+3}) \ldots (1-q^{u+v})}$$

$$= \Sigma \nu_{u,\, u+v}(n) q^n = \sum_1^v \frac{q^{u+m}}{1-q^{u+m}},$$

which is a valuable result.

(i) Put $u = 0$, obtaining

$$\frac{1-q^v}{1-q}\, q - 2\,\frac{(1-q^v)(1-q^{v-1})}{(1-q)(1-q^2)}\, q^3 + 3\,\frac{(1-q^v)(1-q^{v-1})(1-q^{v-2})}{(1-q)(1-q^2)(1-q^3)}\, q^6 + \ldots$$
$$+ (-)^{v+1}\, v\, q^{\binom{v+1}{2}}$$
$$\overline{(1-q)(1-q^2)(1-q^3) \ldots (1-q^v)}$$
$$= \Sigma \nu_{0,\, v}(n)\, q^n,$$

where the divisor $d_{0,\, v}$ is such that

$$\infty > d_{0,\, v} \geqslant \frac{n}{v},$$

i.e. the divisors enumerated are not less than n/v, a single condition.

(ii) In the last formula put $v = \infty$, obtaining

$$\frac{q}{1-q} - 2\frac{q^3}{(1-q)(1-q^2)} + 3\frac{q^6}{(1-q)(1-q^2)(1-q^3)} - 4\frac{q^{10}}{(1-q)(1-q^2)(1-q^3)(1-q^4)} + \dots$$

$$(35) \qquad \frac{}{(1-q)(1-q^2)(1-q^3)(1-q^4)\dots} = \sum_1^\infty \nu(n)\, q^n,$$

where $\nu(n)$ is the number of divisors of n without any restrictions because

$$\infty > d_{0,\infty} \geqslant 0.$$

It will be recalled that $\Sigma \nu(n) q^n$

has two other well known expressions, viz.:

Lambert's $$\frac{q}{1-q} + \frac{q^2}{1-q^2} + \frac{q^3}{1-q^3} + \dots,$$

and Clausen's $$q\frac{1+q}{1-q} + q^4\frac{1+q^2}{1-q^2} + q^9\frac{1+q^3}{1-q^3} + q^{16}\frac{1+q^4}{1-q^4} + \dots.$$

That obtained above, which connects the arithmetical entity $\nu(n)$ with the theory of partitions, is possibly known, though I do not remember meeting with it in other writings.

In the numerator of the left-hand side the general term is

$$(-)^{k+1}\frac{kq^{\binom{k+1}{2}}}{(1-q)(1-q^2)\dots(1-q^k)},$$

where $$\binom{k+1}{2} = 1+2+3+\dots+k.$$

If we denote this number by a lattice of nodes containing in the successive rows k, $k-1$, ..., 2, 1 nodes successively

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and consider the partitions enumerated by

$$\frac{1}{(1-q)(1-q^2)\ldots(1-q^k)},$$

we find that each of the latter involves k or fewer parts, of unrestricted magnitude, which may be graphically represented by k or fewer rows of nodes, the numbers of nodes in the rows being in descending order of magnitude. Adding to each of these the above graph of $\binom{k+1}{2}$, row to row, we obtain the graph of a partition in which there are no repetitions of parts.

Hence we gather that the numerator enumerates, with respect to the number n, the excess of $2j+1$ times the number of partitions into exactly $2j+1$ parts ($j = 0, 1, 2, \ldots$) without repetitions over $2k+2$ times the number of partitions into exactly $2k+2$ parts ($k = 0, 1, 2, \ldots$) without repetitions.

The numerator, to a few terms, is

$$q+q^2-q^3-q^4-3q^5-2q^7+q^8+2q^9+q^{10}+2q^{11}+4q^{12}+\ldots,$$

so that if p_n denote the number of partitions of n we are led to the relations

$$(36)\quad \begin{cases} \nu(n) = p_{n-1}+p_{n-2}-p_{n-3}-p_{n-4}-3p_{n-5}-2p_{n-7}+p_{n-8}+2p_{n-9} \\ \qquad\qquad +p_{n-10}+2p_{n-11}+4p_{n-12}+\ldots, \\ \nu(n)-\nu(n-1)-\nu(n-2)+\nu(n-5)+\nu(n-7)-\nu(n-12)-\ldots \\ \qquad\qquad = \text{coefficient of } q^n \text{ in the numerator,} \end{cases}$$

verified for $n = 10$ by

$$\nu(10) = p_9+p_8-p_7-p_6-3p_5-2p_3+p_2+2p_1+1,$$

$$4 \;= 30+22-15-11-21-6+2+2+1 = 57-53,$$

$$\nu(10)-\nu(9)-\nu(8)+\nu(5)+\nu(3) = 1,$$

$$4 - 3 - 4 + 2 + 2 = 1.$$

In considering the general case

$$\sum_1^\infty m^s \frac{q^{m(u+1)}(1-q^{mv})}{1-q^m} = \Sigma\Sigma d^s_{u,\,u+v}(n)q^n,$$

the left-hand side, as it stands, is a sum of an infinite number of terms and is readily converted into a sum of v terms. Thus, for $s = 1, 2, 3, \ldots$, we readily verify the expressions

$$\sum_1^v \frac{q^{u+m}}{(1-q^{u+m})^2} = \Sigma\Sigma d_{u,\,u+v}(n)q^n,$$

$$\sum_1^v \frac{q^{u+m}(1+q^{u+m})}{(1-q^{u+m})^3} = \Sigma\Sigma d^2_{u,\,u+v}(n)q^n,$$

$$\sum_1^v \frac{q^{u+m}(1+4q^{u+m}+q^{2u+2m})}{(1-q^{u+m})^4} = \Sigma\Sigma d^3_{u,\,u+v}(n)q^n,$$

$$\sum_1^v \frac{q^{u+m}(1+11q^{u+m}+11q^{2u+2m}+q^{3u+3m})}{(1-q^{u+m})^5} = \Sigma\Sigma d^4_{u,\,u+v}(n)q^n,$$

where the numerical coefficients in the series of numerators are for identification with the scheme of numbers met with in § 3.

In general we have

$$\sum_1^v \frac{\sum_1^s c_{s,t}\, q^{t(u+m)}}{(1-q^{u+m})^{s+1}} = \Sigma\Sigma d^s_{u,\,u+v}(n)q^n. \tag{37}$$

When $v = \infty$, so that $\dfrac{n}{u} > d_{u,\,\infty}$,

the only condition,

$$\sum_1^\infty \frac{\sum_1^s c_{s,t}\, q^{t(u+m)}}{(1-q^{u+m})^{s+1}} = \Sigma\Sigma d^s_{u,\,\infty}(n)q^n.$$

When $v = 1$,

$$\frac{\sum_1^s c_{s,t}\, q^{t(u+1)}}{(1-q^{u+1})^{s+1}} = \Sigma\Sigma d^s_{u,\,u+1}(n)\, q^n.$$

The coefficient of $q^{n(u+1)}$ in the left-hand side is

$$c_{s,1}\binom{s+u-1}{n-1} + c_{s,2}\binom{s+u-2}{n-2} + \ldots + c_{s,n} = \Sigma d^s_{u,\,u+1}(nu+n).$$

Ex. gr., $s = 2$, $u = 3$, $n = 9$.

The divisors of 36 which are such that

$$\tfrac{36}{3} > d \geqslant \tfrac{36}{4}$$

are simply the one divisor 9 and

$$c_{2,1}\binom{10}{8}+c_{2,2}\binom{9}{7}=45+36=9^2.$$

5. A similar theory exists for other classes of divisors.
Thus, for those whose conjugates are uneven, we start with

$$\Sigma\frac{a^m q^m}{1-q^{2m}}=\Sigma\Sigma a^\delta q^n,$$

and, differentiating s times with $a\frac{d}{da}$, we derive

$$\Sigma\frac{m^s a^m q^m}{1-q^{2m}}=\Sigma\Sigma\delta^s a^\delta q^n,$$

and putting $a=q^u$, $\quad \Sigma\frac{m^s q^{m(u+1)}}{1-q^{2m}}=\Sigma\Sigma\delta^s q^{n+\delta u},$

and changing n so that

$$\Sigma\frac{m^s q^{m(u+1)}}{1-q^{2m}}=\Sigma\Sigma\delta_u^s q^n,$$

we find that $\qquad \frac{n}{u}>\delta_u.$

Substituting $u+2v$ for u and subtracting

$$\Sigma\frac{m^s q^{m(u+1)}(1-q^{2mv})}{1-q^{2m}}=\Sigma\Sigma\delta^s_{u,u+2v}(n)q^n, \tag{38}$$

where $\qquad \frac{n}{u}>\delta_{u,u+2v}\geqslant\frac{n}{u+2v}.$

Put $s=0$, obtaining

$$\Sigma\frac{q^{m(u+1)}(1-q^{2mv})}{1-q^{2m}}=\Sigma\Sigma\delta^0_{u,u+2v}(n)q^n,$$

where now $\Sigma\delta^0_{u,u+2v}(n)$ is the number of divisors with uneven conjugates such that the above limits are satisfied.

For $v = 1$, $$\frac{q^{u+1}}{1-q^{u+1}} = \Sigma\Sigma\, \delta^0_{u,\, u+2}(n) q^n,$$

a trivial result.

For $v = 2$, $$\frac{\frac{1-q^4}{1-q^2}\, q^{u+1} - 2q^{2u+4}}{(1-q^{u+1})(1-q^{u+3})} = \Sigma\Sigma\, \delta^0_{u,\, u+4}(n) q^n,$$

and, in general,

$$\frac{\frac{1-q^{2v}}{1-q^2}\, q^{u+1} - 2\,\frac{(1-q^{2v})(1-q^{2v-2})}{(1-q^2)(1-q^4)}\, q^{2u+4} + 3\,\frac{(1-q^{2v})(1-q^{2v-2})(1-q^{2v-4})}{(1-q^2)(1-q^4)(1-q^6)}\, q^{3u+9} - \ldots + (-)^{v+1} v q^{(u+v)v}}{(1-q^{u+1})(1-q^{u+3}) \ldots (1-q^{u+2v-1})}$$

$$= \Sigma\Sigma\, \delta^0_{u,\, u+2v}(n) q^n;$$

put $u = 0$, then

$$(39)\quad \frac{\frac{1-q^{2v}}{1-q^2}\, q - 2\,\frac{(1-q^{2v})(1-q^{2v-2})}{(1-q^2)(1-q^4)}\, q^4 + 3\,\frac{(1-q^{2v})(1-q^{2v-2})(1-q^{2v-4})}{(1-q^2)(1-q^4)(1-q^6)}\, q^9 - \ldots + (-)^{v+1} v q^{v^2}}{(1-q)(1-q^3)(1-q^5) \ldots (1-q^{2v-1})}$$

$$= \Sigma\Sigma\, \delta^0_{0,\, 2v}(n) q^n,$$

where $$\infty > \delta_{0,\, 2v} \geqslant \frac{n}{2v},$$

i.e. the divisors enumerated are not less than $n/2v$ and have uneven conjugates.

If we now put $v = \infty$,

$$(40)\quad \frac{\frac{q}{1-q^2} - 2\,\frac{q^4}{(1-q^2)(1-q^4)} + 3\,\frac{q^9}{(1-q^2)(1-q^4)(1-q^6)} - 4\,\frac{q^{16}}{(1-q^2)(1-q^4)(1-q^6)(1-q^8)} - \ldots}{(1-q)(1-q^3)(1-q^5) \ldots}$$

$$= \Sigma\Sigma\, \delta_{0,\, \infty}(n) q^n,$$

where $$\infty > \delta_{0,\, \infty} \geqslant 0,$$

and we have a new function which generates numbers which enumerate the whole of the divisors which are before us.

The numerator is interpretable.

Since q, q^4, q^9, q^{16}, ... are graphically representable as

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and $(1-q^2, 1-q^4, \ldots, 1-q^{2m})^{-1}$ enumerates the partitions of numbers into n or fewer even parts, we can add on to the m-th of these graphs the graphs of such partitions, thus obtaining graphs of partitions into exactly m uneven parts, which involve neither *sequences nor repetitions of parts*.

Hence we gather that the numerator enumerates, with respect to the number n, the excess of $2j+1$ times the number of partitions into exactly $2j+1$ uneven parts which involve neither sequences nor repetitions over $2k+2$ times the number of partitions into exactly $2k+2$ uneven parts which involve neither sequences nor repetitions of parts, where

$$j = 0, 1, 2, \ldots, \qquad k = 0, 1, 2, \ldots .$$

The numerator, to a few terms, is

$$q+q^2+2q^3+q^5-q^6-4q^8-4q^{10}-6q^{12}+q^{13}+\ldots,$$

and the coefficient in this of q^n is

$$\delta(n)-\delta(n-1)-\delta(n-3)+\delta(n-4)-\delta(n-5)+\delta(n-6)-\delta(n-7)$$

$$+2\delta(n-8)-2\delta(n-9)+2\delta(n-10)-2\delta(n-11)+3\delta(n-12)-\ldots .$$

For $n = 12$, we verify that

$$2-2-3+1-2+2-2$$

$$+2-4+2-2 \qquad\qquad = -6.$$

Returning to the general case the left-hand side is expressible as the sum

of v terms so that,

for $s = 1$, $\sum_1^v \frac{q^{u+2m-1}}{(1-q^{u+2m-1})^2} = \Sigma\Sigma\delta_{u,\,u+2v}(n)\,q^n$,

„ $s = 2$, $\sum_1^v \frac{q^{u+2m-1}(1+q^{u+2m-1})}{(1-q^{u+2m-1})^3} = \Sigma\Sigma\delta^2_{u,\,u+2v}(n)q^n$,

„ $s = 3$, $\sum_1^v \frac{q^{u+2m-1}(1+4q^{u+2m-1}+q^{2u+4m-2})}{(1-q^{u+2m-1})^4} = \Sigma\Sigma\delta^3_{u,\,u+2v}(n)q^n$,

&c.,

the numerator numbers being those that have already appeared in § 3.

In general

$$\sum_1^v \frac{\sum_1^s c_{s,t}\, q^{t(u+2m-1)}}{(1-q^{u+2m-1})^{s+1}} = \Sigma\Sigma\delta^s_{u,\,u+2v}(n)q^n. \tag{41}$$

THE INFLUENCE OF DIFFUSION ON THE PROPAGATION OF SOUND WAVES IN AIR

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Introduction.

1. The ordinary expression for the velocity of sound in a gas $[c = \sqrt{(5p/3\rho)}]$ is derived on the assumption that the sound wave spreads outwards without loss of energy, and that the alternate compressions and expansions are performed adiabatically. The influence of internal friction or viscosity was first considered by Stokes,* who found that to a first approximation the waves are still propagated with velocity c, but that the amplitude of the oscillations at a distance x cms. from the source is reduced in the ratio $e^{-\alpha x}$. In the case of plane waves in air, the coefficient of decay α is approximately $1{\cdot}4.10^{-13}n^2$, n being the frequency of the oscillations in the sound. The amplitude of the oscillations in sound of frequency 3300 (wave length 10 cms.) would thus be reduced in the ratio e^{-1} after travelling 8·8 kms.†

It was subsequently pointed out by Kirchhoff‡ that for consistency in the general theory it is also necessary to take account of heat conduction, because the influence of the latter is of the same order of importance as that of viscosity. His result corresponds in the case of air to a coefficient of decay equal to about half that obtained from viscosity.§

It has been remarked by one of the present writers‖ that in a complete theory the one other "first order" mean-free-path property of a

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* *Cambridge Transactions*, Vol. 9 (1851), § 49. Cf. also Rayleigh, *Theory of Sound*, Vol. 2, § 346.

† These figures are given by Rayleigh, *Theory of Sound*, Vol. 2, § 346. The corresponding note on the musical scale is high, roughly four octaves above the violin G.

‡ *Pogg. Ann.*, Vol. 134 (1868), p. 177. Cf. also his *Vorlesungen über die Theorie der Wärme*, Ch. XI.

§ Cf. Lamb, *Hydrodynamics*, 3rd ed., p. 589.

‖ S. Chapman, *Phil. Trans.*, A, Vol. 217 (1917), p. 159.

gas, viz. diffusion, should be similarly taken into account in discussing sound propagation in a mixed gas. It is the main object of this paper to supply this deficiency in the theory. Since air is itself a mixed gas the result is of some interest, though the influence of diffusion is much less than that of conduction or viscosity.* The velocity of propagation is, to the first order, unaffected, while the amplitude of the oscillations diminishes with the distance (x) from the source according to the law $e^{-\beta x}$. For air under normal conditions of temperature and pressure β is $1{\cdot}2.10^{-15}n^2$, so that the intensity of the sound is reduced in the ratio e^{-1} at a distance $8.10^{13}n^{-2}$ cms.; for $n = 3300$ this is 8000 kms.

It appeared also that the general results obtained might possibly be applied to estimate the effect of diffusion on the propagation of sound in an atmosphere laden with fog, by treating the air as a simple gas and taking the fog particles as the second set of molecules. It was found, however, that unless the particles of water are extremely minute our approximations break down, and the results obtained suggested an exaggerated fictitious degree of damping, even for sounds of moderate pitch. The statistical methods involved in the gas theory are, in fact, inapplicable when, as in the case of fog particles, the "molecules" of one set have a diameter comparable with the mean free path of the other set.

The General Equations for a Mixed Gas.

2. We assume that there are but two kinds of molecules in the gas, of masses m_1 and m_2,† and present in numbers ν_1, ν_2 per unit volume. The total number of molecules (ν), the mean molecular mass (m), and the total density (ρ) will then be given by

$$\nu = \nu_1+\nu_2, \quad m\nu = m_1\nu_1+m_2\nu_2, \quad \rho = \nu m. \tag{2.1}$$

The pressure and absolute temperature will be denoted, as usual, by p and T; R being the usual gas constant, we have

$$p = R\nu T. \tag{2.2}$$

We also introduce other quantities λ_1, λ_2, λ', m', ρ' which are defined by the equations

$$\nu\lambda_1 = \nu_1, \quad \nu\lambda_2 = \nu_2, \tag{2.3}$$

* This is partly because the molecular weights of oxygen and nitrogen are so nearly equal.

† For definiteness we will suppose $m_1 > m_2$.

so that $$\lambda_1+\lambda_2 = 1,$$

(2.4) $$2\lambda' = \lambda_1-\lambda_2, \quad m' = m_1-m_2, \quad \rho' = \nu(m_1-m_2).$$

The mean velocity components of the molecules of the two kinds at the point considered will be denoted by (u_1, v_1, w_1), (u_2, v_2, w_2). The motion of the composite gas is analysed into a certain "mean" velocity (u, v, w), together with a motion of interdiffusion in which equal *numbers* of molecules m_1, m_2 are transferred per unit time across a surface moving with the mean velocity (u, v, w). The diffusive motion may be represented by the mean velocities $(u', v', w')/\lambda_1$ and $-(u', v', w')/\lambda_2$, for the two sets of molecules, superposed on their mean motion (u, v, w). Thus

(2.4) $$(u_1, v_1, w_1) = (u, v, w)+(u', v', w')/\lambda_1,$$

(2.5) $$(u_2, v_2, w_2) = (u, v, w)-(u', v', w')/\lambda_2.$$

Then we have also*

(2.6) $$(u, v, w) = \lambda_1(u_1, v_1, w_1)+\lambda_2(u_2, v_2, w_2),$$

(2.7) $$(u', v', w') = (u_1-u_2, v_1-v_2, w_1-w_2)/\lambda_1\lambda_2.$$

3. The equations of continuity of *number* for the two sets of molecules are, as usual,

(3.1) $$\frac{\partial \nu_1}{\partial t} + \sum_{x, y, z} \frac{\partial(\nu_1 u_1)}{\partial x} = 0,$$

(3.2) $$\frac{\partial \nu_2}{\partial t} + \sum_{x, y, z} \frac{\partial(\nu_2 u_2)}{\partial x} = 0.$$

By addition we get the corresponding equation for the gas as a whole

(3.3) $$\frac{\partial \nu}{\partial t} + \sum_{x, y, z} \frac{\partial(\nu u)}{\partial x} = 0.$$

The ordinary hydrodynamical equation of continuity relates to the continuity of *mass*, not number of molecules; it can be derived from the above by writing in it $\nu = \rho/m$. The mobile operator $\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}+w\frac{\partial}{\partial z}$ being denoted by $\frac{D}{Dt}$, the equation becomes

(3.4) $$\frac{1}{\rho}\frac{D\rho}{Dt} - \frac{1}{m}\frac{Dm}{Dt} + \sum_{x, y, z}\frac{\partial u}{\partial x} = 0.$$

* The notation here adopted is the same as that employed in the memoir by S. Chapman, with the exception that the mean total quantities p, ρ, u, v, w, &c., are written without their suffix $(_0)$.

This differs from the ordinary equation of continuity

$$\frac{1}{\rho}\frac{D\rho}{Dt} + \sum_{x,y,z}\frac{\partial U}{\partial x} = 0,$$

because the mean velocity here used (u, v, w) differs from the velocity (U, V, W) ordinarily employed, which is that of the mean *mass* of the gas at the point. Diffusion is measured by the rate of transfer of molecules in equal *numbers* in opposite directions. There is consequently a resultant flow of mass in the direction of motion of the heavier molecules m_1: this is represented by the additional term $-\frac{1}{m}\frac{Dm}{Dt}$ in the equation (3.4).

4. The equations of motion and energy for a mixed gas have been given in the memoir last cited,* to the first order of small quantities depending on viscosity, thermal conduction and diffusion. For our present purpose, since the influence on sound propagation exerted by the two former factors has been already adequately considered, and since first order effects are additive, it will be sufficient to reproduce these equations without the terms depending on viscosity and thermal conduction. The equations of motion and energy are then

$$(4.1)\quad \nu\frac{D}{Dt}(mu+m'u') + \frac{\partial p}{\partial x} + \frac{\partial(\rho' u' u)}{\partial x} + \frac{\partial(\rho' v' u)}{\partial y} + \frac{\partial(\rho' w' u)}{\partial z} = 0,$$

with two similar equations for v and w: and

$$(4.2)\quad \rho C_v T\left(\frac{1}{T}\frac{DT}{Dt} + \tfrac{2}{3}\sum_{x,y,z}\frac{\partial u}{\partial x}\right)$$

$$= \frac{\nu m'}{J}\left\{\tfrac{1}{2}(u^2+v^2+w^2)\left(\sum_{x,y,z}\frac{\partial u'}{\partial x}\right) - \sum_{x,y,z} u'\frac{Du}{Dt}\right\} - \tfrac{2}{3}\sum_{x,y,z}\frac{\partial}{\partial x}\left[\frac{k_T}{\lambda_1\lambda_2}\rho C_v T u'\right].$$

Here J denotes the mechanical equivalent of heat, C_v the specific heat of the gas at constant volume, and k_T is the ratio of the coefficient of thermal diffusion to the ordinary coefficient of diffusion D_{12}.

5. The velocity (u', v', w') which measures the rate of diffusion is given by the following equation,† wherein it is assumed that there are no external forces—acting unequally per unit mass of the two sets of molecules —causing diffusion:

$$(5.1)\qquad u' = D_{12}\left[-\frac{\partial\lambda'}{\partial x} + k_p\frac{1}{p}\frac{\partial p}{\partial x} - k_T\frac{1}{T}\frac{\partial T}{\partial x}\right].$$

* Cf. equations (10.14), (11.05), (12.05), (12.12), (12.23), (19.04) of this memoir.

† Cf. the memoir cited: equations (10.02), (10.11) and the table on p. 185.

The three terms on the right represent respectively the diffusion due to variations in the relative concentration of the component gases and to gradients in the pressure and temperature. The three coefficients of diffusion D_{12}, $D_p = k_p D_{12}$, $D_T = k_T D_{12}$ can be calculated from expressions given in the memoir.

The variations of λ' [cf. (2 . 4)] and of (u', v', w') cannot, of course, be mutually independent. By subtracting (3 . 2) from (3 . 1), we get

$$\frac{\partial(\nu\lambda')}{\partial t} + \sum_{x,\,y,\,z} \frac{\partial(\lambda'\nu u)}{\partial x} + \sum_{x,\,y,\,z} \frac{\partial(\nu u')}{\partial x} = 0,$$

which, on simplification by means of the equation of continuity, becomes

$$\frac{\nu D\lambda'}{Dt} + \sum_{x,\,y,\,z} \frac{\partial(\nu u')}{\partial x} = 0. \tag{5 . 2}$$

The Propagation of Sound.

6. In applying these equations to the discussion of the propagation of sound we take the simplest case of plane waves travelling in the x-direction. The gas being assumed stationary as a whole, the y, z components of motion entirely vanish, while the x-components will be small quantities. All the variable characteristics of the gas (velocity, and divergences of pressure, temperature and density from the mean) will depend on x and t through a factor of the type $e^{2\pi i n(t - x/c)}$. Here n is the frequency and c the velocity of propagation of the sound waves. Hence whenever the operator $\partial/\partial t$ occurs in our equations we may replace it by $-c\,\partial/\partial x$.

We may first consider what will be the actual value of the diffusion velocity u' in a typical case of sound propagation. Neglecting small quantities of the second order, and eliminating $\partial/\partial t$ as just explained, equation (4 . 4) becomes

$$\frac{\partial\lambda'}{\partial x} = \frac{1}{c}\frac{\partial u'}{\partial x} = -\frac{2\pi i n}{c^2} u'. \tag{6 . 1}$$

Hence equation (4 . 3) may be written in the form

$$u'\left(1 - \frac{2\pi i n D_{12}}{c^2}\right) = D_{12}\left(k_p \frac{1}{p}\frac{\partial p}{\partial x} - k_T \frac{1}{T}\frac{\partial T}{\partial x}\right). \tag{6 . 2}$$

The coefficient of u' in this equation differs very slightly from unity; $\frac{2\pi n D_{12}}{c^2}$ is of the order 10^{-7} in the case of ordinary sound waves in air, and merely represents a slight reflex effect on the velocity of diffusion (itself

considered as due mainly to the gradients of pressure and temperature) owing to the variation in the relative concentration which it produces. We may neglect this second order effect, and on eliminating $\frac{1}{T}\frac{\partial T}{\partial x}$ by the equation (cf. 2 . 2)

$$\frac{1}{p}\frac{\partial p}{\partial x} = \frac{1}{\nu}\frac{\partial \nu}{\partial x} + \frac{1}{T}\frac{\partial T}{\partial x}, \tag{6 . 3}$$

equation (6 . 2) becomes

$$u' = D_{12}\left\{(k_p - k_T)\frac{1}{p}\frac{\partial p}{\partial x} + k_T\frac{1}{\nu}\frac{\partial \nu}{\partial x}\right\}. \tag{6 . 4}$$

Considering air as a mixture of three parts (by volume) of nitrogen and one part of oxygen, the values of D_{12}, k_p, k_T are* 0·174, 0·026, 0·008 respectively. Since $\frac{1}{\nu}\frac{\partial \nu}{\partial x}$ is approximately equal to $\frac{2}{3}\frac{1}{p}\frac{\partial p}{\partial x}$, we may write

$$u' = D_{12}(k_p - \tfrac{2}{5}k_T)\frac{1}{p}\frac{\partial p}{\partial x}.$$

For a sound wave of moderate loudness and pitch (frequency 256) the value of $\frac{1}{p}\frac{\partial p}{\partial x}$, according to Rayleigh,† is of the order 10^{-8}; the value of u' will therefore be of the order 10^{-11} cms. per sec. The number of molecules of the two kinds which would cross 1 sq. cm. in opposite directions in unit time owing to the diffusion is of the order 10^8. Since a particular condensation only lasts for about 2.10^{-3} secs., the number of molecules actually transferred during any such condensation is roughly 10^5.

7. We next simplify the equations of continuity, motion and energy (3·1, 4·1, and 4·2) by the methods already described. The first of these, the equation of continuity, becomes

$$\frac{c-u}{\nu}\frac{\partial \nu}{\partial x} = \frac{\partial u}{\partial x},$$

or, since u is negligible in comparison with c,

$$\frac{1}{\nu}\frac{\partial \nu}{\partial x} = \frac{1}{c}\frac{\partial u}{\partial x}. \tag{7 . 1}$$

* The first figure is given by Jeans, *Dynamical Theory of Gases*, 2nd ed., p. 337. The second and third are taken from the table on p. 185 of the memoir cited above.

† *Theory of Sound*, Vol. 2, §§ 245, 384.

Neglecting terms of the second order in the equation of motion (4 . 1) likewise, this becomes

$$\frac{1}{p}\frac{\partial p}{\partial x} = \frac{\nu c}{p}\left(m\frac{\partial u}{\partial x}+m'\frac{\partial u'}{\partial x}\right). \tag{7 . 2}$$

Making use of equations (6 . 4) and (7 . 1) to eliminate u' and $\frac{1}{\nu}\frac{\partial \nu}{\partial x}$, and performing the differentiation with respect to x in (6 . 4), we have

$$\frac{1}{p}\frac{\partial p}{\partial x}\left[1+\frac{2\pi i n\nu m'}{p}D_{12}(k_p-k_T)\right] = \frac{\nu mc}{p}\frac{\partial u}{\partial x}\left[1-\frac{2\pi i n m'}{mc^2}D_{12}k_T\right]. \tag{7 . 3}$$

Further, omitting terms of order higher than that of those (of the same kind) retained, and eliminating u', $\frac{1}{T}\frac{\partial T}{\partial x}$ and $\frac{1}{\nu}\frac{\partial \nu}{\partial x}$ by means of equations (6 . 4), (6 . 3) and (7 . 1), the equation of energy (4 . 2) becomes successively

$$\frac{1}{T}\frac{\partial T}{\partial x} = \frac{2}{3c}\frac{\partial u}{\partial x}-\frac{1}{\rho c C_v T}\frac{\partial}{\partial x}\left(\frac{k_T}{\lambda_1\lambda_2}\rho C_v T u'\right)$$

$$= \frac{2}{3c}\frac{\partial u}{\partial x}+\frac{2}{3}\frac{k_T}{\lambda_1\lambda_2}\frac{2\pi i n}{c^2}D_{12}\left\{(k_p-k_T)\frac{1}{p}\frac{\partial p}{\partial x}+k_T\frac{1}{\nu}\frac{\partial \nu}{\partial x}\right\}$$

and

$$\frac{1}{p}\frac{\partial p}{\partial x}\left[1-\frac{2}{3}\frac{k_T}{\lambda_1\lambda_2}\frac{2\pi i n D_{12}}{c^2}(k_p-k_T)\right]$$

$$= \frac{5}{3c}\frac{\partial u}{\partial x}\left[1+\frac{2}{3}\frac{k_T^2}{\lambda_1\lambda_2}\frac{6\pi i n}{5c^2}D_{12}\right]. \tag{7 . 4}$$

Now equations (7 . 3) and (7 . 4) give two expressions for the ratio $\frac{1}{p}\frac{\partial p}{\partial x}\Big/\frac{\partial u}{\partial x}$ which may conveniently be written in the form

$$\frac{\nu mc}{p}[1-a] \quad \text{and} \quad \frac{5}{3c}[1+a'], \tag{7 . 5, 7 . 6}$$

where

$$a = 2\pi i n m' D_{12}\left[\frac{\nu}{p}(k_p-k_T)+\frac{1}{mc^2}k_T\right] \tag{7 . 7}$$

and

$$a' = \frac{4\pi i n k_T}{3\lambda_1\lambda_2 c^2}D_{12}[k_p-\tfrac{2}{5}k_T], \tag{7 . 8}$$

and squares of the small quantities a, a' are neglected. Thus in order

that they may be consistent we must have

$$\frac{\nu m c}{p}[1-a] = \frac{5}{3c}[1+a]$$

or

$$\frac{1}{c^2} = \frac{3\nu m}{5p}[1-a-a'] = \frac{1}{c_0^2}(1-a-a'),$$

where

$$c_0^2 = \frac{5p}{3\nu m} = \frac{5p}{3\rho},$$

so that c_0 is the velocity of sound as ordinarily calculated. Hence to the same order of approximation, we have

$$\frac{1}{c} = \frac{1}{c_0}[1-\tfrac{1}{2}(a+a')].$$

The small term $\frac{1}{2}(a+a')$ represents the influence of diffusion on the propagation of sound. It introduces into the ordinary periodic factor associated with the propagation, viz. $e^{2\pi i n(t-x/c)}$, an additional factor $e^{\pi i n(a+a')x/c_0}$, which, since $(a+a')$ is purely imaginary [cf. (7 . 7) and (7 . 8)], is a negative exponential factor $e^{-\beta x}$ indicating damping or loss of intensity. Thus to the first order of approximation the (real) velocity of transmission is unaffected by the diffusion.

Evidently

$$\beta = -\pi i n(a+a')/c_0,$$

and substituting $5p/3\nu m$ for c^2 on the right of (7 . 7), we find the following expression for β,

(7 . 5)

$$\beta = \frac{2\pi^2 n^2}{c_0^3} D_{12}(k_p - \tfrac{2}{5}k_T)\left(\frac{5m'}{3m} + \frac{2k_T}{3\lambda_1\lambda_2}\right).$$

Now for air, regarded as a mixture of three parts by volume of nitrogen and one part of oxygen, we have, as in § 6, $D_{12} = 0{\cdot}174$, $k_p = 0{\cdot}026$, and $k_T = 0{\cdot}008$. Further, $\nu_1/\nu_2 = 1/3$, and $m_1/m_2 = 16/14$, so that

$$\frac{5m'}{3m} = \frac{(m_1-m_2)(\nu_1+\nu_2)}{m_1\nu_1+m_2\nu_2} = \frac{20}{87} = 0{\cdot}23,$$

whilst, since $\lambda_1 = 1/4$, $\lambda_2 = 3/4$, we have

$$\frac{2k_T}{3\lambda_1\lambda_2} = \frac{32}{9}k_T = 0{\cdot}028.$$

Thus, since $c = 33000$ cms. per sec., we have

$$\beta = 1{\cdot}2\,.\,10^{-15}n^2.$$

This means that the amplitude of the oscillations is reduced in the ratio e^{-1} in a distance equal to $8 . 10^{10} n^{-2}$ kms. For $n = 3300$ this is 8000 kms. The influence of diffusion on the damping of sound waves in air is consequently quite inappreciable, being only about one-hundredth as large as that due to viscosity. If the disproportion of the masses of the two kinds of molecules had been greater (say $m_1/m_2 = 16$) and under the more favourable condition of equal numbers of the two molecules ($\lambda_1 = \lambda_2 = \frac{1}{2}$), the magnitude of the diffusion effect might be considerably increased. Using these values of m_1/m_2 and λ_1, λ_2, and the value 1 for D_{12} (as for the case of oxygen and hydrogen) β becomes $4 . 10^{-13} n^2$. The effects due to viscosity and diffusion are therefore of the same order of magnitude in a gas of this type.

PROOFS OF CERTAIN IDENTITIES AND CONGRUENCES ENUNCIATED BY S. RAMANUJAN

By H. B. C. Darling.

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The object of this paper is to furnish proofs of certain identities and congruences which have been enunciated from time to time by Mr. S. Ramanujan, but of which he has not yet furnished complete demonstrations. There are six of these relations in the present paper, viz. those given in equations (27), (51), see also (51*a*), (53), (80), (91), and (99). The results do not appear to admit of being readily established, and the writer of this paper has found it necessary in the first instance to introduce certain relations in connection with theta functions.

Let
$$u_a = 1-q^{ai+n}-q^{-ai+2-n}+q^{2ai+2n+2}+q^{-2ai+6-2n} -q^{3ai+3n+6}-q^{-3ai+12-3n}+\ldots, \tag{1}$$

in which the index involving rai is $r(ai+n)+r(r-1)$, and that involving $-rai$ is $r(-ai+2-n)+r(r-1)$; then

$$\begin{aligned} u_{a-i} &= 1-q^{ai+n+1}-q^{-ai+1-n}+q^{2ai+2n+4}+q^{-2ai+4-2n}-q^{3ai+3n+9}-\ldots \\ &= 1-q(q^{ai+n}+q^{-ai-n})+q^4(q^{2ai+2n}+q^{-2ai-2n}) \\ &\quad -q^9(q^{3ai+3n}+q^{-3ai-3n})+\ldots; \end{aligned}$$

so that if $q = e^{-\pi K'/K}$, as in elliptic functions, we have

$$\begin{aligned} u_{a-i} &= 1-2q\cos\frac{\pi K'}{K}(a-in)+2q^4\cos 2\frac{\pi K'}{K}(a-in)-\ldots \\ &= \Theta\{K'(a-in)\}. \end{aligned}$$

Hence
$$u_a = \Theta\{K'(a+i-in)\};$$

and putting $a=0$ in (1), we have

$$1-q^n-q^{2-n}+q^{2n+2}+q^{6-2n}-\ldots = \Theta\{(n-1)K'i\}.$$

But the left-hand side of this equation is

$$(1-q^n)(1-q^{2-n})(1-q^{n+2})(1-q^{4-n})(1-q^{n+4})(1-q^{6-n})\ldots$$
$$\ldots\times(1-q^2)(1-q^4)(1-q^6)\ldots,$$

so that $\Theta\{(n-1)K'i\} = (1-q^n)(1-q^{2-n})(1-q^{n+2})(1-q^{4-n})\ldots$
$$\ldots\times(1-q^2)(1-q^4)(1-q^6)\ldots. \qquad (2)$$

In (2) putting successively $n = \frac{4}{5}, \frac{2}{5}, \frac{3}{5}, \frac{1}{5}$, we obtain

$$\left.\begin{aligned}
\Theta\left(\frac{K'i}{5}\right) &= (1-q^{\frac{1}{5}})(1-q^{\frac{2}{5}})(1-q^{\frac{11}{5}})(1-q^{\frac{12}{5}})\ldots \\
&\qquad\ldots\times(1-q^2)(1-q^4)(1-q^6)\ldots \\
\Theta\left(\frac{3K'i}{5}\right) &= (1-q^{\frac{3}{5}})(1-q^{\frac{4}{5}})(1-q^{\frac{13}{5}})(1-q^{\frac{14}{5}})\ldots \\
&\qquad\ldots\times(1-q^2)(1-q^4)(1-q^6)\ldots \\
\Theta\left(\frac{2K'i}{5}\right) &= (1-q^{\frac{2}{5}})(1-q^{\frac{3}{5}})(1-q^{\frac{12}{5}})(1-q^{\frac{13}{5}})\ldots \\
&\qquad\ldots\times(1-q^2)(1-q^4)(1-q^6)\ldots \\
\Theta\left(\frac{4K'i}{5}\right) &= (1-q^{\frac{4}{5}})(1-q^{\frac{1}{5}})(1-q^{\frac{14}{5}})(1-q^{\frac{11}{5}})\ldots \\
&\qquad\ldots\times(1-q^2)(1-q^4)(1-q^6)\ldots
\end{aligned}\right\}. \qquad (3)$$

Hence

$$\frac{\Theta\left(\frac{K'i}{5}\right)}{\Theta\left(\frac{3K'i}{5}\right)} = \frac{(1-q^{\frac{1}{5}})(1-q^{\frac{2}{5}})(1-q^{\frac{12}{5}})(1-q^{\frac{13}{5}})\ldots}{(1-q^{\frac{3}{5}})(1-q^{\frac{4}{5}})(1-q^{\frac{13}{5}})(1-q^{\frac{14}{5}})\ldots}. \qquad (4)$$

But

$$\Theta\left(\frac{3K'i}{5}\right)\Theta\left(\frac{K'i}{5}\right) = \frac{\Theta^2\left(\frac{2K'i}{5}\right)\Theta^2\left(\frac{K'i}{5}\right)}{\Theta^2(0)}\left(1-k^2\,\mathrm{sn}^2\frac{K'i}{5}\,\mathrm{sn}^2\frac{2K'i}{5}\right), \qquad (5)$$

also
$$\mathrm{sn}\frac{4K'i}{5} = -\frac{1}{k\,\mathrm{sn}\frac{K'i}{5}}, \qquad \mathrm{sn}\frac{2K'i}{5} = -\frac{1}{k\,\mathrm{sn}\frac{3K'i}{5}}. \qquad (6)$$

We shall find it convenient to write simply 1 for $K'i/5$, 2 for $2K'i/5$, and so on. Thus, by (6), equation (5) becomes

$$\Theta 3\,.\,\Theta 1 = \frac{\Theta^2 2\,.\,\Theta^2 1}{\Theta^2 0}\left(1-\frac{\mathrm{sn}^2 1}{\mathrm{sn}^2 3}\right),$$

whence
$$\frac{\Theta 1}{\Theta 3} = \frac{\Theta^2 0}{\Theta^2 2}\,\frac{\mathrm{sn}^2 3}{\mathrm{sn}^2 3-\mathrm{sn}^2 1}. \qquad (7)$$

Further $$\operatorname{sn} 4 \operatorname{sn} 2 = \frac{\operatorname{sn}^2 3 - \operatorname{sn}^2 1}{1 - k^2 \operatorname{sn}^2 1 \operatorname{sn}^2 3},$$

hence, by (6),

$$\left.\begin{aligned} 1 - k^2 \operatorname{sn}^2 1 \operatorname{sn}^2 3 &= k^2 \operatorname{sn} 1 \operatorname{sn} 3 (\operatorname{sn}^2 3 - \operatorname{sn}^2 1) \\ \text{whence also} \\ 1 - k^2 \operatorname{sn}^4 1 &= k^2 \operatorname{sn} 1 (\operatorname{sn}^2 3 - \operatorname{sn}^2 1)(\operatorname{sn} 3 + \operatorname{sn} 1) \\ 1 - k^2 \operatorname{sn}^4 3 &= k^2 \operatorname{sn} 3 (\operatorname{sn}^2 3 - \operatorname{sn}^2 1)(\operatorname{sn} 1 - \operatorname{sn} 3) \\ 1 + k^2 \operatorname{sn}^3 1 \operatorname{sn} 3 &= k^2 \operatorname{sn} 1 \operatorname{sn}^2 3 (\operatorname{sn} 3 + \operatorname{sn} 1) \\ \frac{1}{k^2} &= \operatorname{sn} 1 \operatorname{sn}^3 3 - \operatorname{sn}^3 1 \operatorname{sn} 3 + \operatorname{sn}^2 1 \operatorname{sn}^2 3 \end{aligned}\right\}. \quad (8)$$

Again $$\Theta 4 . \Theta 0 = \frac{\Theta^4 2}{\Theta^2 0}(1 - k^2 \operatorname{sn}^4 2),$$

hence, by (8) and (6),

$$\Theta 4 . \Theta 2 = \frac{\Theta^5 2}{\Theta^3 0} \frac{(\operatorname{sn}^2 3 - \operatorname{sn}^2 1)(\operatorname{sn} 3 - \operatorname{sn} 1)}{\operatorname{sn}^3 3}. \quad (9)$$

But $$\Theta 4 . \Theta 2 = \frac{\Theta^2 3 . \Theta^2 1}{\Theta^2 0}(1 - k^2 \operatorname{sn}^2 1 \operatorname{sn}^2 3),$$

which, by (8), $$= \frac{\Theta^2 3 . \Theta^2 1}{\Theta^2 0} k^2 \operatorname{sn} 1 \operatorname{sn} 3 (\operatorname{sn}^2 3 - \operatorname{sn}^2 1). \quad (10)$$

Thus, by (9), $$\frac{\Theta^2 3 . \Theta^2 1}{\Theta^2 0} = \frac{\Theta^5 2}{\Theta^3 0} \frac{(\operatorname{sn} 3 - \operatorname{sn} 1)}{k^2 \operatorname{sn} 1 \operatorname{sn}^4 3}. \quad (11)$$

Further $$\Theta 8 . \Theta 2 = \frac{\Theta^2 5 . \Theta^2 3}{\Theta^2 0}(1 - k^2 \operatorname{sn}^2 5 \operatorname{sn}^2 3),$$

which may be written

$$\Theta 8 . \Theta 2 = -\frac{\Theta^2 3}{\Theta^2 0} k \operatorname{sn}^2 3 . H^2(K'i), \quad (12)$$

since $$\Theta 5 = \Theta(K'i) = 0.$$

But* $$H(K'i) = iq^{-\frac{1}{4}} \sqrt{\left(\frac{2k'K}{\pi}\right)} = iq^{-\frac{1}{4}} \Theta(0);$$

therefore (12) becomes

$$\Theta 8 . \Theta 2 = \Theta^2 3 . kq^{-\frac{1}{2}} \operatorname{sn}^2 3. \quad (13)$$

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* Cayley's *Elliptic Functions*, 1895, p. 161.

Also, by (2), $$\Theta\{(n-2)K'i\} = -q^{n-1}\Theta(nK'i);$$

thus, putting $n = \frac{2}{5}$, we have

$$\Theta\left(\frac{8K'i}{5}\right) = -q^{-\frac{3}{5}}\Theta\left(\frac{2K'i}{5}\right),$$

that is $$\Theta 8 = -q^{-\frac{3}{5}}\Theta 2.$$

Substituting in (13) we find

$$\Theta^2 2 = -\Theta^2 3 . kq^{\frac{1}{5}} \operatorname{sn}^2 3. \tag{14}$$

Hence (7) may be written

$$\frac{\Theta 3 . \Theta 1}{\Theta^2 0} = -\frac{k^{-1}q^{-\frac{1}{5}}}{\operatorname{sn}^2 3 - \operatorname{sn}^2 1}. \tag{15}$$

Again, squaring (11) and dividing by $\Theta^4 0$, we have

$$\frac{\Theta^4 3 . \Theta^4 1}{\Theta^8 0} = \frac{\Theta^{10} 2}{\Theta^{10} 0} \frac{(\operatorname{sn} 3 - \operatorname{sn} 1)^2}{k^4 \operatorname{sn}^2 1 \operatorname{sn}^8 3},$$

which, by (14), $$= -\frac{\Theta^{10} 3}{\Theta^{10} 0} kq^{\frac{1}{2}} \frac{\operatorname{sn}^2 3}{\operatorname{sn}^2 1} (\operatorname{sn} 3 - \operatorname{sn} 1)^2. \tag{16}$$

Multiplying together (15) and (16), we obtain

$$\Theta^5 3 . \Theta^5 1 = \Theta^{10} 3 . q \frac{\operatorname{sn}^2 3 (\operatorname{sn} 3 - \operatorname{sn} 1)}{\operatorname{sn}^2 1 (\operatorname{sn} 3 + \operatorname{sn} 1)},$$

that is, $$\frac{\Theta^5 1}{\Theta^5 3} = q^{\frac{2}{5}} \frac{\operatorname{sn}^2 3 (\operatorname{sn} 3 - \operatorname{sn} 1)}{\operatorname{sn}^2 1 (\operatorname{sn} 3 + \operatorname{sn} 1)}. \tag{17}$$

In (4) let $$C_2 = \frac{\Theta 1}{\Theta 3} = \frac{(1-q^4)(1-q^8)(1-q^{14})(1-q^{16})\dots}{(1-q^2)(1-q^6)(1-q^{12})(1-q^{18})\dots}, \tag{18}$$

and let C_1 be what C_2 becomes when $q^{\frac{1}{2}}$ is written in place of q; thus

$$C_1 = \frac{(1-q^2)(1-q^3)(1-q^7)(1-q^8)\dots}{(1-q^1)(1-q^4)(1-q^5)(1-q^9)\dots}. \tag{19}$$

Hence, by (3), $$C_1 = \frac{\Theta 2 . \Theta 3}{\Theta 1 . \Theta 4}. \tag{20}$$

But, by (9), $$\frac{\Theta^2 2}{\Theta^2 4} = \frac{\Theta^6 0}{\Theta^6 2} \frac{\operatorname{sn}^6 3}{(\operatorname{sn}^2 3 - \operatorname{sn}^2 1)^2 (\operatorname{sn} 3 - \operatorname{sn} 1)^2},$$

which, by (7), $$= \frac{\Theta^3 1}{\Theta^3 3} \frac{(\operatorname{sn}^2 3 - \operatorname{sn}^2 1)^3}{\operatorname{sn}^6 3} \frac{\operatorname{sn}^6 3}{(\operatorname{sn}^2 3 - \operatorname{sn}^2 1)^2 (\operatorname{sn} 3 - \operatorname{sn} 1)^2}$$

$$= \frac{\Theta^3 1}{\Theta^3 3} \frac{\operatorname{sn} 3 + \operatorname{sn} 1}{\operatorname{sn} 3 - \operatorname{sn} 1}.$$

Thus, by (20), $C_1^2 = \dfrac{\Theta^2 2}{\Theta^2 4}\dfrac{\Theta^2 3}{\Theta^2 1} = \dfrac{\Theta 1}{\Theta 3}\dfrac{\operatorname{sn} 3+\operatorname{sn} 1}{\operatorname{sn} 3-\operatorname{sn} 1}$;

therefore $C_1^{10} = \dfrac{\Theta^5 1}{\Theta^5 3}\left(\dfrac{\operatorname{sn} 3+\operatorname{sn} 1}{\operatorname{sn} 3-\operatorname{sn} 1}\right)^5$;

so that, by (17),

$$C_1^{10} = q^{\frac{1}{2}}\frac{\operatorname{sn}^2 3\,(\operatorname{sn} 3+\operatorname{sn} 1)^4}{\operatorname{sn}^2 1\,(\operatorname{sn} 3-\operatorname{sn} 1)^4},$$

that is, $$C_1^5 = q^{\frac{1}{2}}\frac{\operatorname{sn} 3\,(\operatorname{sn} 3+\operatorname{sn} 1)^2}{\operatorname{sn} 1\,(\operatorname{sn} 3-\operatorname{sn} 1)^2}. \tag{21}$$

Since, by (18), $C_2 = \dfrac{\Theta 1}{\Theta 3}$, equation (17) may be written

$$C_2^5 = q^{\frac{2}{5}}\frac{\operatorname{sn}^2 3\,(\operatorname{sn} 3-\operatorname{sn} 1)}{\operatorname{sn}^2 1\,(\operatorname{sn} 3+\operatorname{sn} 1)}. \tag{22}$$

From (21) and (22), $$C_1^5 C_2^{10} = q\frac{\operatorname{sn}^5 3}{\operatorname{sn}^5 1},$$

therefore $$C_1 C_2^2 = q^{\frac{1}{5}}\frac{\operatorname{sn} 3}{\operatorname{sn} 1}. \tag{23}$$

Again, from (21) and (22), $$\frac{C_1^2}{C_2} = \frac{\operatorname{sn} 3+\operatorname{sn} 1}{\operatorname{sn} 3-\operatorname{sn} 1}. \tag{24}$$

Hence $$\frac{\operatorname{sn} 1}{\operatorname{sn} 3} = \frac{C_1^2-C_2}{C_1^2+C_2},$$

and therefore, by (23), $$q^{\frac{1}{5}} = C_1 C_2^2\frac{C_1^2-C_2}{C_1^2+C_2}. \tag{25}$$

Mr. Ramanujan writes

$$f(x) = x^{\frac{1}{5}}\frac{(1-x)(1-x^4)(1-x^6)(1-x^9)\dots}{(1-x^2)(1-x^3)(1-x^7)(1-x^8)\dots}; \tag{26}$$

so that if we make $q = x^5$, we have, by (18) and (19),

$$C_1 = x^{\frac{1}{5}}/f(x), \qquad C_2 = x^{\frac{2}{5}}/f(x^2),$$

and then (25) may be written

$$\{f(x)\}^2 - f(x^2) + f(x)\{f(x^2)\}^2[\{f(x)\}^2 + f(x^2)] = 0, \tag{27}$$

which is the form in which the result in (25) was enunciated by Mr. Ramanujan.

It may be mentioned that Prof. L. J. Rogers* has shown that the expression (26) is identical with the continued fraction

$$\cfrac{x^{\frac{1}{5}}}{1+\cfrac{x}{1}+\cfrac{x^2}{1}+\cfrac{x^3}{1}+\cdots}$$

and that a second proof of this identity by Prof. Rogers as well as an independent proof by Mr. Ramanujan have been published recently.†

Now, by (15),

$$\mathrm{sn}^2 1-\mathrm{sn}^2 3 = k^{-1}q^{-\frac{1}{2}}\frac{\Theta^2 0}{\Theta 1.\Theta 3} = 2A, \tag{28}$$

suppose, and from (10) and (15),

$$\mathrm{sn}^2 1\,\mathrm{sn}^2 3 = k^{-2}q^{\frac{1}{2}}\frac{\Theta^2 2.\Theta^2 4}{\Theta^2 1.\Theta^2 3} = 4B, \tag{29}$$

suppose. Then, multiplying (28) by $-\mathrm{sn}^2 3$, we obtain

$$\mathrm{sn}^4 3+2A\,\mathrm{sn}^2 3-4B = 0,$$

whence $$\mathrm{sn}^2 3 = -A \pm \sqrt{(A^2+4B)},$$

and therefore, by (29), $\mathrm{sn}^2 1 = A \pm \sqrt{(A^2+4B)}$.

Since sn 3 and sn 1 are both imaginary, we see that the lower sign must be taken; thus

$$\mathrm{sn}^2 3 = -A-\sqrt{(A^2+4B)},$$

$$\mathrm{sn}^2 1 = \quad A-\sqrt{(A^2+4B)},$$

whence

$$\mathrm{sn}\,3 = i\left[\sqrt{\tfrac{1}{2}}\left\{\sqrt{(A^2+4B)}+\sqrt{(4B)}\right\}+\sqrt{\tfrac{1}{2}}\left\{\sqrt{(A^2+4B)}-\sqrt{(4B)}\right\}\right],$$

$$\mathrm{sn}\,1 = i\left[\sqrt{\tfrac{1}{2}}\left\{\sqrt{(A^2+4B)}+\sqrt{(4B)}\right\}-\sqrt{\tfrac{1}{2}}\left\{\sqrt{(A^2+4B)}-\sqrt{(4B)}\right\}\right],$$

and from these equations it follows that

$$\frac{\mathrm{sn}^2 3}{\mathrm{sn}^2 1}\,\frac{\mathrm{sn}\,3-\mathrm{sn}\,1}{\mathrm{sn}\,3+\mathrm{sn}\,1} = \frac{-A-\sqrt{(A^2+4B)}}{+A-\sqrt{(A^2+4B)}}\,\frac{\sqrt{\left\{\sqrt{(A^2+4B)}-\sqrt{(4B)}\right\}}}{\sqrt{\left\{\sqrt{(A^2+4B)}+\sqrt{(4B)}\right\}}}$$

$$= \frac{\left\{\sqrt{(A^2+4B)}+A\right\}^2\left\{\sqrt{(A^2+4B)}-\sqrt{(4B)}\right\}}{4AB}. \tag{30}$$

Now, by (28) and (29),

$$\frac{A^2}{4B} = \tfrac{1}{4}q^{-\frac{1}{2}}\frac{\Theta^4 0}{\Theta^2 2.\Theta^2 4}; \tag{31}$$

* *Proc. London Math. Soc.*, Ser. 1, Vol. 25 (1894), p. 329.

† *Proc. Camb. Phil. Soc.*, Vol. 19, p. 211.

so that by (3), since

$$\Theta(0) = \{(1-q)(1-q^3)(1-q^5)\ldots\}^2(1-q^2)(1-q^4)(1-q^6)\ldots,$$

we have $$\frac{A^2}{4B} = \tfrac{1}{4}q^{-\frac{2}{3}}\frac{\{(1-q)(1-q^3)(1-q^5)\ldots\}^{10}}{\{(1-q^{\frac{1}{3}})(1-q^{\frac{3}{3}})(1-q^{\frac{5}{3}})\ldots\}^2}.$$

Let $$A^2/4B = \nu^2, \tag{32}$$

so that $$\nu = \tfrac{1}{2}q^{-\frac{1}{3}}\frac{\{(1-q)(1-q^3)(1-q^5)\ldots\}^5}{\{(1-q^{\frac{1}{3}})(1-q^{\frac{3}{3}})(1-q^{\frac{5}{3}})\ldots\}}; \tag{33}$$

then the right-hand side of (30) becomes

$$\frac{\{\sqrt{(1+\nu^2)}+\nu\}^2\{\sqrt{(1+\nu^2)}-1\}}{\nu},$$

and therefore, by (17),

$$\frac{\Theta^5 1}{\Theta^5 3} = q^{\frac{2}{3}}\frac{\{\sqrt{(1+\nu^2)}+\nu\}^2\{\sqrt{(1+\nu^2)}-1\}}{\nu}. \tag{34}$$

Adopting Mr. Ramanujan's notation as shown in (26), we see by (4) that (34) may be written

$$\{f(q^{\frac{2}{3}})\}^{-5} = \frac{\{\sqrt{(1+\nu^2)}+\nu\}^2\{\sqrt{(1+\nu^2)}-1\}}{\nu}, \tag{35}$$

and therefore

$$\{f(q^{\frac{2}{3}})\}^{-5} - \{f(q^{\frac{1}{3}})\}^5$$

$$= \frac{\{\sqrt{(1+\nu^2)}+\nu\}^2\{\sqrt{(1+\nu^2)}-1\}}{\nu} - \frac{\{\sqrt{(1+\nu^2)}-\nu\}^2\{\sqrt{(1+\nu^2)}+1\}}{\nu}$$

$$= \frac{1}{\nu}(4\nu^3-4\nu^2+4\nu-2) = \frac{1}{\nu}\{(\nu-2)(1+2\nu)^2+11\nu\},$$

whence $$\{f(q^{\frac{2}{3}})\}^{-5} - \{f(q^{\frac{1}{3}})\}^5 - 11 = \frac{(\nu-2)(1+2\nu)^2}{\nu}. \tag{36}$$

Again, by (28) and (29), the last equation of (8) may be written

$$\sqrt{(4B)}\,2A+4B = \frac{1}{k^2},$$

that is, by (32), $$2\nu+1 = \frac{1}{4k^2B},$$

whence $$4B = 1/k^2(1+2\nu), \tag{37}$$

and therefore, by (32), $$A^2 = \frac{\nu^2}{k^2(1+2\nu)}. \tag{38}$$

Since $$\operatorname{sn} 4 = \frac{2 \operatorname{sn} 2 \operatorname{cn} 2 \operatorname{dn} 2}{1-k^2 \operatorname{sn}^4 2}, \tag{39}$$

and $\operatorname{sn} 2 = -\operatorname{ns} 3/k$, $\operatorname{cn} 2 = i \operatorname{ds} 3/k$, $\operatorname{dn} 2 = i \operatorname{cs} 3$, also $\operatorname{sn} 4 = -\operatorname{ns} 1/k$, we have, from (39) and (8),

$$2 \operatorname{sn} 1 \operatorname{cn} 3 \operatorname{dn} 3 = k(\operatorname{sn}^2 3 - \operatorname{sn}^2 1)(\operatorname{sn} 1 - \operatorname{sn} 3),$$

whence, squaring,

$$4 \operatorname{sn}^2 1 (1-\operatorname{sn}^2 3)(1-k^2 \operatorname{sn}^2 3) = k^2 (\operatorname{sn}^2 3 - \operatorname{sn}^2 1)^2 (\operatorname{sn} 1 - \operatorname{sn} 3)^2. \tag{40}$$

Again, $$\frac{2 \operatorname{sn} 1 \operatorname{cn} 1 \operatorname{dn} 1}{1-k^2 \operatorname{sn}^4 1} = \operatorname{sn} 2 = -\frac{1}{k} \operatorname{ns} 3,$$

and therefore, by the second equation of (8),

$$2 \operatorname{sn} 3 \operatorname{cn} 1 \operatorname{dn} 1 = -k(\operatorname{sn}^2 3 - \operatorname{sn}^2 1)(\operatorname{sn} 1 + \operatorname{sn} 3), \tag{41}$$

whence, squaring,

$$4 \operatorname{sn}^2 3 (1-\operatorname{sn}^2 1)(1-k^2 \operatorname{sn}^2 1) = k^2 (\operatorname{sn}^2 3 - \operatorname{sn}^2 1)^2 (\operatorname{sn} 1 + \operatorname{sn} 3)^2. \tag{42}$$

Adding (40) and (42), we have

$$4\{\operatorname{sn}^2 1 + \operatorname{sn}^2 3 - 2 \operatorname{sn}^2 1 \operatorname{sn}^2 3 - 2k^2 \operatorname{sn}^2 1 \operatorname{sn}^2 3 + k^2 \operatorname{sn}^2 1 \operatorname{sn}^2 3 (\operatorname{sn}^2 3 + \operatorname{sn}^2 1)\}$$
$$= 2k^2 (\operatorname{sn}^2 3 - \operatorname{sn}^2 1)^2 (\operatorname{sn}^2 3 + \operatorname{sn}^2 1). \tag{43}$$

Now since $\operatorname{sn}^2 1$ and $\operatorname{sn}^2 3$ are both negative, we have by (28) and (29),

$$\sqrt{(A^2+4B)} = -\tfrac{1}{2}(\operatorname{sn}^2 3 + \operatorname{sn}^2 1),$$

and consequently (43) may be written

$$-2\sqrt{(A^2+4B)}\{1+4k^2B\} - 8(1+k^2)B = -4k^2A^2\sqrt{(A^2+4B)};$$

that is, substituting for A and B from (37) and (38),

$$\frac{1}{k}\sqrt{\left(\frac{1+\nu^2}{1+2\nu}\right)}\frac{2+2\nu}{1+2\nu} + \frac{1+k^2}{k^2(1+2\nu)} = \frac{2\nu^2}{1+2\nu}\frac{1}{k}\sqrt{\left(\frac{1+\nu^2}{1+2\nu}\right)},$$

whence $$(\nu^2-\nu-1)\sqrt{\left(\frac{1+\nu^2}{1+2\nu}\right)} = \frac{1+k^2}{2k}; \tag{44}$$

so that $$(1+\nu^2)(1+\nu-\nu^2)^2 = \frac{(1+k^2)^2}{4k^2}(1+2\nu);$$

that is, $$(1+2\nu)+\nu^5(\nu-2) = \frac{(1+k^2)^2}{4k^2}(1+2\nu),$$

and therefore $$\nu^5(\nu-2) = (1+2\nu)\,\frac{k'^4}{4k^2}, \tag{45}$$

whence $$\frac{(\nu-2)(1+2\nu)^2}{\nu} = \frac{(1+2\nu)^3}{\nu^6}\,\frac{k'^4}{4k^2}, \tag{46}$$

which, by (38), $$= k'^4/4A^6k^8. \tag{47}$$

Also, by (28), $$4A^2 = k^{-2}q^{-\frac{1}{2}}\frac{\Theta^4 0}{\Theta^2 1\,.\,\Theta^2 3};$$

that is, by aid of (3),

$$4A^2 = k^{-2}q^{-\frac{1}{2}}\,\frac{\{(1-q)(1-q^3)(1-q^5)\dots\}^8\,\{(1-q^2)(1-q^4)(1-q^6)\dots\}^2}{\{(1-q^{\frac{1}{2}})(1-q^{\frac{3}{2}})(1-q^{\frac{5}{2}})\dots\}^2};$$

or, writing for brevity,

$$\Pi_n = (1-q^n)(1-q^{2n})(1-q^{3n})\dots \tag{48}$$

$$4A^2 = k^{-2}q^{-\frac{1}{2}}\,\frac{\Pi_2^2}{\Pi_{\frac{1}{2}}^2}\,\{(1-q)(1-q^3)(1-q^5)\dots\}^8. \tag{49}$$

Again,* $$\frac{k'^4}{k^2} = \tfrac{1}{16}q^{-1}\{1-q)(1-q^3)(1-q^5)\dots\}^{24}, \tag{50}$$

and, by (49), $$\frac{1}{4A^6k^6} = 16q^3\,\frac{\Pi_{\frac{1}{2}}^6}{\Pi_2^6}\,\frac{1}{\{(1-q)(1-q^3)(1-q^5)\dots\}^{24}};$$

hence, by (50), the right-hand side of (47),

$$= q^{-\frac{3}{2}}\,\frac{\Pi_{\frac{1}{2}}^6}{\Pi_2^6};$$

so that

$$\frac{(\nu-2)(1+2\nu)^2}{\nu} = q^{-\frac{3}{2}}\,\frac{\Pi_{\frac{1}{2}}^6}{\Pi_2^6},$$

and therefore, by (36),

$$\{f(q^{\frac{1}{2}})\}^{-5} - \{f(q^{\frac{1}{2}})\}^5 - 11 = q^{-\frac{3}{2}}\,\frac{\Pi_{\frac{1}{2}}^6}{\Pi_2^6},$$

whence, writing x for $q^{\frac{1}{2}}$, and taking account of (48),

$$\{f(x)\}^{-5} - \{f(x)\}^5 - 11 = \frac{1}{x}\left\{\frac{(1-x)(1-x^2)(1-x^3)\dots}{(1-x^5)(1-x^{10})(1-x^{15})\dots}\right\}^6, \tag{51}$$

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* Cayley, *l.c.*, p. 290.

where $f(x)$ is given by (26); and this is the form in which the identity was enunciated by Mr. Ramanujan in 1914. Quite recently* he has stated the same identity in a different form, viz.,

$$H(x)\{G(x)\}^{11}-x^2G(x)\{H(x)\}^{11}=1+11x\{G(x)H(x)\}^6, \qquad (51a)$$

where
$$G(x)=\frac{1}{(1-x)(1-x^4)(1-x^6)(1-x^9)\ldots},$$

$$H(x)=\frac{1}{(1-x^2)(1-x^3)(1-x^7)(1-x^8)\ldots},$$

which is readily seen to be equivalent to (51).

Next, taking the log of (35), we have

$$-5\log f(q^5)=2\log\{\sqrt{(1+\nu^2)}+\nu\}+\log\{\sqrt{(1+\nu^2)}-1\}-\log\nu,$$

and therefore, differentiating with respect to ν,

$$-2q^{-5}\frac{f'(q^5)}{f(q^5)}\frac{dq}{d\nu}=\frac{2}{\sqrt{(1+\nu^2)}}+\frac{\nu}{\sqrt{(1+\nu^2)}\{\sqrt{(1+\nu^2)}-1\}}-\frac{1}{\nu}=\frac{2\nu+1}{\nu\sqrt{(1+\nu^2)}};$$

so that
$$\frac{f'(q^5)}{f(q^5)}=-\tfrac{1}{2}q^5\frac{1+2\nu}{\nu\sqrt{(1+\nu^2)}}\frac{d\nu}{dk}\frac{dk}{dq}. \qquad (52)$$

Again (45) may be written

$$\frac{\nu^6-2\nu^5}{1+2\nu}=\tfrac{1}{4}\left(\frac{1}{k^2}-2+k^2\right),$$

whence, differentiating, we have

$$\frac{10\nu^4(\nu^2-\nu-1)}{(1+2\nu)^2}\frac{d\nu}{dk}=-\tfrac{1}{2}\frac{1-k^4}{k^3};$$

and therefore, by (44),

$$\frac{d\nu}{dk}=-\tfrac{1}{10}\frac{k'^5}{k^2}\frac{\sqrt{\{(1+\nu^2)(1+2\nu)^3\}}}{\nu^4}.$$

Also since
$$\log q=-\pi\frac{K'}{K},$$

we have
$$\frac{dk}{dq}=\frac{2kk'^2K^2}{\pi^2q}.$$

* *Proc. London Math. Soc.*, Ser. 2, Vol. 18, p. xx.

Hence (52) becomes

$$\frac{f'(q^{\frac{2}{5}})}{f(q^{\frac{2}{5}})} = \frac{q^{-\frac{2}{5}}}{10\pi^2}\,\frac{k'^4K^2}{k}\,\frac{(1+2\nu)^{\frac{5}{2}}}{\nu^5},$$

which, by (38),

$$= \frac{q^{-\frac{2}{5}}}{10\pi^2}\,\frac{k'^4K^2}{k^6A^5} = \frac{q^{-\frac{2}{5}}}{40}\,\frac{k'^2\Theta^4 0}{k^6A^5},$$

and this, by (28),

$$= \tfrac{4}{5}q^{\frac{1}{10}}\,\frac{k'^2}{k}\,\frac{\Theta^5 1.\Theta^5 3}{\Theta^6 0},$$

or, by (3) and (50),

$$= \frac{q^{-\frac{2}{5}}}{5}\,\frac{\{(1-q)(1-q^3)(1-q^5)\ldots\}^{12}\,\Pi_{\frac{2}{5}}^5\,\Pi_2^5}{\{(1-q)(1-q^3)(1-q^5)\ldots\}^{12}\,\Pi_2^6};$$

so that

$$\frac{f'(q^{\frac{2}{5}})}{f(q^{\frac{2}{5}})} = \tfrac{1}{5}q^{-\frac{2}{5}}\,\frac{\{(1-q^{\frac{2}{5}})(1-q^{\frac{4}{5}})(1-q^{\frac{6}{5}})\ldots\}^5}{(1-q^2)(1-q^4)(1-q^6)\ldots};$$

whence, writing x for $q^{\frac{2}{5}}$,

$$\frac{f'(x)}{f(x)} = \frac{1}{5x}\,\frac{\{(1-x)(1-x^2)(1-x^3)(1-x^4)\ldots\}^5}{(1-x^5)(1-x^{10})(1-x^{15})(1-x^{20})\ldots}, \tag{53}$$

where $f(x)$ is given by (26); and this is the form in which the identity was enunciated by Mr. Ramanujan.

Next, consider the well-known expansion, due to Euler :—

$$(1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5)\ldots$$
$$= 1-x-x^2+x^5+x^7-x^{12}-x^{15}+x^{22}+x^{26}-x^{35}-x^{40}+x^{51}+x^{57}-x^{70}-x^{77}$$
$$+x^{92}+x^{100}-x^{117}-x^{126}+x^{145}+x^{155}-x^{176}-x^{187}+x^{210}+x^{222}-\ldots,$$

the law of the series being that the sign of the m-th pair of terms is $(-)^m$, the sum of their indices is $3m^2$, and the difference between their indices is m. It is readily seen that the series may be arranged in the form

$$1-x^{35}-x^{40}+x^{145}+x^{155}-x^{330}-x^{345}+\ldots$$
$$+x^5(1-x^{10}-x^{65}+x^{95}+x^{205}-x^{255}-x^{420}+\ldots)$$
$$-x(1-x^{25}-x^{50}+x^{125}+x^{175}-x^{300}-x^{375}+\ldots)$$
$$-x^2(1-x^{20}-x^{55}+x^{115}+x^{185}-x^{285}-x^{390}+\ldots)$$
$$+x^7(1-x^5-x^{70}+x^{85}+x^{215}-x^{240}-x^{435}+\ldots).$$

Thus writing $x = q^{\frac{1}{5}}$, and adopting Cayley's notation, viz.

$$(1-q^a)(1-q^b)(1-q^c)\ldots \equiv (a)(b)(c)\ldots,$$

we have

$$\begin{aligned}
&(\tfrac{1}{5})(\tfrac{2}{5})(\tfrac{3}{5})(\tfrac{4}{5})\ldots \\
&= \{(7)(8)(22)(23)(37)(38)\ldots + q(2)(13)(17)(28)(32)\ldots\}(15)(30)(45)\ldots \\
&\quad -q^{\frac{1}{5}}(5)(10)(15)\ldots \\
&\quad -q^{\frac{2}{5}}\{(4)(11)(19)(26)\ldots - q(1)(14)(16)(29)\ldots\}(15)(30)(45)\ldots ; \qquad (54)
\end{aligned}$$

which we will write

$$(\tfrac{1}{5})(\tfrac{2}{5})(\tfrac{3}{5})(\tfrac{4}{5})\ldots = A - q^{\frac{1}{5}}B - q^{\frac{2}{5}}C, \qquad (55)$$

where A, B, and C involve no fractional powers of q.

Cubing both sides of (55), we obtain

$$\begin{aligned}
&\{(\tfrac{1}{5})(\tfrac{2}{5})(\tfrac{3}{5})(\tfrac{4}{5})\ldots\}^3 \\
&= (A^3 - 3qBC^2) - q^{\frac{1}{5}}(3A^2B + qC^3) + 3q^{\frac{2}{5}}(AB^2 - A^2C) + q^{\frac{3}{5}}(6ABC - B^3) \\
&\qquad + 3q^{\frac{4}{5}}(AC^2 - B^2C), \qquad (56)
\end{aligned}$$

But by Jacobi's expansion

$$\begin{aligned}
&\{(1-x)(1-x^2)(1-x^3)(1-x^4)\ldots\}^3 \\
&= 1 - 3x + 5x^3 - 7x^6 + 9x^{10} - 11x^{15} + 13x^{21} - 15x^{28} + 17x^{36} - 19x^{45} \\
&\quad + 21x^{55} - \ldots,
\end{aligned}$$

in which the indices of x are of the form $\frac{m(m+1)}{2}$. Writing $x = q^{\frac{1}{5}}$, we obtain from this expansion

$$\begin{aligned}
&\{(\tfrac{1}{5})(\tfrac{2}{5})(\tfrac{3}{5})(\tfrac{4}{5})\ldots\}^3 \\
&= 1 + 9q^2 - 11q^3 - 19q^9 + 21q^{11} + 29q^{21} - 31q^{24} - \ldots \\
&\quad - q^{\frac{1}{5}}(3 + 7q - 13q^4 - 17q^7 + 23q^{13} + 27q^{18} - \ldots) \\
&\quad + 5q^{\frac{2}{5}}(1 - 3q^5 + 5q^{15} - 7q^{30} + 9q^{50} - 11q^{75} + \ldots). \qquad (57)
\end{aligned}$$

Comparing (56) and (57), we see that

$$AC = B^2, \qquad (58)$$

while, from (54) and (55),

$$B = (5)(10)(15)\ldots \qquad (59)$$

Now denote the first series on the right-hand side of (57) by r, so that

$$r = 1+9q^2-11q^3-19q^9+21q^{11}+29q^{21}-31q^{24}-\ldots, \tag{60}$$

and let $$s = 1-q^2-q^3+q^9+q^{11}-q^{21}-q^{24}+\ldots; \tag{61}$$

then $$r-s = 10q^2-10q^3-20q^9+20q^{11}+30q^{21}-30q^{24}-\ldots:$$

and therefore

$$s-r = 10\{q^{\frac{1}{2}}(q^{\frac{1}{2}}-q^{-\frac{1}{2}})-2q^{\frac{9}{2}}(q^{\frac{3}{2}}-q^{-\frac{3}{2}})+3q^{\frac{25}{2}}(q^{\frac{5}{2}}-q^{-\frac{5}{2}})-\ldots\},$$

and if in the terms within single brackets we write

$$q^{\frac{1}{2}} = e^{2xi}, \tag{62}$$

this becomes

$$s-r = 10i\,(2q^{\frac{1}{2}}\sin 2x-4q^{\frac{9}{2}}\sin 4x+6q^{\frac{25}{2}}\sin 6x-\ldots),$$

or, putting $$p = q^{\frac{5}{2}} = e^{10xi}, \tag{63}$$

$$s-r = 5i\,\frac{d}{dx}\,\Theta\left(\frac{2Kx}{\pi}\right)$$

where $$p = e^{-\pi K'/K}, \tag{64}$$

and thus $$s-r = \frac{10iK}{\pi}\,\Theta\left(\frac{2Kx}{\pi}\right)\,Z\left(\frac{2Kx}{\pi}\right);$$

that is, by (63) and (64),

$$s-r = \frac{10iK}{\pi}\,\Theta\left(\frac{iK'}{5}\right)\,Z\left(\frac{iK'}{5}\right). \tag{65}$$

Again* $$Zu+Zv-Z(u+v) = k^2\,\mathrm{sn}\,u\,\mathrm{sn}\,v\,\mathrm{sn}(u+v) \tag{66}$$

and

$$\frac{\mathrm{cn}\,u\,\mathrm{dn}\,u}{\mathrm{sn}\,u} = \frac{i\pi}{2K}+Z(u+iK')-Zu; \tag{67}$$

thus, by (66),

$$-2Zu-Z(2u) = k^2\,\mathrm{sn}^2\,u\,\mathrm{sn}\,2u,$$

and $$2Z(2u)-Z(4u) = k^2\,\mathrm{sn}^2\,2u\,\mathrm{sn}\,4u,$$

whence $$4Zu-Z(4u) = 2k^2\,\mathrm{sn}^2\,u\,\mathrm{sn}\,2u+k^2\,\mathrm{sn}^2\,2u\,\mathrm{sn}\,4u. \tag{68}$$

* Cayley, *l.c.*, pp. 145 and 149.

Now, in (67), let $u = -4iK'/5$, then

$$-\frac{\operatorname{cn}\frac{4iK'}{5}\operatorname{dn}\frac{4iK'}{5}}{\operatorname{sn}\frac{4iK'}{5}} = \frac{i\pi}{2K} + Z\left(\frac{iK'}{5}\right) + Z\left(\frac{4iK'}{5}\right);$$

and in (68), let $u = iK'/5$, then

$$2k^2\operatorname{sn}^2\frac{iK'}{5}\operatorname{sn}\frac{2iK'}{5} + k^2\operatorname{sn}^2\frac{2iK'}{5}\operatorname{sn}\frac{4iK'}{5} = 4Z\left(\frac{iK'}{5}\right) - Z\left(\frac{4iK'}{5}\right),$$

and therefore, by addition, we have, adopting our earlier notation,

$$5Z\left(\frac{iK'}{5}\right) + \frac{i\pi}{2K} = -\frac{\operatorname{cn} 4\operatorname{dn} 4}{\operatorname{sn} 4} + 2k^2\operatorname{sn}^2 1\operatorname{sn} 2 + k^2\operatorname{sn}^2 2\operatorname{sn} 4,$$

which, since

$$\operatorname{sn}\frac{4iK'}{5} = -\frac{1}{k}\operatorname{ns}\frac{iK'}{5}, \quad \operatorname{cn}\frac{4iK'}{5} = \frac{i}{k}\operatorname{ds}\frac{iK'}{5}, \quad \operatorname{dn}\frac{4iK'}{5} = i\operatorname{cs}\frac{iK'}{5}, \quad \ldots,$$

becomes

$$5Z\left(\frac{iK'}{5}\right) + \frac{i\pi}{2K} = -\frac{\operatorname{cn} 1\operatorname{dn} 1}{\operatorname{sn} 1} - 2k\frac{\operatorname{sn}^2 1}{\operatorname{sn} 3} - \frac{1}{k\operatorname{sn} 1\operatorname{sn}^2 3},$$

which, by (41),

$$= \frac{k(\operatorname{sn}^2 3 - \operatorname{sn}^2 1)(\operatorname{sn} 1 + \operatorname{sn} 3)}{2\operatorname{sn} 1\operatorname{sn} 3} - 2k\frac{\operatorname{sn}^2 1}{\operatorname{sn} 3} - \frac{1}{k\operatorname{sn} 1\operatorname{sn}^2 3}$$

$$= \frac{k\left(\operatorname{sn} 1\operatorname{sn}^3 3 - \operatorname{sn} 3\operatorname{sn}^3 1 + \operatorname{sn}^4 3 - \operatorname{sn}^2 1\operatorname{sn}^2 3 - 4\operatorname{sn} 3\operatorname{sn}^3 1 - \frac{2}{k^2}\right)}{2\operatorname{sn} 1\operatorname{sn}^2 3}$$

and this, by (8),

$$= \frac{k}{2\operatorname{sn} 1\operatorname{sn}^2 3}(-\operatorname{sn} 1\operatorname{sn}^3 3 + \operatorname{sn}^4 3 - 3\operatorname{sn}^2 1\operatorname{sn}^2 3 - 3\operatorname{sn} 3\operatorname{sn}^3 1)$$

$$= \frac{k}{2\operatorname{sn} 1\operatorname{sn} 3}\{\operatorname{sn}^2 3(\operatorname{sn} 3 - \operatorname{sn} 1) - 3\operatorname{sn}^2 1(\operatorname{sn} 1 + \operatorname{sn} 3)\};$$

so that

$$5Z\left(\frac{iK'}{5}\right) + \frac{i\pi}{2K} = \frac{k\operatorname{sn} 1(\operatorname{sn} 3 + \operatorname{sn} 1)}{2\operatorname{sn} 3}\left\{\frac{\operatorname{sn}^2 3(\operatorname{sn} 3 - \operatorname{sn} 1)}{\operatorname{sn}^2 1(\operatorname{sn} 3 + \operatorname{sn} 1)} - 3\right\}. \quad (69)$$

But, multiplying together (15) and (17), we have

$$\frac{\Theta^6 1}{\Theta^2 0.\Theta^4 3} = -\frac{\mathrm{sn}^2 3}{\mathrm{sn}^2 1}\,\frac{k^{-1}p^{\frac{3}{10}}}{(\mathrm{sn}\,3+\mathrm{sn}\,1)^2},$$

whence

$$\frac{\mathrm{sn}\,1\,(\mathrm{sn}\,3+\mathrm{sn}\,1)}{\mathrm{sn}\,3} = ik^{-\frac{1}{2}}p^{\frac{3}{20}}\frac{\Theta(0).\Theta^2 3}{\Theta^3 1};$$

also by (22) and (18),

$$\frac{\mathrm{sn}^2 3(\mathrm{sn}\,3-\mathrm{sn}\,1)}{\mathrm{sn}^2 1(\mathrm{sn}\,3+\mathrm{sn}\,1)} = p^{-\frac{2}{5}}\frac{\Theta^5 1}{\Theta^5 3};$$

thus (69) becomes

$$5Z\left(\frac{iK'}{5}\right)+\frac{i\pi}{2K} = \frac{i}{2}k^{\frac{1}{2}}p^{\frac{3}{20}}\frac{\Theta(0)}{\Theta 1}\frac{\Theta^2 3}{\Theta^2 1}\left(p^{-\frac{2}{5}}\frac{\Theta^5 1}{\Theta^5 3}-3\right),$$

and therefore since $\Theta 1 = \Theta(iK'/5)$, we see that

$$\frac{10iK}{\pi}\Theta\left(\frac{iK'}{5}\right)Z\left(\frac{iK'}{5}\right) = -\frac{k^{\frac{1}{2}}K}{\pi}p^{\frac{3}{20}}\Theta(0)\frac{\Theta^2 3}{\Theta^2 1}\left(p^{-\frac{2}{5}}\frac{\Theta^5 1}{\Theta^5 3}-3\right)+\Theta\left(\frac{iK'}{5}\right).$$

Comparing this with (65), we have

$$s-r = \Theta\left(\frac{iK'}{5}\right)-\frac{k^{\frac{1}{2}}K}{\pi}p^{\frac{3}{20}}\Theta(0)\frac{\Theta^2 3}{\Theta^2 1}\left(p^{-\frac{2}{5}}\frac{\Theta^5 1}{\Theta^5 3}-3\right).$$

Now $\Theta(0) = \sqrt{(2k'K/\pi)}$, hence

$$r = s-\Theta\left(\frac{iK'}{5}\right)+\frac{2^{\frac{1}{2}}k^{\frac{1}{2}}k'^{\frac{1}{2}}K^{\frac{3}{2}}}{\pi^{\frac{3}{2}}}p^{\frac{3}{20}}\frac{\Theta^2 3}{\Theta^2 1}\left(p^{-\frac{2}{5}}\frac{\Theta^5 1}{\Theta^5 3}-3\right),$$

and substituting for $k^{\frac{1}{2}}k'^{\frac{1}{2}}K^{\frac{3}{2}}$,* we obtain

$$r = s-\Theta\left(\frac{iK'}{5}\right)+p^{\frac{3}{4}}\Pi_2^3\frac{\Theta^2 3}{\Theta^2 1}\left(p^{-\frac{2}{5}}\frac{\Theta^5 1}{\Theta^5 3}-3\right), \tag{70}$$

where $$\Pi_2 = (1-p^2)(1-p^4)(1-p^6)\ldots.$$

Again, by (3),

$$\Theta\left(\frac{iK'}{5}\right) = \Pi_2(1-p^{\frac{1}{5}})(1-p^{\frac{2}{5}})(1-p^{\frac{3}{5}})(1-p^{\frac{4}{5}})\ldots,$$

and also

$$\frac{\Theta 1}{\Theta 3} = \frac{(1-p^{\frac{1}{2}})(1-p^{\frac{2}{2}})(1-p^{\frac{3}{2}})(1-p^{\frac{4}{2}})\ldots}{(1-p^{\frac{1}{1}})(1-p^{\frac{2}{1}})(1-p^{\frac{3}{1}})(1-p^{\frac{4}{1}})\ldots};$$

* Cayley, *l.c.*, p. 290.

whence replacing p by q^5 from (63), and putting

$$\mu = \frac{(1-q^2)(1-q^3)(1-q^7)(1-q^8)\dots}{(1-q)(1-q^4)(1-q^6)(1-q^9)\dots}, \tag{71}$$

we see that (70) becomes

$$r = s - \Pi_5(1-q^2)(1-q^3)(1-q^7)(1-q^8)\dots\dots + \Pi_5^3 \frac{q}{\mu^2}\left(\frac{\mu^5}{q} - 3\right), \tag{72}$$

where

$$\Pi_n = (1-q^n)(1-q^{2n})(1-q^{3n})\dots \textit{ad inf.} \tag{73}$$

Now, by (61),

$$s = 1-q^2-q^3+q^9+q^{11}-\dots = \Pi_5(1-q^2)(1-q^3)(1-q^7)(1-q^8)\dots.$$

Hence we have, by (72) and (60),

$$r = 1+9q^2-11q^3-19q^9+21q^{11}+\dots = \Pi_5^3 \frac{q}{\mu^2}\left(\frac{\mu^5}{q} - 3\right). \tag{74}$$

But, comparing the two identities (56) and (57), we see that

$$r = A^3 - 3qBC^2 = A^3 - 3q\Pi_5^5/A^2,$$

by (58) and (59). Hence, by (74),

$$A^3 - 3q\frac{\Pi_5^5}{A^2} = \mu^3\Pi_5^3 - 3q\frac{\Pi_5^3}{\mu^2}, \tag{75}$$

which may be written

$$\frac{A-\mu\Pi_5}{A^2\mu^2}(A^4\mu^2 + A^3\mu^3\Pi_5 + A^2\mu^4\Pi_5^2 + 3qA\Pi_5^3 + 3q\mu\Pi_5^4) = 0.$$

Now regarding this as an equation in A, we see that the expression within the brackets is always positive for positive values of A, if q be positive and less than unity; and therefore this expression cannot vanish for positive values of A. But, by (54) and (55), A is essentially positive: hence the only appropriate solution of (75) is

$$\left.\begin{aligned} A &= \mu\Pi_5, \\ \text{which, by (71) and (73),} \\ &= \frac{(1-q^2)(1-q^3)(1-q^7)(1-q^8)\dots}{(1-q)(1-q^4)(1-q^6)(1-q^9)\dots}(1-q^5)(1-q^{10})(1-q^{15})\dots \end{aligned}\right\}. \tag{76}$$

Again, comparing (56) and (57), we see that

$$3+7q-13q^4-17q^7+23q^{13}+\dots = 3A^2B + qC^3,$$

which, by (58), (59), and (76), is equal to

$$\mu^2\Pi_5^3\left(\frac{q}{\mu^5}+3\right). \tag{77}$$

Hence, by aid of (74), we see that (57) becomes

$$\{(\tfrac{1}{5})(\tfrac{2}{5})(\tfrac{3}{5})\ldots\}^3=\Pi_5^3\left\{\frac{q}{\mu^2}\left(\frac{\mu^5}{q}-3\right)-q^{\frac{1}{5}}.\mu^2\left(\frac{q}{\mu^5}+3\right)+5q^{\frac{2}{5}}\right\}. \tag{78}$$

Also substituting in (55) the values of A, B, and C given by (58), (59), and (76), we obtain

$$(\tfrac{1}{5})(\tfrac{2}{5})(\tfrac{3}{5})\ldots=\Pi_5\left(\mu-q^{\frac{1}{5}}-\frac{1}{\mu}\,q^{\frac{2}{5}}\right); \tag{79}$$

that is,

$$\frac{\mu}{q^{\frac{1}{5}}}-\frac{q^{\frac{1}{5}}}{\mu}-1=\frac{(\frac{1}{5})(\frac{2}{5})(\frac{3}{5})\ldots}{q^{\frac{1}{5}}\Pi_5},$$

where μ is given by (71). This identity is enunciated by Mr. Ramanujan in the form

$$\{f(x)\}^{-1}-f(x)-1=\frac{1}{x^{\frac{1}{5}}}\,\frac{(1-x^{\frac{1}{5}})(1-x^{\frac{2}{5}})(1-x^{\frac{3}{5}})\ldots}{(1-x^5)(1-x^{10})(1-x^{15})\ldots}, \tag{80}$$

where $f(x)$ is given by (26).

Now let

$$\frac{1}{(\frac{1}{5})(\frac{2}{5})(\frac{3}{5})\ldots}=a+q^{\frac{1}{5}}b+q^{\frac{2}{5}}c+q^{\frac{3}{5}}d+q^{\frac{4}{5}}e\ldots, \tag{81}$$

where a, b, ... involve no fractional powers of q. Then, multiplying together equations (79) and (81), we obtain

$$\mu^2a \qquad -qd\ -q\mu e=\frac{\mu}{\Pi_5}, \tag{82}$$

$$-\mu\,a+\mu^2b \qquad -qe\ =0, \tag{83}$$

$$-\ a-\mu\,b+\mu^2c \qquad =0, \tag{84}$$

$$-\ b-\mu\,c+\mu^2d \qquad =0, \tag{85}$$

$$-\ c-\mu\,d+\mu^2e=0. \tag{86}$$

Multiplying (82), (84), and (85) by 1, μ^2, $-\mu^3$, respectively, and adding, we find

$$2\mu^4c-(q+\mu^5)d-q\mu e=\frac{\mu}{\Pi_5}. \tag{87}$$

Similarly, multiplying (82), (83), and (85) by 1, μ, μ^3, respectively, and adding, we have

$$-\mu^4c-(q-\mu^5)d-2q\mu e=\frac{\mu}{\Pi_5}. \tag{88}$$

From (86) and (87) we obtain

$$-(q+3\mu^5)d+(2\mu^6-q\mu)e=\frac{\mu}{\Pi_5},$$

while (86) and (88) give

$$(q-2\mu^5)d+(\mu^6+2q\mu)e=-\frac{\mu}{\Pi_5}.$$

From these equations

$$e(4q\mu^6-4\mu^{11}-q^2\mu+7q\mu^6+3\mu^{11}+2q^2\mu)=-\frac{5\mu^6}{\Pi_5},$$

that is

$$e\left(\frac{\mu^5}{q}-\frac{q}{\mu^5}-11\right)=\frac{5}{q\Pi_5},$$

whence

$$d\left(\frac{\mu^5}{q}-\frac{q}{\mu^5}-11\right)=\frac{1}{\mu^4\Pi_5}\left(\frac{3\mu^5}{q}+1\right);$$

and then, from (82), (83), and (84), we obtain successively

$$a\left(\frac{\mu^5}{q}-\frac{q}{\mu^5}-11\right)=\frac{1}{\mu\Pi_5}\left(\frac{\mu^5}{q}-3\right),$$

$$b\left(\frac{\mu^5}{q}-\frac{q}{\mu^5}-11\right)=\frac{1}{\mu^2\Pi_5}\left(\frac{\mu^5}{q}+2\right),$$

$$c\left(\frac{\mu^5}{q}-\frac{q}{\mu^5}-11\right)=\frac{1}{\mu^3\Pi_5}\left(2\,\frac{\mu^5}{q}-1\right).$$

If we put

$$\frac{1}{(1-q)(1-q^2)(1-q^3)\ldots}=\Sigma p_n q^n,$$

the above results may be written

$$\left.\begin{aligned}
a=\Sigma p_{5m}q^m &= \frac{1}{\mu\Pi_5}\left(\frac{\mu^5}{q}-3\right) \div \left(\frac{\mu^5}{q}-\frac{q}{\mu^5}-11\right)\\
b=\Sigma p_{5m+1}q^m &= \frac{1}{\mu^2\Pi_5}\left(\frac{\mu^5}{q}+2\right) \div \quad ,,\\
c=\Sigma p_{5m+2}q^m &= \frac{1}{\mu^3\Pi_5}\left(2\,\frac{\mu^5}{q}-1\right) \div \quad ,,\\
d=\Sigma p_{5m+3}q^m &= \frac{1}{\mu^4\Pi_5}\left(3\,\frac{\mu^5}{q}+1\right) \div \quad ,,\\
e=\Sigma p_{5m+4}q^m &= \frac{5}{q\Pi_5} \div \quad ,,
\end{aligned}\right\} \qquad (89)$$

Substituting q for x in (26), we see by (71) that

$$f(q) = \frac{q^{\frac{1}{5}}}{\mu}$$

and therefore, by (51),

$$\frac{\mu^5}{q} - \frac{q}{\mu^5} - 11 = \frac{1}{q}\left\{\frac{(1-q)(1-q^2)(1-q^3)\ldots}{(1-q^5)(1-q^{10})(1-q^{15})\ldots}\right\}^6; \qquad (90)$$

so that, since by (73),

$$\Pi_5 = (1-q^5)(1-q^{10})(1-q^{15})\ldots,$$

we see that the last line of (89) gives

$$\Sigma p_{5m+4}q^m = 5\,\frac{\{(1-q^5)(1-q^{10})(1-q^{15})\ldots\}^5}{\{(1-q)(1-q^2)(1-q^3)\ldots\}^6}; \qquad (91)$$

a result given by Mr. Ramanujan* as dependent on (80), which he assumes to be known.

Again, multiplying (78) twice by (79), we find that if

$$\frac{1}{(\frac{1}{5})(\frac{2}{5})(\frac{3}{5})\ldots} = a + bq^{\frac{1}{5}} + cq^{\frac{2}{5}} + dq^{\frac{3}{5}} + eq^{\frac{4}{5}},$$

where a, b, ... contain no fractional powers of q, then

$$\{(\tfrac{1}{5})(\tfrac{2}{5})(\tfrac{3}{5})\ldots\}^5 = -5q^{\frac{1}{5}}\Pi_1^6\left(a - bq^{\frac{1}{5}} - cq^{\frac{2}{5}} + dq^{\frac{3}{5}} - \frac{\Delta}{25}\,eq^{\frac{4}{5}}\right); \qquad (92)$$

where

$$\Delta = \frac{\mu^5}{q} - \frac{q}{\mu^5} - 11 \qquad (93)$$

is given by (90). Also, from (78) and (92), we obtain

$$\{(\tfrac{1}{5})(\tfrac{2}{5})(\tfrac{3}{5})\ldots\}^8 = -5\,\frac{q^{\frac{1}{5}}}{\mu}\,\frac{\Pi_1^{12}}{\Pi_5^2}\,(A' + B'q^{\frac{1}{5}} + C'q^{\frac{2}{5}} + D'q^{\frac{3}{5}} + E'q^{\frac{4}{5}}), \qquad (94)$$

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* *Proc. Camb. Phil. Soc.*, Vol. 19 (1919), Pt. 5, pp. 209, 210.

in which A', B', ... involve no fractional powers of q, and are given by

$$\left.\begin{aligned} A' &= \frac{q}{5\mu^7\Delta^2\Pi_5^2}(8s^3-112s^2+56s-1) \\ B' &= \frac{1}{\mu^3\Delta^2\Pi_5^2}(-4s^2+24s+14) \\ C' &= \frac{25s}{\mu^4\Delta^2\Pi_5^2} \\ D' &= \frac{1}{\mu^5\Delta^2\Pi_5^2}(14s^2-24s-4) \\ E' &= -\frac{1}{5\mu^6\Delta^2\Pi_5^2}(s^3+56s^2+112s+8) \end{aligned}\right\}, \qquad (95)$$

where
$$s = \mu^5/q\,; \qquad (96)$$

μ is given by (71), Δ by (93) and (90), and the Π's by (73). Remembering that there is a $q^{\frac{1}{6}}$ outside the brackets in (94), it will be seen from the values of B' and D' that the terms involving $q^{\frac{3}{6}}$ and $q^{\frac{4}{6}}$ in (94) are both 0 (mod 5), and that there is a curious skew symmetry between B' and D' as well as between A' and E'.

Substituting for Δ and s in the value of C', we obtain

$$C' = 25q\mu\,\Pi_5^{10}/\Pi_1^{12},$$

so that the term involving $q^{\frac{1}{6}}$ in (94) is $-125q\,\Pi_5^8 q^{\frac{1}{6}}$; a result which leads to a family of giant congruences. Thus, if

$$\{(1)(2)(3)\ldots\}^8 = \Sigma r_n q^n,$$

then

$$\begin{aligned} \Sigma r_{5n+3}q^n &= -125q\,\{(5)(10)(15)\ldots\}^8 \\ &= -125q(\ldots+\Sigma r_{5m+3}q^{25m+15}), \end{aligned}$$

so that, if $n = 25m+16$, we have, writing q for q^{25},

$$\begin{aligned} \Sigma r_{125m+83}q^m &= -125\Sigma r_{5m+3}q^m \\ &= (-125)^2\,q\,\{(5)(10)(15)\ldots\}^8 \\ &= (-125)^2\,q\,(\ldots+\Sigma r_{5n+3}q^{25n+15}), \end{aligned}$$

so that, if $m = 25n+16$, we have, writing q for q^{25},

$$\Sigma r_{3125n+2083}q^n = (-125)^3\,q\,\{(5)(10)(15)\ldots\}^8.$$

Proceeding thus we find that, if p be any integer, then

$$\sum_{n=0}^{n=\infty} r_{5^{2p-1}n+(5^{2p}-1)/3}\, q^n = (-5)^{3p}\{(5)(10)(15)\ldots\}^8 \equiv 0 \pmod{5^{3p}}.$$

As an example, for $p = 4$, we have

$$\sum_{n=0}^{n=\infty} r_{78125n+130208}\, q^n = 5^{12}\{(5)(10)(15)\ldots\}^8 \equiv 0 \pmod{244140625};$$

a fairly large congruence, but only one of the infants of its big family. A similar series of congruences, but on a smaller scale, can be readily obtained from (78), or otherwise, for the case of $\{(1)(2)(3)\ldots\}^3$.

Next, squaring (94), we find by aid of (95), that

$$\{(\tfrac{1}{5})(\tfrac{2}{5})(\tfrac{3}{5})\ldots\}^{16} = 25\,\frac{q^{\frac{3}{5}}}{\mu^2}\,\frac{\Pi_1^{24}}{\Pi_5^4}(P+Qq^{\frac{1}{5}}+Rq^{\frac{2}{5}}+Sq^{\frac{3}{5}}+Tq^{\frac{4}{5}}), \qquad (97)$$

where P, Q, ... involve no fractional powers of q and are given by

$$P = \frac{q^2}{25\mu^{14}\Delta^4\Pi_5^4}(104s^6+208s^5+21840s^4-76960s^3-19240s^2-1232s+1),$$

$$Q = \frac{q}{5\mu^{10}\Delta^4\Pi_5^4}(-64s^5+2210s^4-11760s^3-3720s^2+2080s+52),$$

$$R = \frac{q}{5\mu^{11}\Delta^4\Pi_5^4}(52s^5-2080s^4-3720s^3+11760s^2+2210s+64),$$

$$S = \frac{q}{25\mu^{12}\Delta^4\Pi_5^4}(s^6+1232s^5-19240s^4+76960s^3+21840s^2-208s+104),$$

$$T = -\frac{q}{25\mu^{13}\Delta^4\Pi_5^4}(16s^6+3472s^5-32240s^4-16115s^3+32240s^2 +3472s-16);$$

whence, remembering that there is a $q^{\frac{2}{5}}$ outside the brackets in (97), it will be noticed that the terms involving $q^{\frac{3}{5}}$ and $q^{\frac{1}{5}}$ in (97) are both 0 (mod 5), and that there is a curious skew symmetry between Q and R as well as between P and S.

Lastly, multiplying together (94) and (97), and remembering that there will now be a $q^{\frac{3}{5}}$ outside the brackets, we find that the term involving

$q^{\frac{1}{5}}$ in $\{(\frac{1}{5})(\frac{2}{5})(\frac{3}{5})\dots\}^{24}$ is

$$q^{\frac{1}{5}}\frac{125}{\mu^3}\frac{\Pi_1^{36}}{\Pi_5^6}\frac{q^2}{25\mu^{17}\Delta^6\Pi_5^6}(966s^8-42504s^7+697452s^6-5015472s^5+2980745s^4 \\ +5015472s^3+697452s^2+42504s+966);$$

which, omitting the initial $q^{\frac{1}{5}}$, and reducing, becomes

$$\frac{5q^2}{\mu^{20}\Delta^6}\frac{\Pi_1^{36}}{\Pi_5^{12}}[966\{(s^2-11s-1)^4-13195s^4\}+2980745s^4];$$

which, by (96), becomes

$$\frac{5}{q^2\Delta^6}\frac{\Pi_1^{36}}{\Pi_5^{12}}\left\{966\left(s-\frac{1}{s}-11\right)^4-9765625\right\};$$

and this, by (96) and (93),

$$=5\frac{\Pi_1^{36}}{\Pi_5^{12}}\left(\frac{966}{q^2\Delta^2}-9765625\frac{q^4}{q^6\Delta^6}\right);$$

which, by (93) and (90), becomes

$$5\frac{\Pi_1^{36}}{\Pi_5^{12}}\left(966\frac{\Pi_5^{12}}{\Pi_1^{12}}-9765625q^4\frac{\Pi_5^{36}}{\Pi_1^{36}}\right)$$

$$=4830\Pi_1^{24}-48828125q^4\Pi_5^{24}=4830\Pi_1^{24}-5^{11}q^4\Pi_5^{24}.$$

Hence, if $$\{(1-q)(1-q^2)(1-q^3)\dots\}^{24}=\Sigma t_n q^n,$$

then

$$\Sigma t_{5n+4}q^n=4830\{(1-q)(1-q^2)(1-q^3)\dots\}^{24} \\ -5^{11}q^4\{(1-q^5)(1-q^{10})(1-q^{15})\dots\}^{24}\dots. \quad (98)$$

Mr. Ramanujan* has recently enunciated the following congruence:—

$$\left.\begin{array}{l}\text{If} \quad \sum_1^\infty \tau(n)x^n=x\{(1-x)(1-x^2)(1-x^3)\dots\}^{24} \\ \text{then} \quad \tau(5n)\equiv 0 \pmod 5\end{array}\right\}. \quad (99)$$

Allowing for the difference in notation, this result is readily seen to be proved by (98).

* *Proc. London Math. Soc.*, Ser. 2, Vol. 18, p. xx.

In view of the long and complicated formulæ from which (98) has been derived, it has appeared desirable to have a numerical check of this identity up to and including the terms involving q^4. The writer has calculated the necessary coefficients, which are as follows :—

$$t_0 = +\ 1, \quad t_3 = -\ 1472, \quad t_{14} = +\ 1217160,$$

$$t_1 = -\ 24, \quad t_4 = +\ 4830, \quad t_{19} = -\ 7109760,$$

$$t_2 = +\ 252, \quad t_9 = -\ 115920, \quad t_{24} = -\ 25499225.$$

These values will be found to be in complete agreement with (98).

THE LONDON MATHEMATICAL SOCIETY.

BALANCE SHEET, 31st October, 1920.

Liabilities.	£	s.	d.
Lieut.-Col. Campbell's Fund	499	7	5
Life Compositions Fund	2474	10	0
Invested Surplus Fund	700	0	0
Reserve Fund	34	18	6
Sundry Creditors	20	15	7
Revenue Account balance in hand	167	14	10
	£3897	6	4

Assets.	£	s.	d.
London & North Western Railway £385 Four per Cent. Guaranteed Stock at cost ...	499	7	5
India Stock £700. 7*s.* 1*d.* 3 per Cent. at cost	700	0	0
War Stock £250 5 per Cent. 1929–47 at cost	236	19	5
National War Bonds £200 5 per Cent., 1st April, 1928, at cost	200	0	0
Funding Loan 4 per Cent., £2500 at cost ...	2000	0	0
Deposit at Bank	200	0	0
Cash at Bank	60	19	6
	£3897	6	4

REVENUE ACCOUNT for the year 1st November, 1919, to 31st October, 1920.

	Year 1918–19.					
	£	s.	d.	£	s.	d.
Balance from last year	77	7	10	*95*	*11*	*0*
Income tax returned (1 year)	33	0	8	*73*	*6*	*1*
Dividends on—						
Lord Rayleigh's Fund ...	37	1	4	*34*	*4*	*0*
The Society's Investments	113	12	7	*120*	*2*	*10*
Entrance Fees (39)	40	19	0	*21*	*0*	*0*
Annual Subscriptions (190)	199	10	0	*147*	*0*	*0*
Sale of *Proceedings*, &c.	213	19	5	*187*	*9*	*2*
Advertisements	2	10	0	*12*	*10*	*0*
Grant made by Royal Society from the Special Publications Fund for the year 1919	100	0	0	—		
Do. for the year 1920	100	0	0	—		
	£918	16	10	*691*	*3*	*1*

	Year 1918–19.					
	£	s.	d.	£	s.	d.
Printing and Circulating *Proceedings*, &c.	722	8	0	*583*	*19*	*9*
Purchases for Library	5	15	3	*3*	*9*	*6*
Rent	20	0	0	*26*	*0*	*0*
Postages and Sundries	2	18	9	*6*	*6*	*0*
Balance in hand on 31st October, 1920...	167	14	10	*77*	*7*	*10*
	£918	16	10	*691*	*3*	*1*

REVENUE ACCOUNT of the DE MORGAN MEDAL FUND.

	£	s.	d.
Balance on 31st October, 1919	5	18	2
Dividends received	2	2	0
Income tax returned (1 year)		18	0
Donation	1	12	4
	£10	10	6

	£	s.	d.
Paid Messrs. Joseph Moore, Ltd., for De Morgan Medal and Bronze Replica ...	10	10	6
	£10	10	6

STATEMENT OF TRUST FUNDS, 31st October, 1920.

Lord Rayleigh's Fund consists of the following Investments:—

Great Indian Peninsula Railway £50. 8*s.* 8*d.* Annuity, Class B.
Do. do. do. do. £200 Guaranteed 3 per Cent. Stock.

This Fund is held on trust to apply the income as part of the general income of the Society.

The *De Morgan Medal Fund* consists of Great Western Railway £60 Five per Cent. Preference Stock.

11 November, 1920.

Signed on behalf of the Council:—
J. E. CAMPBELL,
A. E. WESTERN.

I report to the Members that I have obtained all the information and explanation I required as Auditor, and that I have examined the above Accounts with the books, and that, to the best of my information and of the explanations given to me, such Accounts are properly drawn up so as to exhibit a true and correct view of the state of the affairs of the Society as shown by the books of the Society.

ALLAN J. C. CUNNINGHAM,
Lt.-Col. late R.E., Auditor.

22 November, 1920.

THE THEORY OF A THIN ELASTIC PLATE, BOUNDED BY TWO CIRCULAR ARCS AND CLAMPED

By A. C. DIXON.

[Read January 13th, 1921.]

1. The object of this paper is to find an expression for the deflection at any point P of a horizontal thin isotropic elastic plate bounded by two circular arcs and clamped round the boundary.

Green's function U for this problem is a function of the coordinates such that

(1) U and its first derivatives vanish on the boundary.

(2) The only singularity of U within the boundary is at an arbitrary point P_0, and in the neighbourhood of P_0 the difference

$$U - |PP_0|^2 \log |PP_0|$$

has no singularity.

(3) $\nabla^4 U = 0$ everywhere within the boundary.

U is the deflection produced at P or P_0 by a constant load at P_0 or P, and this constant load may be conveniently taken as the unit.

The determination of U solves the problem in general

(1) If a load is distributed over a part or the whole of the plate the deflection produced may be found by integrating Green's function, multiplied by the surface density of the load, over the area covered, and

(2) If there is no load, but the deflection Z and outward normal gradient $\partial Z/\partial \nu$ are given at each point of the boundary, the deflection Z_0 at P_0 is equal to*

$$\frac{1}{8\pi} \int \left\{ Z \frac{\partial}{\partial \nu} \nabla^2 U - \frac{\partial Z}{\partial \nu} \nabla^2 U \right\} ds, \tag{1}$$

where the integration is round the whole boundary.

* J. Hadamard, *Mémoires présentés par divers savants* ..., t. 33 1908), no. 4, p. 2.

The method of the paper is to find the expression for Z and deduce that for U by comparison with (1).

2. Let the plate be bounded by two circles intersecting in A, B.

Define the position of any point P in the plane by the curvilinear co-ordinates u, v, where

$$u = \log|AP/PB|, \qquad v = \pi - \angle APB. \tag{2.1}$$

Thus

$$AP/PB = e^{u+\iota v},$$

$$AB/PB = 1+e^{u+\iota v} = 2e^{\frac{1}{2}(u+\iota v)}\cosh\tfrac{1}{2}(u+\iota v),$$

$$2PB\cosh\tfrac{1}{2}(u+\iota v) = ABe^{-\frac{1}{2}(u+\iota v)},$$

$$2AP\cosh\tfrac{1}{2}(u+\iota v) = ABe^{\frac{1}{2}(u+\iota v)},$$

$$AP-PB = AB\tanh\tfrac{1}{2}(u+\iota v),$$

$$OP = k\tanh\tfrac{1}{2}(u+\iota v), \tag{2.2}$$

if O is the middle point of AB and $AB = 2k$.

Thus when there is no load the deflection Z, which must satisfy the condition $\nabla^4 Z = 0$, is a function of the form

$$F_1(u+\iota v)+F_2(u+\iota v)\tanh\tfrac{1}{2}(u-\iota v)+F_3(u-\iota v)+F_4(u-\iota v)\tanh\tfrac{1}{2}(u+\iota v).$$

Hence $Z(\cosh u+\cos v)$ may be taken as

$$f_1(u+\iota v)\cosh\tfrac{1}{2}(u-\iota v)+f_2(u+\iota v)\sinh\tfrac{1}{2}(u-\iota v)+f_3(u-\iota v)\cosh\tfrac{1}{2}(u+\iota v)$$
$$+f_4(u-\iota v)\sinh\tfrac{1}{2}(u+\iota v),$$

or, what is equivalent,

$$e^{-\iota v}\phi_1(u+\iota v)+e^{\iota v}\phi_2(u-\iota v)+e^{\iota v}\phi_3(u+\iota v)+e^{-\iota v}\phi_4(u-\iota v). \tag{2.3}$$

As P travels along a circular arc from A to B, v is constant, while u increases from $-\infty$ to $+\infty$.

Suppose α, β to be the values of v on the edges of the plate, so that at interior points $\alpha > v > \beta$.

On the two edges the values of Z and $\partial Z/\partial v$ are given, and the four functions ϕ_1, ϕ_2, ϕ_3, ϕ_4 must satisfy the conditions following.

$$e^{-\iota v}\phi_1(u+\iota v)+e^{\iota v}\phi_2(u-\iota v)+e^{\iota v}\phi_3(u+\iota v)+e^{-\iota v}\phi_4(u-\iota v)$$
$$= Z(\cosh u+\cos v), \tag{2.41}$$

when $$v = \alpha, \beta;$$

$$e^{-\iota v}\phi_1'(u+\iota v)-e^{\iota v}\phi_2'(u-\iota v)+e^{\iota v}\phi_3'(u+\iota v)-e^{-\iota v}\phi_4'(u-\iota v)$$
$$-e^{-\iota v}\phi_1(u+\iota v)+e^{\iota v}\phi_2(u-\iota v)+e^{\iota v}\phi_3(u+\iota v)-e^{-\iota v}\phi_4(u-\iota v)$$
$$= -\iota\frac{\partial Z}{\partial v}(\cosh u+\cos v)+\iota Z\sin v, \quad (2.42)$$

when $$v = \alpha, \beta.$$

Thus, when $v = \alpha, \beta$,

$$e^{-\iota v}\phi_1'(u+\iota v)+e^{\iota v}\phi_2(u-\iota v)+e^{\iota v}\phi_3'(u+\iota v)+e^{\iota v}\phi_3(u+\iota v)$$
$$= \tfrac{1}{2}\left(Z+\frac{\partial Z}{\partial u}-\iota\frac{\partial Z}{\partial v}\right)(\cosh u+\cos v)+\tfrac{1}{2}Z(\sinh u+\iota\sin v), \quad (2.43)$$

and

$$e^{\iota v}\phi_2'(u-\iota v)+e^{-\iota v}\phi_1(u+\iota v)+e^{-\iota v}\phi_4'(u-\iota v)+e^{-\iota v}\phi_4(u-\iota v)$$
$$= \tfrac{1}{2}\left(Z+\frac{\partial Z}{\partial u}+\iota\frac{\partial Z}{\partial v}\right)(\cosh u+\cos v)+\tfrac{1}{2}Z(\sinh u-\iota\sin v). \quad (2.44)$$

Denote by $f(u)$, $f_1(u)$, $g(u)$, $g_1(u)$ the known functions of u which appear on the right in (2.43), (2.44) so that

$$e^{-\iota\alpha}\phi_1'(u+\iota\alpha)+e^{\iota\alpha}\phi_2(u-\iota\alpha)+e^{\iota\alpha}\phi_3'(u+\iota\alpha)+e^{\iota\alpha}\phi_3(u+\iota\alpha) = f(u), \quad (2.5)$$

and so on.

These equations identify $f(u)$, $f_1(u)$, $g(u)$, $g_1(u)$ with certain analytical functions of the complex variable u. If we accept the identification, we have

$$f(u-\iota\alpha) = e^{-\iota\alpha}\phi_1'(u)+e^{\iota\alpha}\phi_2(u-2\iota\alpha)+e^{\iota\alpha}\phi_3'(u)+e^{\iota\alpha}\phi_3(u), \quad (2.51)$$

$$f_1(u-\iota\beta) = e^{-\iota\beta}\phi_1'(u)+e^{\iota\beta}\phi_2(u-2\iota\beta)+e^{\iota\beta}\phi_3'(u)+e^{\iota\beta}\phi_3(u), \quad (2.52)$$

and therefore

$$e^{-\iota\alpha}f(u-\iota\alpha)-e^{-\iota\beta}f_1(u-\iota\beta) = (e^{-2\iota\alpha}-e^{-2\iota\beta})\phi_1'(u)+\phi_2(u-2\iota\alpha)-\phi_2(u-2\iota\beta), \quad (2.53)$$

and, similarly,

$$e^{\iota\alpha}g(u+\iota\alpha)-e^{\iota\beta}g_1(u+\iota\beta) = (e^{2\iota\alpha}-e^{2\iota\beta})\phi_2'(u)+\phi_1(u+2\iota\alpha)-\phi_1(u+2\iota\beta). \quad (2.54)$$

Put $$\phi_1(u)-je^{\iota\alpha+\iota\beta}\phi_2(u-\iota\alpha-\iota\beta) = \psi(u), \quad (2.6)$$

where j has either of the values ± 1.

Then it follows that

$$\psi(u+\iota\alpha-\iota\beta)-\psi(u-\iota\alpha+\iota\beta)-2\iota j\sin(\alpha-\beta)\psi'(u)$$
$$=je^{\iota\beta}f(u-\iota\alpha)-je^{\iota\alpha}f_1(u-\iota\beta)+e^{\iota\alpha}g(u-\iota\beta)-e^{\iota\beta}g_1(u-\iota\alpha). \quad (2.7)$$

3. Now the analytical function whose value is given as $f(u)$ when u is real is

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{\iota\lambda(u-t)}f(t)\,dt\,d\lambda,$$

each integration being over the whole range of real quantities. Thus for the right-hand side of (2.7) we write

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-}^{\infty}e^{\iota\lambda(u-t)}\{je^{\iota\beta+\lambda\alpha}f(t)-je^{\iota\alpha+\lambda\beta}f_1(t)+e^{\iota\alpha+\lambda\beta}g(t)-e^{\iota\beta+\lambda\alpha}g_1(t)\}\,dt\,d\lambda,$$

and since u only occurs in the first exponent, and there linearly, we have as a suggested solution

$$\psi(u)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Theta^{-1}e^{\iota\lambda(u-t)}[\{g_1(t)-jf(t)\}e^{\iota\beta+\lambda\alpha}-\{g(t)-jf_1(t)\}e^{\iota\alpha+\lambda\beta}]$$
$$\times dt\,d\lambda, \quad (3$$

where Θ or $\Theta(\lambda,\alpha,\beta)$ is put for $e^{\lambda(\alpha-\beta)}-e^{-\lambda(\alpha-\beta)}-2j\lambda\sin(\alpha-\beta)$.

Then

$$\psi_1(u)=\frac{1}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sum_j\Theta^{-1}e^{\iota\lambda(u-t)}[\{g_1(t)-jf(t)\}e^{\iota\beta+\lambda\alpha}-\{g(t)-jf_1(t)\}e^{\iota\alpha+\lambda\beta}]$$
$$\times dt\,d\lambda, \quad (3.1)$$

the summation referring to the two values of j, and

$$\psi_2(u)=\frac{1}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sum_j\Theta^{-1}e^{\iota\lambda(u-t)}[\{f(t)-jg_1(t)\}e^{-\iota\alpha-\lambda\beta}$$
$$-\{f_1(t)-jg(t)\}e^{-\iota\beta-\lambda\alpha}]\,dt\,d\lambda. \quad (3.11)$$

Then, from (2.5), since we may put

$$f(u)=\tfrac{1}{2}\sum_j\{f(u)-jg_1(u)\}=\frac{1}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sum_j e^{\iota\lambda(u-t)}\{f(t)-jg_1(t)\}\,dt\,d\lambda,$$

we have

$$\psi_3(u)=\frac{1}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sum_j\Theta^{-1}(1+\iota\lambda)^{-1}e^{\iota\lambda(u-t)}[\{g_1(t)-jf(t)\}(je^{\lambda\beta-\iota\alpha}-\iota\lambda e^{\lambda\alpha-\iota\beta})$$
$$-\{g(t)-jf_1(t)\}(je^{\lambda\alpha-\iota\beta}-\iota\lambda e^{\lambda\beta-\iota\alpha})]\,dt\,d\lambda, \quad (3.12)$$

and, similarly,

$$\phi_4(u) = \frac{1}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sum_j \Theta^{-1}(1+\iota\lambda)^{-1}\, e^{\iota\lambda(u-t)} \left[\{g_1(t)-jf(t)\}\,(j\iota\lambda\, e^{\iota\alpha-\lambda\beta}-e^{\iota\beta-\lambda\alpha}) \right.$$
$$\left. -\{g(t)-jf_1(t)\}\,(j\iota\lambda\, e^{\iota\beta-\lambda\alpha}-e^{\iota\alpha-\lambda\beta})\right] dt\, d\lambda. \quad (3.13)$$

Thus $$Z(\cosh u+\cos v) = \frac{1}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sum_j \Theta^{-1}(1+\iota\lambda)^{-1} e^{\iota\lambda(u-t)}$$

$$\times\left[e^{-\lambda\beta-\iota\alpha}\{f(t)-jg_1(t)\}\,\omega(v,\lambda,\alpha,\beta)-e^{-\lambda\alpha-\iota\beta}\{f_1(t)-jg(t)\}\,\omega(v,\lambda,\beta,\alpha)\right] dt\, d\lambda, \quad (3.2)$$

where $\omega(v,\lambda,\alpha,\beta) = (1+\iota\lambda)\{e^{(\lambda+\iota)v}-je^{(\lambda+\iota)(\alpha+\beta-v)}\}$

$$+(je^{\lambda\beta-\lambda\alpha}-\iota\lambda\, e^{\iota\alpha-\iota\beta})\{e^{\lambda v+\iota(\alpha+\beta-v)}-je^{\iota v+\lambda(\alpha+\beta-v)}\}.$$

If $F(u)$, $G(u)$ are the values of

$$Z(\cosh u+\cos v) \quad \text{and} \quad \frac{\partial}{\partial v}Z(\cosh u+\cos v),$$

when $v=\alpha$, $F_1(u)$, $G_1(u)$ the values of the same functions when $v=\beta$, then from (2.43) and (2.44),

$$f(u) = \tfrac{1}{2}F(u)+\tfrac{1}{2}F'(u)-\tfrac{1}{2}\iota G(u), \quad (3.31)$$

$$g(u) = \tfrac{1}{2}F(u)+\tfrac{1}{2}F'(u)+\tfrac{1}{2}\iota G(u), \quad (3.32)$$

so that in (3.2) we put $\frac{1}{2}(1+\iota\lambda)F(t)\mp\frac{1}{2}\iota G(t)$ for $f(t)$ and $g(t)$, and the like expressions for $f_1(t)$ and $g_1(t)$. Finally, then,

$$Z = \frac{1}{8\pi}(\cosh u+\cos v)^{-1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sum_j \Theta^{-1} e^{\iota\lambda(u-t)}$$

$$\times\left[\{F(t)-jF_1(t)\}\{e^{-\lambda\beta-\iota\alpha}\omega(v,\lambda,\alpha,\beta)+je^{-\lambda\alpha-\iota\beta}\omega(v,\lambda,\beta,\alpha)\}\right.$$

$$\left.-\iota\{G(t)+jG_1(t)\}(1+\iota\lambda)^{-1}\{e^{-\lambda\beta-\iota\alpha}\omega(v,\lambda,\alpha,\beta)-je^{-\lambda\alpha-\iota\beta}\omega(v,\lambda,\beta,\alpha)\}\right] dt\, d\lambda. \quad (3.4)$$

This is the expression for the deflection at (u, v) when there is no load, but when the deflection and gradient are given along the boundary.

4. By comparison of this expression with (1) we can now deduce the value of Green's function U, the deflection at z produced by a unit load at z_0.

Since U depends symmetrically on z, z_0, and since on the edges,

$$\frac{ds}{du} = -\frac{dv}{dv},$$

being positive on $v = \beta$ and negative on $v = \alpha$, we have

$$8\pi Z = \Big[\int_{-\infty}^{\infty} \Big\{ Z_0 \frac{\partial}{\partial v_0} \nabla_0^2 U - \frac{\partial Z_0}{\partial v_0} \nabla_0^2 U \Big\} du_0 \Big]_{v_0=\beta}^{v_0=\alpha}. \tag{4}$$

Now u_0 in (4) is the same variable that is denoted by t in (3.4); also when $v_0 = \alpha$,

$$F(u_0) = Z_0(\cosh u_0 + \cos\alpha), \qquad G(u_0) = \frac{\partial Z_0}{\partial v_0}(\cosh u_0 + \cos\alpha) - Z_0 \sin\alpha.$$

Thus, still when $v_0 = \alpha$,*

$$\nabla_0^2 U = \iota \frac{\cosh u_0 + \cos\alpha}{\cosh u + \cos v} \int_{-\infty}^{\infty} \sum_j \Theta^{-1}(1+\iota\lambda)^{-1} e^{\iota\lambda(u-u_0)}$$

$$\times \{ e^{-\lambda\beta - \iota\alpha}\, \omega(v, \lambda, \alpha, \beta) - j e^{-\lambda\alpha - \iota\beta}\, \omega(v, \lambda, \beta, \alpha) \}\, d\lambda, \tag{4.01}$$

$$\frac{\partial}{\partial v_0} \nabla_0^2 U = \frac{1}{\cosh u + \cos v} \int_{-\infty}^{\infty} \sum_j \Theta^{-1}(1+\iota\lambda)^{-1} e^{\iota\lambda(u-u_0)}$$

$$\times [(\cosh u_0 + \cos\alpha)(1+\iota\lambda) \{ e^{-\lambda\beta-\iota\alpha}\, \omega(v, \lambda, \alpha, \beta) + j e^{-\lambda\alpha-\iota\beta}\, \omega(v, \lambda, \beta, \alpha) \}$$

$$+ \iota \sin\alpha \{ e^{-\lambda\beta-\iota\alpha}\, \omega(v, \lambda, \alpha, \beta) - j e^{-\lambda\alpha-\iota\beta}\, \omega(v, \lambda, \beta, \alpha) \}]\, d\lambda. \tag{4.02}$$

Now, if $\nabla_0^4 V = 0$, and if when $v_0 = \alpha$,

$$\nabla_0^2 V = A e^{\mu u_0}, \qquad \frac{\partial}{\partial v_0} \nabla_0^2 V = \mu B e^{\mu u_0},$$

then, in general,

$$\nabla_0^2 V = A e^{\mu u_0} \cos\mu(v_0 - \alpha) + B e^{\mu u_0} \sin\mu(v_0 - \alpha). \tag{4.1}$$

Applying this rule to the several terms in (4.01) and (4.02) and

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* The boundary conditions when $v_0 = \alpha$, with the equation $\nabla_0^4 U = 0$, are enough to determine completely first $\nabla_0^2 U$ and afterwards U. There are similar boundary conditions when $v_0 = \beta$. They would lead to a similar final expression for U, valid when $v > v_0$, whereas the expression (4.3) given in the text is valid when $v < v_0$. The method of §§ 2-4 is heuristic and by no means rigorous.

putting w_0 for $u_0+\iota v_0$, w_0 for $u_0-\iota v_0$, we have after some rearrangement,

$$\nabla_0^2 U = -\frac{1}{\cosh u+\cos v}\int_{-\infty}^{\infty} \sum_j \Theta^{-1}(1+\iota\lambda)^{-1} e^{\iota\lambda u}$$

$$\times\Big[\{e^{-\lambda\beta-\iota\alpha}\omega(v,\lambda,\alpha,\beta)-je^{-\lambda\alpha-\iota\beta}\omega(v,\lambda,\beta,\alpha)\}$$

$$\times\Big\{\cosh^2\tfrac{1}{2}w_0\frac{d}{dw_0}\frac{e^{-\iota\lambda(w_0-\iota\alpha)}\cosh(\frac{1}{2}w_0-\iota\alpha)}{\lambda\cosh\frac{1}{2}w_0}$$

$$+\cosh^2\tfrac{1}{2}\mathrm{w}_0\frac{d}{d\mathrm{w}_0}\frac{e^{-\iota\lambda(\mathrm{w}_0+\iota\alpha)}\cosh(\frac{1}{2}\mathrm{w}_0+\iota\alpha)}{\lambda\cosh\frac{1}{2}\mathrm{w}_0}\Big\}$$

$$-(1+\iota\lambda)\{e^{-\lambda\beta-\iota\alpha}\omega(v,\lambda,\alpha,\beta)+je^{-\lambda\alpha-\iota\beta}\omega(v,\lambda,\beta,\alpha)\}$$

$$\times\Big\{\cosh^2\tfrac{1}{2}w_0\frac{d}{dw_0}\frac{e^{-\iota\lambda(w_0-\iota\alpha)}}{2\lambda\cosh\frac{1}{2}w_0}\left(\frac{e^{\frac{1}{2}w_0-\iota\alpha}}{1-\iota\lambda}-\frac{e^{-\frac{1}{2}w_0+\iota\alpha}}{1+\iota\lambda}\right)$$

$$-\cosh^2\tfrac{1}{2}\mathrm{w}_0\frac{d}{d\mathrm{w}_0}\frac{e^{-\iota\lambda(\mathrm{w}_0+\iota\alpha)}}{2\lambda\cosh\frac{1}{2}\mathrm{w}_0}\left(\frac{e^{\frac{1}{2}\mathrm{w}_0+\iota\alpha}}{1-\iota\lambda}-\frac{e^{-\frac{1}{2}\mathrm{w}_0-\iota\alpha}}{1+\iota\lambda}\right)\Big\}\Big]d\lambda. \quad (4.2)$$

Now $$\nabla_0^2 = \frac{16}{k^2}\cosh^2\tfrac{1}{2}w_0\cosh^2\tfrac{1}{2}\mathrm{w}_0\frac{\partial^2}{\partial w_0\partial\mathrm{w}_0},$$

and thus U can be found by integration with the conditions that U and $\partial U/\partial v_0$ vanish when $v_0=\alpha$. The value of U is

$$\tfrac{1}{2}k^2(\cosh u+\cos v)^{-1}(\cosh u_0+\cos v_0)^{-1}$$

$$\times\int_0^{\infty}\sum\Theta^{-1}\lambda^{-1}(1+\lambda^2)^{-1}\cosh\iota\lambda(u-u_0)\Omega(\lambda,v,v_0,\alpha,\beta)\,d\lambda, \quad (4.3)$$

where $\Omega(\lambda, v, v_0, \alpha, \beta)$ denotes

$$\{e^{-\lambda\beta-\iota\alpha}\omega(v,\lambda,\alpha,\beta)-je^{-\lambda\alpha-\iota\beta}\omega(v,\lambda,\beta,\alpha)\}\,2\iota(1-\iota\lambda)\sinh\lambda(v_0-\alpha)\sin(v_0-\alpha)$$

$$+\{e^{-\lambda\beta-\iota\alpha}\omega(v,\lambda,\alpha,\beta)+je^{-\lambda\alpha-\iota\beta}\omega(v,\lambda,\beta,\alpha)\}$$

$$\times\{(1-\iota\lambda)\sinh(\iota-\lambda)(v_0-\alpha)-(1+\iota\lambda)\sinh(\iota+\lambda)(v_0-\alpha)\}.$$

The change in the limits of integration is possible because Ω, like $\Theta\lambda$, is an even function of λ.

The following relations are worthy of note:

$$\omega(v,\lambda,\alpha,\beta)-e^{(\iota-\lambda)v+\lambda(\alpha+\beta)}\Theta = \omega(v,\lambda,\beta,\alpha)-je^{(\lambda-\iota)v+\iota(\alpha+\beta)}\Theta$$

$$=2(1+\iota\lambda)\{e^{(\lambda+\iota)\alpha}-je^{(\lambda+\iota)\beta}\}\sinh\lambda(v-\alpha)\sinh\iota(v-\alpha)$$

$$+\{e^{(\lambda+\iota)\alpha}+je^{(\lambda+\iota)\beta}\}\{(1+\iota\lambda)\sinh(\iota+\lambda)(v-\alpha)-(1-\iota\lambda)\sinh(\iota-\lambda)(v-\alpha)\};$$

this vanishes to the first order when $\lambda = 0$ or ι, and if $j = +1$, also when $\lambda = -\iota$.

Again, Θ vanishes to the first order when $\lambda = 0$, and if $j = +1$, also when $\lambda = \pm\iota$. Ω vanishes to the second order when $\lambda = 0$ and to the order $\frac{1}{2}(3+j)$ when $\lambda = \pm\iota$.

$$[\Omega(\lambda, v, v_0, \alpha, \beta) - \Theta\{(1+\iota\lambda)\sinh(\iota+\lambda)(v-v_0) - (1-\iota\lambda)\sinh(\iota-\lambda)(v-v_0)\}$$

is a symmetric function of v, v_0, namely,

$$\tfrac{1}{2}j e^{-\iota\alpha-\iota\beta}\{\omega(v, \lambda, \alpha, \beta) - \Theta e^{(\iota-\lambda)v+\lambda(\alpha+\beta)}\}\{\omega(v_0, -\lambda, \alpha, \beta) + \Theta e^{(\iota+\lambda)v_0-\lambda(\alpha+\beta)}\}$$
$$+\tfrac{1}{2}j e^{-\iota\alpha-\iota\beta}\{\omega(v_0, \lambda, \alpha, \beta) - \Theta e^{(\iota-\lambda)v_0+\lambda(\alpha+\beta)}\}\{\omega(v, -\lambda, \alpha, \beta) + \Theta e^{(\iota+\lambda)v-\lambda(\alpha+\beta)}\}$$
$$-\Theta\{2\iota\lambda\sinh\iota(v+v_0-2\alpha)\cosh\lambda(v-v_0) + 2\sinh\lambda(v+v_0-2\alpha)\cosh\iota(v-v_0)\}. \tag{4.5}$$

When $\Theta = 0$,

$$\frac{\omega(v, -\lambda, \alpha, \beta)}{\omega(v, \lambda, \alpha, \beta)} = \frac{1-\iota\lambda}{\iota j\lambda e^{(\lambda+\iota)\alpha+(\lambda-\iota)\beta} - e^{2\lambda\beta}} = -\frac{\iota j\lambda e^{(\iota-\lambda)\alpha-(\iota+\lambda)\beta} + e^{-2\lambda\beta}}{1+\iota\lambda}.$$

5. The reasoning of §§ 2–4 is not rigorous, and it is therefore necessary to examine and verify the result, and first to test the expression (4.3) for convergency.

The subject of integration has no poles on the path; even at the lower limit it is finite, for Ω vanishes to the second order. The question of convergency only arises at the upper limit.

Suppose $\alpha > v_0 > v > \beta$; then when λ is great and positive the commanding terms in Ω are those in which the exponent contains

$$\lambda(2\alpha - v - v_0) \quad \text{or} \quad \lambda(\alpha - v_0 + v - \beta).$$

But on account of the summation with respect to j the result is unaltered if we add to Ω

$$j\Theta e^{\lambda(\alpha+\beta-v-v_0)}\{(1+\iota\lambda)e^{\iota(\alpha+\beta-v-v_0)} + (1-\iota\lambda)e^{-\iota(\alpha+\beta-v-v_0)}\},$$

which destroys the terms in which the exponent contains

$$\lambda(2\alpha - v - v_0).$$

The highest exponents left are now

$$\lambda(\alpha+\beta-v-v_0) \quad \text{and} \quad \lambda(\alpha - v_0 + v - \beta),$$

which are both lower than $\lambda(\alpha-\beta)$ the exponent in the commanding term of Θ, that is, of the denominator.

It follows that so long as $\alpha > v_0 > v > \beta$, the integral represents a function which is finite and continuous and has differential coefficients of all orders.

The exponents in the various terms of Ω contain, besides finite terms that do not affect the convergency,

$$\pm\lambda(v-\beta) \pm \lambda(v_0-\alpha)$$

and

$$\pm\lambda(v-\alpha) \pm \lambda(v_0-\alpha),$$

but in the latter case the term is multiplied by j, and the summation with respect to j has the effect of reducing the coefficient of λ by $\alpha-\beta$. So long as the effective exponents are less than $\lambda(\alpha-\beta)$ the conclusion holds that the integral represents a continuous function having differential coefficients of all orders.

Thus, if $\alpha > v_0 > \beta$, v may have any value between v_0 and $2\beta-v_0$, or on the other hand if $\alpha > v > \beta$, v_0 may range between v and $2\alpha-v$. In particular v_0 may take the value α, and v the value β.

It is necessary to find what happens when v, v_0 approach each other. The terms in Ω that have $\lambda(v-v_0+\alpha-\beta)$ in the exponent are

$$e^{\lambda(v-v_0+\alpha-\beta)}\{(1+\iota\lambda)e^{\iota(v-v_0)}+(1-\iota\lambda)e^{\iota(v_0-v)}\}, \tag{5.1}$$

and, if $v = v_0$, these terms are no longer small in comparison with $e^{\lambda(\alpha-\beta)}$ when λ is great. They could be destroyed by adding, to Ω,

$$-\Theta e^{\lambda(v-v_0)}\{(1+\iota\lambda)e^{\iota(v-v_0)}+(1-\iota\lambda)e^{\iota(v_0-v)}\}, \tag{5.2}$$

or to the subject of integration in (4.3),

$$-\{e^{\lambda(v-v_0+\iota u-\iota u_0)}+e^{\lambda(v-v_0-\iota u+\iota u_0)}\}\left\{\frac{2}{\lambda}\cos(v-v_0)-\frac{e^{\iota(v_0-v)}}{\lambda-\iota}-\frac{e^{\iota(v-v_0)}}{\lambda+\iota}\right\}. \tag{5.3}$$

If we add further

$$\frac{4}{\lambda}e^{-\lambda}\cos(v-v_0)-\frac{2}{\lambda-\iota}e^{\iota-\lambda}\cosh(u-u_0)-\frac{2}{\lambda+\iota}e^{-\iota-\lambda}\cosh(u-u_0), \tag{5.4}$$

we avoid a failure of convergency, since the whole expression added is finite when $\lambda = 0$ or $\pm\iota$.

Now, by Frullani's theorem, if the real parts of a, b are not positive,

$$\int_0^\infty \frac{e^{a\lambda}-e^{b\lambda}}{\lambda}d\lambda = \log\frac{b}{a}; \tag{5.5}$$

it follows that

$$\int_{\mp\iota}^{\infty} \frac{e^{a(\lambda\pm\iota)}-e^{b(\lambda\pm\iota)}}{\lambda\pm\iota}d\lambda = \log\frac{b}{a}. \tag{5.51}$$

Thus the integral from 0 to ∞ of the expression added to the subject of integration in (4 . 3) is

$$2\cos(v-v_0)\{\log(v_0-v+\iota u_0-\iota u)+\log(v_0-v-\iota u_0+\iota u)\}$$
$$-e^{u_0-u}\log(v_0-v+\iota u_0-\iota u)-e^{u-u_0}\log(v_0-v-\iota u_0+\iota u)$$
$$-e^{u_0-u}\log(v_0-v-\iota u_0+\iota u)-e^{u-u_0}\log(v_0-v+\iota u_0-\iota u)$$
$$+\int_0^{\iota}\frac{1}{\lambda-\iota}e^{\iota(v_0-v)}\{e^{\lambda(v-v_0+\iota u-\iota u_0)}+e^{\lambda(v-v_0-\iota u+\iota u_0)}-2e^{\iota(v-v_0)+\iota-\lambda}\cosh(u-u_0)\}\,d\lambda$$
$$+\int_0^{-\iota}\frac{1}{\lambda+\iota}e^{\iota(v-v_0)}\{e^{\lambda(v-v_0-\iota u+\iota u_0)}+e^{\lambda(v-v_0+\iota u-\iota u_0)}$$
$$-2e^{-\iota(v-v_0)-\iota-\lambda}\cosh(u-u_0)\}\,d\lambda,$$

or $$-2\{\cosh(u-u_0)-\cos(v-v_0)\}\log\{(v_0-v)^2+(u_0-u)^2\}$$

together with the two integrals last written.

The two integrals do not affect the singularity in U. Hence the singular part of U where $v=v_0$ is

$$\frac{k^2\{\cosh(u-u_0)-\cos(v-v_0)\}}{(\cosh u+\cos v)(\cosh u_0+\cos v_0)}\log\{(v_0-v)^2+(u_0-u)^2\},$$

or $$\tfrac{1}{2}|PP_0|^2\log\{(v_0-v)^2+(u_0-u)^2\}.$$

This differs from $|PP_0|^2\log|PP_0|$ only by a function without singularity in the neighbourhood, and therefore U has the singularity of the deflection produced by unit load at P_0.

6. Now the singular part of U is a familiar function which exists everywhere. The above expression for it by means of Frullani integrals only holds when $v\leqslant v_0$, but when $v>v_0$ it is represented by a similar expression, differing only in the sign of $v-v_0$.

Hence we can "continue" the function U across the arc $v=v_0$ by adding the terms (5 . 3) and subtracting similar terms differing only in the sign of $v-v_0$: the terms (5 . 4) disappear, being symmetrical in v, v_0. The effect is to add to $\Omega(\lambda, v, v_0, \alpha, \beta)$, the expression

$$-2\Theta\{(1+\iota\lambda)\sinh(\lambda+\iota)(v-v_0)+(1-\iota\lambda)\sinh(\lambda-\iota)(v-v_0)\},$$

which is $\Omega(\lambda, v_0, v, \alpha, \beta)-\Omega(\lambda, v, v_0, \alpha, \beta)$ by (4 . 5):

that is, the effect is to interchange P and P_0 in (4 . 3), as it should be.

Hence U has no singularity on the plate except at the load point P_0.

It remains to prove that U and $\partial U/\partial v$ vanish on the edges.

When $v_0 = \alpha$, this follows by the construction of $\Omega(\lambda, v, v_0, \alpha, \beta)$.

When $v = \beta$,

$$\Omega(\lambda, v, v_0, \alpha, \beta) = 2j\Theta\{(1-\iota\lambda)\sinh(\lambda-\iota)(v_0-\alpha)+(1+\iota\lambda)\sinh(\lambda+\iota)(v_0-\alpha)\},$$

$$\frac{\partial}{\partial v}\Omega(\lambda, v, v_0, \alpha, \beta) = 4\iota j\Theta(1+\lambda^2)\sinh\lambda(v_0-\alpha)\sinh\iota(v_0-\alpha).$$

Each of these disappears on summation with respect to j, so that the result (4 . 3) is completely established.

7. The expression for U may be written in other forms by taking account of complex values of λ, which has hitherto been a real variable.

In the first place, when $v < v_0$,

$$U = \tfrac{1}{4}k^2(\cosh u+\cos v)^{-1}(\cosh u_0+\cos v_0)^{-1}\int_{-\infty}^{\infty}\sum_j\{\Theta\lambda(1+\lambda^2)\}^{-1}e^{\iota\lambda(u-v_0)} \times\Omega(\lambda, v, v_0, \alpha, \beta)d\lambda, \quad (7)$$

where the integration is over the whole real axis. It may however be taken over an infinite closed contour, enclosing when $u > u_0$ the first two quadrants, or when $u < u_0$ the other two.

For when the real part of λ is great, and for instance positive, it follows as in § 5 that Ω/Θ is effectively at most of the same order as $\lambda^2 e^{\lambda(v-v_0)}$, unless λ is so near a zero of Θ that $2\sinh\lambda(\alpha-\beta)$ ceases to be the commanding term in Θ.

On the other hand, the factor $e^{\iota\lambda(u-v_0)}$ is less than 1 in absolute value if λ is in the first or second quadrant (when $u > u_0$) and it decreases indefinitely when the lateral part of λ is increased.

To avoid the difficulty of the zeros of Θ, let the contour cross the lateral axis in such a way that for a suitable distance on each side

$$\lambda(\alpha-\beta) = (n+\tfrac{1}{2})\iota\pi+\theta,$$

where n is a positive integer and θ is real. Then

$$\Theta = (-1)^n 2\iota\cosh\theta - j\{(2n+1)\iota\pi+2\theta\}\sin(\alpha-\beta)/(\alpha-\beta).$$

The absolute value of this is at least of the order of n, if n is so taken that $(-1)^n$ and $j\sin(\alpha-\beta)$ have opposite signs.

When λ has a considerable real part, the inequality

$$|\sinh(\pm a \pm \iota b)| > \sinh a,$$

shows that, if the real part of $\lambda(\alpha-\beta)$ is greater than $2\log n$, $\sinh\lambda(\alpha-\beta)$

is of the order of n^2, and therefore Θ is of the order of n^2 if λ is not above the order of n.

From these considerations it appears that the integral tends to zero when $n \to \infty$ if $\lambda(\alpha-\beta)$ travels along the following path, where R is written for $\{(n+\frac{1}{2})^2\pi^2+(2\log n)^2\}^{\frac{1}{2}}$: from $-R$ to $(n+\frac{1}{2})\iota\pi-2\log n$ along the circle of radius R about the origin as centre, then from

$$(n+\tfrac{1}{2})\iota\pi-2\log n \quad \text{to} \quad (n+\tfrac{1}{2})\iota\pi+2\log n$$

along the chord of this circle and then on to $+R$ along the circumference again.

The path in (7) may then be taken as closed, consisting of the diameter from $-R$ to $+R$ and the path as just described but reversed.

Hence the integral is equal to $2\iota\pi$ multiplied by the sum of the series of residues of the subject at those poles which lie within the contour, that is, those values of λ in the first two quadrants for which $\Theta = 0$. There are no contributions from the apparent poles 0 and $\pm\iota$, for the numerator vanishes at these points to as high an order as the denominator.

Lastly, if $\Theta = 0$,

$$\Omega(\lambda, v, v_0, \alpha, \beta) = \Omega(\lambda, v_0, v, \alpha, \beta) = j e^{-\iota\alpha-\iota\beta}\,\omega(v, \lambda, \alpha, \beta)\,\omega(v_0, -\lambda, \alpha, \beta).$$

Conclusion.

8. Thus, if the rectangular coordinates x, y are connected with two parameters u, v by the relations

$$x \pm \iota y = k \tanh \tfrac{1}{2}(u \pm \iota v),$$

and if a thin isotropic plate is bounded by the circular arcs on which v has the values α, β $(\alpha > \beta)$ while u takes all values between $\pm\infty$, then unit load at (u_0, v_0) produces at (u, v) a deflection

$$\frac{\iota\pi k^2}{2(\cosh u+\cos v)(\cosh u_0+\cos v_0)} \Sigma \frac{j e^{\iota\lambda(u-u_0)-\iota\alpha-\iota\beta}\,\omega(v, \lambda, \alpha, \beta)\omega(v_0, -\lambda, \alpha, \beta)}{(\lambda+\lambda^3)\,\partial\Theta/\partial\lambda}, \tag{8}$$

where $j = \pm 1$, $\quad \Theta = 2\sinh\lambda(\alpha-\beta)-2j\lambda\sin(\alpha-\beta)$,

and the summation extends over both values of j and over those values of λ in the first two quadrants (or if $u < u_0$, in the third and fourth) for which $\Theta = 0$: $\omega(v, \lambda, \alpha, \beta)$ denotes

$$(1+\iota\lambda)\{e^{\lambda v+\iota v}-j e^{(\lambda+\iota)(\alpha+\beta-v)}\}$$
$$+(j e^{-\lambda\alpha+\lambda\beta}-\iota\lambda e^{\iota\alpha-\iota\beta})\{e^{\lambda v+\iota(\alpha+\beta-v)}-j e^{\iota v+\lambda(\alpha+\beta-v)}\}.$$

Other expressions for this deflection have been given in (4 . 3) and (7).

The deflection which has given values and first derivatives on the perimeter of the plate, supposed unloaded, can be deduced from the last result by the formula (1). One form of the expression has been given in (3 . 4).

The values of λ for which $\Theta = 0$ are all imaginary except 0 itself, and the sequence of them tends to infinity in the neighbourhood of the imaginary axis.

9. If $\alpha - \beta = \pi$, the plate becomes the complete circle.

If $\alpha - \beta$ is decreased indefinitely the limiting form of the plate is that of an infinite strip bounded by two parallel lines or the figure bounded by two circles touching each other.

In the case of the infinite strip if the bounding lines are $x = 0$, a, the solution, when $y > y_0$, is

$$\begin{aligned}\text{Green's function} = \Sigma\big[&\{e^{\theta x}(2x\theta - 1 - je^{\theta\alpha}) + e^{-\theta x}(1 + je^{\theta\alpha} + 2j\theta x e^{\theta\alpha})\} \\ &\times \{e^{-\theta x_0}(-2x_0\theta - 1 - je^{-\theta\alpha}) + e^{\theta x_0}(1 + je^{-\theta\alpha} - 2j\theta x_0 e^{-\theta\alpha})\} \\ &\times \tfrac{1}{2}j\pi e^{i\theta(y-y_0)} \div 2a\theta^3(e^{\theta\alpha} + e^{-\theta\alpha} - 2j)\big].\end{aligned}$$

where j is ± 1 and the summation extends over both values of j and over all the values θ in the first two quadrants such that

$$e^{\theta\alpha} - e^{-\theta\alpha} - 2j\theta a = 0.$$

This is the deflection produced at (x, y) by unit load at (x_0, y_0). The deflection Z such that

$$Z = f(y) \text{ when } x = 0, \quad f_1(y) \text{ when } x = a,$$

$$\frac{\partial Z}{\partial x} = g(y) \text{ when } x = 0, \quad g_1(y) \text{ when } x = a,$$

is* $\dfrac{i}{4a}\displaystyle\int_{-\infty}^{\infty} \Sigma \frac{e^{\theta x}(2x\theta - 1 - je^{\theta\alpha}) + e^{-\theta x}(1 + je^{\theta\alpha} + 2j\theta x e^{\theta\alpha})}{\theta(e^{\theta\alpha} + e^{-\theta\alpha} - 2j)}$

$$\times \left[(1 - je^{-\theta\alpha})\{g_1(t) + jg(t)\} - \theta(1 + je^{-\theta\alpha})\{f_1(t) - jf(t)\}\right] e^{i\theta(t-y)}\, dt.$$

10. There are two noteworthy points in the investigation. One is the

* For expansions of this nature see L. N. G. Filon (*Proc. London Math. Soc.*, 1906, p. 422), and compare the expressions given by J. Dougall (*Trans. R. S. E.*, Vol. 41, 1904) for an infinite thick plate.

part played by the functional equation (2.7).* In the problem of the rectangular plate there are three such equations, and the difficulty of that problem is to satisfy all the three.

This functional equation indicates the kind of singularity to be expected in Green's function at a corner of *any* plate.

A second point is the form of the result (8) in which the different terms are all products, one factor depending only on P, and the other only on P_0. The typical factor is $\omega(v, \lambda, \alpha, \beta)$, a function which vanishes to the second order along both edges and has no singularity except at the corners. This function expresses what may be called a "mode" of strain of the clamped plate, corresponding to the modes of vibration of a string.

* In the ordinary potential theory there is a similar, but simpler, functional equation, namely,

$$\psi(u + \iota\alpha - \iota\beta) - \psi(u - \iota\alpha + \iota\beta) = \text{a known function.}$$

In each case the right side vanishes when Green's function is in question.

ON A TYPE OF MODULAR RELATION

By L. J. ROGERS.

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1. *Definitions and Notation.*—In these *Proceedings*, Ser. 2, Vol. 18, p. xx, a statement is made by Ramanujan with reference to the functions $G(x)$, $H(x)$, defined as basic series or as infinite products.* From the basic series forms, it follows that

$$H(x)/G(x) = \frac{1}{1+}\,\frac{x}{1+}\,\frac{x^2}{1+}\,\frac{x^3}{1+}\cdots;$$

from the product forms it follows that

$$G(x) = \frac{1-x^2-x^3+x^9+x^{11}-\ldots}{1-x-x^2+x^5+x^7-\ldots},$$

$$H(x) = \frac{1-x-x^4+x^7+x^{13}-\ldots}{1-x-x^2+x^5+x^7}.$$

It will be convenient to make a slight alteration in the use of the argument-symbol x, by writing q^2 for x, to bring the series into line with the usual elliptic function notation. Moreover, it will be better to adopt what we may call a *standard* form of ϑ-function, in which the numerators of all indices are perfect squares. This is easily done by multiplying by a suitable power of x in each case, which we may call the *standardizing* power with standardizing index. Thus the numerators of $G(x)$ and $H(x)$ require indices $\frac{1}{40}$, $\frac{9}{40}$ respectively, and the common denominator $\frac{1}{24}$.

I write then

$$g = \frac{q^{\frac{1}{40}}(1-q^4-q^6+\ldots)}{q^{\frac{1}{24}}(1-q^2-q^4+\ldots)}, \qquad h = \frac{q^{\frac{9}{40}}(1-q^2-q^8+\ldots)}{q^{\frac{1}{24}}(1-q^2-q^4+\ldots)}, \qquad (1.1$$

* In the numerator of the second term of the series for $G(x)$, *for* 1, *read* x.

so that Ramanujan's identity now takes the form

$$gh(g^{10}-11g^5h^5-h^{10}) = 1. \tag{1.2}$$

The quotient h/g will be written μ (1.3), and the results of replacing q by q^p in g, h, μ will be written g_p, h_p, μ_p (1.31). The continued fraction form for μ is now

$$\mu = \frac{q^{\frac{1}{5}}}{1+}\,\frac{q^2}{1+}\,\frac{q^4}{1+}\,\frac{q^6}{1+}\,\ldots, \tag{1.4}$$

which has the advantage of conciseness in form, but is otherwise, as the basic series are, irrelevant to the present investigation, which is based purely on ϑ-function identities.

I may point out that Ramanujan's other identity (*loc. cit.*)

$$H(x)\,G(x^{11})-x^2G(x)\,H(x^{11}) = 1,$$

now becomes

$$hg_{11}-gh_{11} = 1, \tag{1.5}$$

which, with (1.2), has the advantage of containing no extraneous powers of the argument.

The denominators of g and h are

$$\frac{1}{\sqrt{3}}\,\vartheta_1\left(\frac{\pi}{3},\ q^{\frac{1}{3}}\right) = q^{\frac{1}{12}}\prod_1^\infty(1-q^{2n}),$$

and will be written P,* with P_p for

$$\frac{1}{\sqrt{3}}\,\vartheta_1\left(\frac{\pi}{3},\ q^{\frac{1}{3}p}\right).$$

Dashes attached to g, h, μ, will denote like functions of the complementary modulus q' (1.6).

2. *Proof of identity* (1.2).—Writing c_1 for $\cos\frac{1}{10}\pi$, c_3 for $\cos\frac{3}{10}\pi$, and

$$a = c_3/c_1 = \tfrac{1}{2}(\sqrt{5}-1), \tag{2.1}$$

we have

$$\vartheta_2(\tfrac{1}{10}\pi,\ q^{\frac{1}{5}}) = 2c_1(q^{\frac{1}{20}}+aq^{\frac{9}{20}}-aq^{\frac{49}{20}}-q^{\frac{81}{20}}-\ldots),$$

or, replacing the ϑ_2-function by its equivalent product,

$$P(g+ah) = q^{\frac{1}{20}}\prod_1^\infty(1+q^{\frac{2n}{5}}e^{\frac{1}{5}\pi i})(1+q^{\frac{2n}{5}}e^{-\frac{1}{5}\pi i})q^{-\frac{n}{5}}P_{\frac{1}{5}}, \tag{2.2}$$

* The use of G as in Whittaker and Watson's *Modern Analysis*, p. 465, is at present untenable, and is also unstandardised.

while

$$\vartheta_2(\tfrac{3}{10}\pi,\ q^{\frac{1}{2}}) = 2c_3 P\left(g - \frac{h}{a}\right)$$

$$= 2c_3 q^{\frac{1}{40}} \prod_1^\infty (1+q^{\frac{2}{5}n} e^{\frac{2}{5}\pi i})(1+q^{\frac{2}{5}n} e^{-\frac{2}{5}\pi i})\, q^{-\frac{1}{40}} P_{\frac{1}{2}}. \qquad (2.3)$$

Hence

$$P^2(g+ah)\left(g - \frac{1}{a}h\right) = P^2(g^2 - gh - h^2) = q^{\frac{1}{10}} \prod_1^\infty \left(\frac{1-q^{2n}}{1-q^{5n}}\right) q^{-\frac{1}{10}} P_{\frac{1}{2}}^2$$

$$= P P_{\frac{1}{2}}. \qquad (2.4)$$

Moreover $$P^2 gh = q^{\frac{1}{2}}(1-q^4)(1-q^6)\ldots(1-q^{10})(1-q^{20})\ldots$$

$$\times (1-q^2)(1-q^8)\ldots(1-q^{10})(1-q^{20})\ldots$$

$$= P P_5. \qquad (2.5)$$

But, by changing q into $q\omega$, $q\omega^2$, ..., where $\omega = e^{\frac{2}{5}\pi i}$, and multiplying the results (2.2), we have on the left-hand side a factor $g^5 + a^5 h^5$, since h/g is altered to $\omega^2 h/g$, &c., while on the right-hand side the product arising from any factor $1+q^{\frac{2}{5}n} e^{\frac{1}{5}\pi i}$, *i.e.* $1-q^{\frac{2}{5}n}\omega^2$, is $1-q^{2n}$, if n is not a multiple of 5, but is otherwise $(1-q^{2m}\omega^2)^5$, where $n = 5m$.

Hence

$$P^5(g^5+a^5h^5)$$

$$= q^{\frac{1}{2}} \prod (1-q^{2n})^2 \prod_1^{\prime\infty} (1-q^{2m}\omega^2)^5 (1-q^{2m}\omega^3)^5 \prod (1-q^{2n}) \prod_1^\infty (1-q^{2m})^5,$$

where n has all positive integral values except multiples of 5, *i.e.*

$$P^5(g^5+a^5h^5) = q^{\frac{1}{2}} \frac{\prod' (1-q^{2n})^8}{\prod (1-q^{10m})^3} \prod_1^\infty (1+q^{2n} e^{\frac{1}{5}\pi i})^5 (1+q^{2n} e^{-\frac{1}{5}\pi i})^5$$

$$= \frac{P^3}{P_5^3} \frac{\vartheta_2(\frac{1}{10}\pi,\ q)^5}{(2c_1)^5} = \frac{P^3}{P_5^3}(g_5 + a h_5)^5 P_5^5, \qquad (2.6)$$

i.e. $$g^5 + a^5 h^5 = P_5^2 (g_5 + a h_5)^5 / P^2. \qquad (2.7)$$

Similarly, from (2.3), or by changing $\sqrt{5}$ to $-\sqrt{5}$ in a, we have

$$g^5 - a^{-5} h^5 = P_5^2 (g_5 - a^{-1} h_5)^5 / P^2. \qquad (2.8)$$

Changing q to q^5, (2.4) becomes

$$g_5^2 - g_5 h_5 - h_5^2 = P/P_5, \tag{2.9}$$

while (2.5) becomes

$$gh = P_5/P; \tag{2.10}$$

whence, seeing that $a^5 = \frac{1}{2}(5\sqrt{5}-11)$, we have

$$gh\,(g^{10} - 11g^5h^5 - h^{10}) = P_5^5\,(g_5^2 - g_5 h_5 - h_5^2)^5/P^5 = 1.$$

It may be here observed that a similar relation exists between the ϑ-series derived from $\vartheta_2(x, q^{1/p})$, where

$$x = \frac{\pi}{2p},\ \frac{3\pi}{2p},\ \ldots,\ \frac{p-2}{p}\pi,$$

each being divided by P. The relation asserts the equality to unity of a homogeneous algebraic function of degree $\frac{1}{2}(p^2-1)$ in the ϑ-quotients corresponding to g and h when $p = 5$.

3. The main object of the present memoir is to establish algebraic relations connecting μ and μ_p when $p = 2, 3, 5, 11$.

It is easy to see that such algebraic relations exist, For, by (2.4), (2.5), and (1.3), (1.7),

$$1 - \mu - \mu^2 = \mu\, P_5/P_5.$$

But P_5/P and P_5/P are known each to be connected algebraically with the moduli and multipliers in the quintic transformation of elliptic functions, and hence with the modulus k, so that μ is connected algebraically with k; and hence the modular equation of any order p implies an algebraic relation between μ and μ_p.

4. *Complementary relation.*—From the formula

$$\sqrt{w}\,\vartheta_2(x, q) = \sum_{n=-\infty}^{\infty} (-1)^n\, e^{-\pi/w\,(n+x/\pi)^2},$$

where $e^{-\pi w} = q$, we have

$$\frac{\vartheta_2(\frac{3}{10}\pi, q^5)}{\vartheta_2(\frac{1}{10}\pi, q^5)} = \frac{\Sigma\,(-1)^n\, e^{-5\pi/w\,(n+\frac{3}{10})^2}}{\Sigma\,(-1)^n\, e^{-5\pi/w\,(n+\frac{1}{10})^2}},$$

which, with the notation of (1 . 3), (1 . 6), and (2 . 1), gives

$$\frac{a-\mu}{1+a\mu} = \mu'.^{*} \tag{4 . 1}$$

5. *Quadratic relation.*—Writing u_n for $\vartheta_2(\frac{1}{10}n, q)$, where

$$n = 0,\ 1,\ 2,\ 3,\ 4,$$

we have from the formula

$$\vartheta_2(x+y+z)\,\vartheta_2(x)\,\vartheta_2(y)\,\vartheta_2(z)+\vartheta_1(x+y+z)\,\vartheta_1(x)\,\vartheta_1(y)\,\vartheta_1(z)$$
$$= \vartheta_2(0)\,\vartheta_2(y+z)\,\vartheta_2(z+x)\,\vartheta_2(x+y),$$

when $x = y = \frac{1}{10}\pi,\ z = \frac{1}{5}\pi,$

$$u_1u_4(u_1u_2+u_3u_4) = u_0u_2u_3^2, \tag{5 . 1}$$

and when $x = y = \frac{3}{10}\pi,\ z = \frac{3}{5}\pi,$

$$u_2u_3(u_1u_2-u_3u_4) = u_0u_1^2u_4. \tag{5 . 2}$$

But
$$\frac{\vartheta_1(\frac{1}{5}\pi, q^2)}{\vartheta_1(\frac{3}{5}\pi, q^2)} = \frac{\vartheta_1(\frac{1}{10}\pi)\,\vartheta_2(\frac{1}{10}\pi)}{\vartheta_1(\frac{3}{10}\pi)\,\vartheta_2(\frac{3}{10}\pi)},$$

from a known formula for $\vartheta_1(2x, q^2)$, *i.e.*

$$\frac{\vartheta_2(\frac{3}{10}\pi, q^2)}{\vartheta_2(\frac{1}{10}\pi, q^2)} = \frac{u_1u_4}{u_3u_2}. \tag{5 . 3}$$

By § 4, $u_3/u_1 = \mu'$, so that (5 . 3) gives $u_4/u_2 = \mu'\mu_1'$, and (5 . 1), (5 . 2) give by division

$$\frac{u_4^2}{u_2^2}\,\frac{1+\mu' u_4/u_2}{1-\mu' u_4/u_2} = \mu'^2.$$

Suppressing dashes and changing μ, μ_1 to μ_2, μ we get, finally,

$$\mu^3\mu_2^2+\mu^2+\mu\mu_2^2-\mu_2 = 0. \tag{5 . 4}$$

6. *Cubic relation.*—Since

$$Q\vartheta_1(3x, q^3) = \vartheta_1(x)\,\{\vartheta_2^2(x)\,\vartheta_1^2(\tfrac{1}{3}\pi)-\vartheta_1^2(x)\,\vartheta_2^2(\tfrac{1}{3}\pi)\}/\vartheta_2^2(0),$$

where
$$Q = \vartheta_1^2(\tfrac{1}{3}\pi)\,\vartheta_1'(0)/3\vartheta_1'(0, q^3),$$

* This gives a simple numerical value for μ when $q = e^{-\pi}$, for $a/(1-a^2) = 1$, so that $a = \tan(\frac{1}{2}\tan^{-1}2)$, and $\mu = \tan(\frac{1}{4}\tan^{-1}2)$.

we easily deduce that

$$\vartheta_1(3x, q^3)\vartheta_1^3(2x) - \vartheta_1(6x, q^3)\vartheta_1(x)$$

$$= 3\frac{\vartheta_1'(0, q^3)}{\vartheta_1'(0)\vartheta_2^2(0)}\vartheta_1(x)\vartheta_1(2x)\{\vartheta_2^2(x)\vartheta_1^2(2x) - \vartheta_1^2(x)\vartheta_2^2(2x)\}$$

$$= 3\frac{\vartheta_1'(0, q^3)}{\vartheta_1'(0)}\vartheta_1^2(x)\vartheta_1(2x)\vartheta_1(3x). \qquad (6.1)$$

By changing x into xi/w, we have a relation connecting ϑ_1-functions for moduli $q'^{\frac{1}{3}}$ and q', which after suppressing dashes and changing q to q^3, gives

$$\vartheta_1(x)\vartheta_1^3(2x, q^3) - \vartheta_1(2x)\vartheta_1^3(x, q^3) = \frac{\vartheta_1'(0)}{\vartheta_1'(0, q^3)}\vartheta_1^2(x, q^3)\vartheta_1(2x, q^3)\vartheta_1(3x, q^3). \quad (6.2)$$

Multiplying (6.1) and (6.2), the factors independent of x cancel out. If $x = \frac{1}{5}\pi$, $\vartheta_1(3x) = \vartheta_1(2x)$, and

$$\frac{\vartheta_1(x)}{\vartheta_1(2x)} = \frac{u_3}{u_1} = \mu'.$$

Also $\vartheta_1(6x, q^3) = -\vartheta_1(x, q^3)$, so that in the resultant equation we connect $\vartheta_1(x)/\vartheta_1(2x)$ with $\vartheta_1(x, q^3)/\vartheta_2(2x, q^3)$, *i.e.* μ' with μ_3'. Writing then μ_3 for the former and μ for the latter, we get

$$(1+\mu\mu_3^3)(\mu_3-\mu^3) = 3\mu^2\mu_3^2,$$

or

$$\mu^4\mu_3^3+\mu^3+3\mu^2\mu_3^2-\mu\mu_3^4-\mu_3 = 0. \qquad (6.2)$$

7. *Quintic relation.*—From (2.7), (2.8), we have

$$\frac{a^5-\mu^5}{1+a^5\mu^5} = \left(\frac{a-\mu_5}{1+a\mu_5}\right)^5;$$

i.e.

$$\mu^5 = \mu_5\frac{1-2\mu_5+4\mu_5^2-3\mu_5^3+\mu_5^4}{1+3\mu_5+4\mu_5^2+2\mu_5^3+\mu_5^4}. \qquad (7.1)$$

8. *Relation when* $p = 11$.—This is immediately deduced from Ramanujan's formulæ (1.2) and (1.5), viz.

$$\mu\mu_{11}(1-11\mu^5-\mu^{10})(1-11\mu_{11}^5-\mu_{11}^{10}) = (\mu-\mu_{11})^{12}. \qquad (8.1)$$

9. The identity (1.5), together with others of the same type,* may be

* These were communicated privately to me in February 1919, but, as I understand that Ramanujan has left no proof, I suggest the proof given in this section.

proved by Schröter's formulæ,* connected with the multiplication of two ϑ_3-series of different orders. As his formulæ require some modification and specialisation for the present problem, it will be simpler to give what is virtually his method in detail, and to employ summational forms instead of ϑ-function notation.

Let p be a prime number, and α, β any integers, such that

$$\alpha m^2+\beta = \lambda p, \qquad (9.1)$$

m being odd. The indicial letters r, s, t, σ following the symbol Σ will denote summation extending through all values $0, \pm 1, \pm 2, \ldots, \pm \infty$. Let

$$\Sigma(-1)^r q^{p\alpha(r+mv)^2}.\Sigma(-1)^s q^{p\beta(s+v)^2} = \Sigma\Sigma(-1)^{r+s} q^I, \qquad (9.2)$$

so that

$$I = p\alpha(r+mv)^2+p\beta(s+v)^2.$$

Put $r = ms+t$, so that for any given value of s, t is equally general with r, and

$$\begin{aligned} I &= p\alpha\{m(s+v)+t\}^2+p\beta(s+v)^2 \\ &= \lambda p^2(s+v)^2+2p\alpha mt(s+v)+p\alpha t^2 \quad [\text{by } (9.1)] \\ &= \lambda\left\{p(s+v)+\frac{\alpha mt}{\lambda}\right\}^2+\frac{\alpha\beta}{\lambda}t^2, \end{aligned}$$

while

$$(-1)^{r+s} = (-1)^{(m+1)s+t} = (-1)^t.$$

Now let v have the p values $\frac{1}{2p}, \frac{3}{2p}, \ldots, \frac{2p-1}{2p}$, and add together all the equations (9.2) so obtained. The series on the left-hand side will be equal in pairs, while their values for $v = p/2p$ will be zero. On the right-hand side we have $\Sigma\Sigma(-1)^p q^I$, where now

$$\begin{aligned} I &= \lambda\left\{p\left(s+\frac{2n-1}{2p}\right)+\frac{\alpha mt}{\lambda}\right\}^2+\frac{\alpha\beta}{\lambda}t^2 \quad \left(n = 1, 2, \ldots, \frac{p-1}{2}\right) \\ &= \lambda\left\{\frac{2\sigma+1}{2}+\frac{\alpha mt}{\lambda}\right\}^2+\frac{\alpha\beta}{\lambda}t^2. \qquad (9.3) \end{aligned}$$

Let $p = 5$, $m = 1$, so that $\alpha+\beta = 5\lambda$. If $v = \frac{1}{10}$,

$$\Sigma(-1)^r q^{5\alpha(r+\frac{1}{10})^2} = q^{\alpha/20}(1-q^{4\alpha}-q^{9\alpha}+\ldots) = g_\alpha P_\alpha,$$

according to the notation of § 1. If $v = \frac{3}{10}$,

$$\Sigma(-1)^r q^{5\alpha(r+\frac{3}{10})^2} = h_\alpha P_\alpha.$$

* Enneper, *Elliptische Functionen*, Zweite Auflage, p. 474.

Hence $$2(g_\alpha g_\beta + h_\alpha h_\beta) P_\alpha P_\beta = \Sigma\Sigma(-1)^t q^I, \tag{9.4}$$

as in (9.3). Again, if $p = 5$, $m = 3$, so that $9\alpha+\beta = 5\lambda$, then when $v = \frac{1}{10}$,

$$\Sigma(-1)^r q^{5\alpha(r+\frac{3}{10})^2} = h_\alpha,$$

and when $v = \frac{3}{10}$, $$\Sigma(-1)^r q^{5\alpha(r+\frac{3}{10})^2} = -g_\alpha;$$

so that $$2(h_\alpha g_\beta - h_\beta g_\alpha) = \Sigma\Sigma q^I(-1)^t, \tag{9.41}$$

as in (9.3).

Now suppose $p = 3$, $m = 1$, and use letters a, b, l instead of α, β, λ. We shall get the same value of I as in (9.3), provided $l = \lambda$, $ab = \alpha\beta$, and

$$\frac{am}{\lambda} \pm \frac{a}{l} \text{ is an integer.} \tag{9.5}$$

When $v = \frac{1}{6}$ or $\frac{5}{6}$,

$$\Sigma(-1)^r q^{3\alpha(r+v)^2} = q^{\frac{1}{12}\alpha}(1 - q^{2\alpha} - q^{4\alpha} + \ldots) = P_\alpha,$$

so that the left-hand side is

$$2P_a P_b. \tag{9.6}$$

Hence, by (9.4), $$g_\alpha g_\beta + h_\alpha h_\beta = P_a P_b / P_\alpha P_\beta, \tag{9.7}$$

where $$\alpha+\beta = 5\lambda, \quad a+b = 3l = 3\lambda;$$

while, by (9.5), $$h_\alpha g_\beta - h_\beta g_\alpha = P_a P_b / P_\alpha P_\beta, \tag{9.71}$$

where $$\alpha+\beta = 5\lambda, \quad a+b = 3l = 3\lambda.$$

Thus, if $\alpha = 1$, $\beta = 11$, $m = 3$, $\lambda = 4$, then $a = 1$, $b = 11$, $l = 4$, so that (9.5) is satisfied; and, by (9.71),

$$hg_{11} - h_{11} g = PP_{11}/PP_{11} = 1,$$

as in (1.5).

If $\alpha = 1$, $\beta = 9$, $m = 1$, $\lambda = 2$, $a = 3$, $b = 3$, $l = 2$, we have

$$gg_9 + hh_9 = P_3^2/PP_9. \tag{9.8}$$

When $\alpha = 1$, $\beta = 14$, $m = 1$, $\lambda = 3$, $a = 2$, $b = 7$, $l = 3$, and

$$gg_{14} + hh_{14} = P_2 P_7 / PP_{14}. \tag{9.81}$$

When $\alpha = 2$, $\beta = 7$, $m = 3$, $\lambda = 5$, $a = 1$, $b = 14$, $l = 5$, and $3am/\lambda - a/l = 1$, and

$$h_2 g_7 - h_7 g_2 = PP_{14}/P_2 P_7. \tag{9.82}$$

10. Certain cases of (9 . 4) and (9 . 41) may be treated without the help of the results for $p = 3$. For instance, if $\lambda = 1$, when of course $m = 1$, then I in (9 . 3) is equivalent to $\left(\frac{2\sigma+1}{2}\right)^2 + \alpha\beta t^2$; for amt is an integer. and may be merged in the general symbol σ. In this case

$$2(g_\alpha g_\beta + h_\alpha h_\beta) = \Sigma q^{(\sigma+\frac{1}{2})^2} \Sigma (-1)^t q^{\alpha\beta t^2}/P_\alpha P_\beta$$
$$= \vartheta_2(0)\, \vartheta(0, q^{\alpha\beta})/P_\alpha P_\beta.$$

Thus
$$gg_4 + hh_4 = \tfrac{1}{2}\vartheta_2(0)\, \vartheta(0, q^4)/PP_4,$$
$$g_2 g_3 + h_2 h_3 = \tfrac{1}{2}\vartheta_2(0)\, \vartheta(0, q^5)/PP_6. \qquad (10 . 1)$$

Again, when $\alpha = 1$, $\beta = 6$, $m = 3$, $\lambda = 3$,

$$I = 3(\sigma+\tfrac{1}{2}+t)^2 + It^2 \equiv 3(\sigma+\tfrac{1}{2})^2 + It^2.$$

Hence
$$hg_6 - h_6 g = \tfrac{1}{2}\Sigma q^{3(\sigma+\frac{1}{2})^2} \Sigma q^{2t^2} (-1)^t/PP_6$$
$$= \tfrac{1}{2}\vartheta_2(0, q^3)\, \vartheta(0, q^2)/PP_6. \qquad (10 . 2)$$

Again, when $p = 2$, $m = 1$, the left-hand series in (9 . 2) have only one form, derived from $v = \frac{1}{4}$ or $v = \frac{3}{4}$, viz.

$$\Sigma(-1)^r q^{2(r+\frac{1}{4})^2} = q^{\frac{1}{2}}(1 - q - q^3 + \ldots)$$
$$= q^{\frac{1}{2}}(1-q)(1-q^3)(1-q^5)(1-q^7)\ldots(1-q^4)(1-q^8)$$
$$= P_{\frac{1}{2}} P_2/P. \qquad (10 . 3)$$

Thus, if $\alpha = 3$, $\beta = 8$, $m = 3$, $\lambda = 7$, $a = 2$, $b = 12$, $p = 2$, $l = 7$, so that

$$\frac{ma}{\lambda} - \frac{a}{l} = 1,$$

we have
$$h_3 g_8 - h_8 g_3 = \frac{PP_4}{P_2 P_8} \frac{P_6 P_{24}}{P_3 P_{12}}. \qquad (10 . 4)$$

When $\alpha = 1$, $\beta = 16$, $m = 3$, $\lambda = 5$, $a = 2$, $b = 12$, $p = 2$, $l = 7$, so that

$$\frac{ma}{\lambda} + \frac{a}{l} = 1,$$

$$hg_{16} - h_{16} g = \frac{PP_4}{P_2} \frac{P_4 P_{16}}{P_8} \frac{1}{PP_{16}} = \frac{P_4^2}{P_2 P_8}. \qquad (10 . 5)$$

These results, as well as many others, have all been given by Ramanujan.

11. To resume the theory of the modular connection between μ and μ_p, there still remains the case of $p = 7$, which presents great difficulties when treated by the methods of § 5 and § 6. The relation

$$(gg_{14}+hh_{14})(h_2g_7-h_7g_2) = 1,$$

derived from (9 . 81) and (9 . 82), combined with (1 . 2), with its extension to suffixes 2, 7, 14, would, in connection with (5 . 4), give a relation between μ and μ_7, but the method is impracticable.

It is to be observed, however, that in the cases of $p = 2, 3, 11$ the relations are of degree $p+1$, just as the Jacobian modular equation in $\sqrt[4]{k}$ and $\sqrt[4]{l}$ is of degree $p+1$, except in the quadratic cases. Though it is not obvious how we may set forth a general hypothesis, we may at least see what the roots of μ are when $\mu_p (p = 2, 3, 11)$ is supposed given. Writing $\mu(q^{2p})$ for μ_p, we see that p of the roots of the equation are $\mu(q^2)$, $\mu(q^2\omega)$, $\mu(q^2\omega^2)$, ..., where $\omega = e^{2\pi i/p}$. Now

$$\mu(q^2) = q^{\frac{2}{5}}\Pi(1-q^{2n})^{\pm 1},$$

according as $n \equiv \pm 1$, or $\pm 2 \pmod 5$. The product of these p roots is therefore

$$q^{\frac{2}{5}p}\Pi(1-q^{2np})^{\pm 1}\,\Pi(1-q^{2m})^{\pm p},$$

where $n \not\equiv 0 \pmod p$, but $m \equiv 0 \pmod p$; except that, in the case of $p = 2$, $\mu(-q^2)$ is negative, and a negative sign must be placed before the expression.

Now $\Pi(1-q^{2np})^{\pm 1}$ includes all the binomial factors of μ_p, except those for which $n \equiv 0 \pmod p$. These can all be supplied by μ_{p^2}, either by multiplication or division.

Thus, when $p = 3$, we have $(1-q^6)(1-q^{24})\ldots/(1-q^{12})\ldots$, where $1-q^{36}$ fails in the numerator and $1-q^{18}$ in the denominator. Hence the required product of the p roots is $\mu_3\mu_9$; and in general, when $p = \pm 2 \pmod 5$, it is $\mu_p\mu_{p^2}$. If, however, $p = \pm 1 \pmod 5$, as when $p = 11$, we have $(1-q^{22})(1-q^{88})\ldots/(1-q^{44})\ldots$, where $1-q^{242}$ fails in the numerator and $1-q^{484}$ in the denominator, so that the product is μ_{11}/μ_{121}, or in general μ_p/μ_{p^2}.

Similarly in $\Pi(1-q^{2m})^{\pm p}$, when $m \equiv 0 \pmod p$, we have all the binomial factors of μ_p^{-p}, if $p \equiv \pm 2 \pmod 5$, but all the binomial factors of μ_p^p, if $p \equiv \pm 1 \pmod 5$. Hence the product of the p roots is $\mu_p^{-p+1}\mu_{p^2}$, when $p \equiv \pm 2 \pmod 5$, and μ_p^{p+1}/μ_{p^2} when $p \equiv \pm 1 \pmod 5$. Thus in the quadratic case, where, by (5 . 4), the product of all the roots in μ is $1/\mu_2$, it follows, since $\mu(-q^2)$ is negative, that the third root is $-1/\mu_4$.

In the cubic case, by (6 . 2), the product of all the roots in μ is $-1/\mu_3^2$, so that the fourth root is $-1/\mu_9$.

In the quintic case (see § 7), the above considerations do not apply, and μ^5 is explicit in μ_5.

In the 11-ic case, the product of all the roots is μ_{11}^{12}, so that the 12th root is μ_{121}.

In conclusion we may notice that if $\mu_p = a$ then $\mu'_{1/p} = 0$, by § 4; *i.e.* $q'_{1/p} = 0$, which by the modular theory of Jacobi and Sohncke implies that p roots of q' are zero, *i.e.* p roots of μ are a. Thus, when $\mu_2 = a$, (5 . 4) reduces to

$$(\mu - a)^2 \left(\mu - \frac{1}{a}\right) = 0;$$

when $\mu_3 = a$, (6 . 2) reduces to

$$(\mu - a)^3 \left(\mu + \frac{1}{a}\right) = 0:$$

when $\mu_{11} = a$, (8 . 1) reduces to

$$(\mu - a)^{12} = 0.$$

FUNCTIONS OF LIMITING MATRICES

By F. B. Pidduck.

[Read June 10th, 1920.]

1. Interesting problems arise in the theory of matrices when two or more of the roots are equal. This case has been discussed by Frobenius,* Sylvester,† Buchheim,‡ and Taber,§ by different methods. Sylvester‖, ¶ gave the formula for any function of a matrix with unequal roots, and suggested that the case of equal roots might be treated by passing to the limit. Sylvester's suggestion has to be modified before it can be put into practice, as the purely symbolic method leads to difficulties when we come to consider the non-scalar fractional powers of a scalar.¶, §

It appears that several lines of investigation can be coordinated by reverting to Grassmann's treatment of a matrix as an open product.** The limiting process can then be carried out in full generality, and thus we have convenient explicit formulæ in terms of the scalar coefficients of the degenerate matrix.

If all the roots $\lambda_1, \lambda_2, \ldots, \lambda_n$ of a matrix of order n are distinct there are n distinct (generalised) axes $u_1, u_2, \ldots, u_n$. For simplicity we shall denote Grassmann's external multiplication by simple juxtaposition as in the latter part of A1, avoiding the square brackets of A2. Write

$$u_1' = (-)^{n-1}\frac{u_2u_3\ldots u_n}{u_1u_2\ldots u_n}, \quad u_2' = (-)^{n-2}\frac{u_1u_3\ldots u_n}{u_1u_2\ldots u_n}, \quad \ldots, \quad u_n' = \frac{u_1u_2\ldots u_{n-1}}{u_1u_2\ldots u_n}. \tag{1}$$

* G. Frobenius, *Jour. f. Math.* (*Crelle*), Vol. 84 (1878), p. 1.

† J. J. Sylvester, *Johns Hopkins Univ. Circulars*, Vol. 3 (1884), p. 9 (*Math. Papers*, Vol. 4, p. 133); *Amer. Jour. Math.*, Vol. 6 (1884), p. 270 (*Math. Papers*, Vol. 4, p. 208).

‡ A. Buchheim, *Proceedings*, Vol. 16 (1884), p. 63; *Phil. Mag.* [5], Vol. 22 (1886), p. 173.

§ H. Taber, *Amer. Jour. Math.* Vol. 12 (1890), p. 337.

‖ J. J. Sylvester, *Comptes Rendus*, Vol. 94 (1882), p. 55 (*Math. Papers*, Vol. 3, p. 562).

¶ J. J. Sylvester, *Phil. Mag.* [5], Vol. 16 (1883), p. 267 (*Math. Papers*, Vol. 4, p. 110).

** H. Grassmann, *Die lineale Ausdehnungslehre*, p. 266, 1844 [referred to as A1; *Ges. Werke*, Vol. 1 (1), p. 284]; *Die Ausdehnungslehre*, p. 245, 1862 [referred to as A2; *Ges. Werke*, Vol. 1 (2), p. 243].

Then the matrix is expressed as an open product in the form

$$\Phi = \lambda_1 u_1 . u_1' + \lambda_2 u_2 . u_2' + \ldots + \lambda_n u_n . u_n'.$$

The notation is a combination of Grassmann's first notation with Gibbs' dyadic notation as modified by Heaviside. Thus

$$\Phi x = \lambda_1 u_1 . u_1' x + \lambda_2 u_2 . u_2' x + \ldots + \lambda_n u_n . u_n' x,$$

where $u_1' x$ is the external product

$$(-)^{n-1} \frac{u_2 \ldots u_n x}{u_1 u_2 \ldots u_n}$$

and may be described without impropriety as the scalar product of u_1' and x.* Following Gibbs, u' is said to be the extensive quantity reciprocal to u.

2. Let $s+1$ axes coalesce with u_r. In addition, let u_t be a typical axis which remains distinct from u_r while the corresponding root tends to λ_r, and u_m a typical axis distinct from u_r with a root λ_m distinct from λ_r. We have therefore four sets of quantities to be considered in the first place, represented by the scheme

$$\begin{array}{c|c|c|c} \lambda_r & \lambda_r+\epsilon_1 \ \ldots \ \lambda_r+\epsilon_s & \lambda_r+\epsilon_t & \lambda_m, \\ u_r & u_r+x_1 \ \ldots \ u_r+x_s & u_t & u_m, \end{array}$$

where the ϵ's are ultimately indefinitely small scalars and the x's ultimately indefinitely small extensive quantities. For the present they are finite. Write P for the complete external product

$$u_r (u_r+x_1) \ldots (u_r+x_s) \Pi u_t \Pi u_m,$$

and PEu for the product omitting any factor u, with such a sign that $PEu'P$ is the extensive quantity reciprocal to u in the n-ad P. Write Q similarly for the complete external product

$$u_r x_1 \ldots x_s \Pi u_t \Pi u_m.$$

Then

$$P = Q, \quad PE(u_r+x_\sigma) = QEx_\sigma, \quad PEu_t = QEu_t, \quad PEu_m = QEu_m,$$

and

$$PEu_r = QEu_r - \sum_{\sigma=1}^{s} QEx_\sigma,$$

* See A2, Abschnitt 1, Kap. 4: the scalar product is used in this sense in A1, pp. 268 *et seq.* The quantities u' are practically Grassmann's complementary quantities.

a case of a general theorem of Grassmann's to be used later. Hence

$$\Phi = \lambda_r u_r . \frac{PEu_r}{P} + \sum_{\sigma=1}^{s} (\lambda_r + \epsilon_\sigma)(u_r + x_\sigma) . \frac{PE(u_r + x_\sigma)}{P}$$
$$+ \sum_t (\lambda_r + \epsilon_t) u_t . \frac{PEu_t}{P} + \sum_m \lambda_m u_m . \frac{PEu_m}{P}$$
$$= \lambda_r \left[u_r . \frac{QEu_r}{Q} + \sum_{\sigma=1}^{s} x_\sigma . \frac{QEx_\sigma}{Q} + \sum_t u_t . \frac{QEu_t}{Q} \right] + \sum_m \lambda_m u_m . \frac{QEu_m}{Q}$$
$$+ \sum_{\sigma=1}^{s} \epsilon_\sigma (u_r + x_\sigma) . \frac{QEx_\sigma}{Q} + \sum_t \epsilon_t u_t . \frac{QEu_t}{Q} .$$

Since $1 = \Sigma u \,.\, QEu/Q$ in any n-ad Q, we have more simply

$$\Phi = \lambda_r + \sum_m (\lambda_m - \lambda_r) u_m . \frac{QEu_m}{Q} + \sum_{\sigma=1}^{s} \epsilon_\sigma (u_r + x_\sigma) . \frac{QEx_\sigma}{Q} + \sum_t \epsilon_t u_t . \frac{QEu_t}{Q} . \quad (2)$$

The problem before us is to find the limiting form of Φ when the scalars ϵ and the extensive quantities x tend to zero. Take a set of extensive quantities $u_{r+1}, u_{r+2}, \ldots, u_{r+s}$ arbitrarily, but fixed. Let R be the complete external product

$$u_r u_{r+1} \ldots u_{r+s} \text{H} u_t \text{H} u_m ,$$

and let $$x_\sigma = (\sigma 0) u_r + (\sigma 1) u_{r+1} + \ldots + (\sigma s) u_{r+s} + \Sigma(\sigma t) u_t + \Sigma(\sigma m) u_m . \quad (3)$$

Thus

$$Q = u_r \{ (10) u_r + (11) u_{r+1} + \ldots + (1s) u_{r+s} + \Sigma(1t) u_t + \Sigma(1m) u_m \}$$
$$\{ (20) u_r + (21) u_{r+1} + \ldots + (2s) u_{r+s} + \Sigma(2t) u_t + \Sigma(2m) u_m \}$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$
$$\{ (s0) u_r + (s1) u_{r+1} + \ldots + (ss) u_{r+s} + \Sigma(st) u_t + \Sigma(sm) u_m \} \text{H} u_t \text{H} u_m .$$

Write Δ for the determinant

$$\begin{vmatrix} (11) & (12) & \ldots & (1s) \\ (21) & (22) & \ldots & (2s) \\ \ldots & \ldots & \ldots & \\ (s1) & (s2) & \ldots & (ss) \end{vmatrix} .$$

Then leaving QEx_σ undetermined for the moment,

$$Q = \Delta R, \quad QEu_r = \Delta REu_r - \sum_{\sigma=1}^{s} (\sigma 0) QEx_\sigma ,$$

$$QEu_t = \Delta REu_t - \sum_{\sigma=1}^{s} (\sigma t) QEx_\sigma , \quad QEu_m = \Delta REu_m - \sum_{\sigma=1}^{s} (\sigma m) QEx_\sigma .$$

Write $u' = REu/R$ for the quantity reciprocal to any factor u of the n-ad R, in accordance with (1). Then equation (2) gives

$$\Phi = \Phi_1 + \Phi_2,$$

where

$$\Phi_1 = \lambda_r \left[u_r \,.\, u'_r + \sum_{\rho=1}^{s} u_{r+\rho} \,.\, u'_{r+\rho} \right] + \sum_t (\lambda_r + \epsilon_t)\, u_t \,.\, u'_t + \sum_m \lambda_m u_m \,.\, u'_m, \quad (4)$$

$$\Phi_2 = \sum_{\sigma=1}^{s} \left[\epsilon_\sigma (1+\sigma 0)\, u_r + \sum_{\rho=1}^{s} \epsilon_\sigma (\sigma\rho) u_{r+\rho} + \sum_t (\epsilon_\sigma - \epsilon_t)(\sigma t)\, u_t \right.$$
$$\left. + \sum_m (\epsilon_\sigma - \lambda_m + \lambda_r)(\sigma m)\, u_m \right] . \frac{QEx_\sigma}{Q}. \quad (5)$$

No new terms arise in Φ_1 in the limit. As regards Φ_2, we have*

$$\frac{QEx_\sigma}{Q} = \sum_{\alpha=1}^{s} (\sigma\alpha)'\, u'_{r+\alpha},$$

where $(pq)'$ is the minor of (pq) in Δ, divided by Δ. The coefficient of $u_r \,.\, u'_{r+\alpha}$ $(\alpha = 1 \text{ to } s)$ in Φ_2 is

$$\mu_\alpha = \sum_{\sigma=1}^{s} \epsilon_\sigma (1+\sigma 0)(\sigma\alpha)'. \quad (6)$$

The coefficient of $u_{r+\rho} \,.\, u'_{r+\alpha}$ $(\rho, \alpha = 1 \text{ to } s)$ is

$$\nu_{\rho\alpha} = \sum_{\sigma=1}^{s} \epsilon_\sigma (\sigma\rho)(\sigma\alpha)'. \quad (7)$$

The coefficient of $u_t \,.\, u'_{r+\alpha}$ $(\alpha = 1 \text{ to } s)$ is

$$\eta_{t\alpha} = \sum_{\sigma=1}^{s} (\epsilon_\sigma - \epsilon_t)(\sigma t)(\sigma\alpha)'. \quad (8)$$

The coefficient of $u_m \,.\, u'_{r+\alpha}$ $(\alpha = 1 \text{ to } s)$ is

$$\xi_{m\alpha} = \sum_{\sigma=1}^{s} (\epsilon_\sigma - \lambda_m + \lambda_r)(\sigma m)(\sigma\alpha)'. \quad (9)$$

Since the vanishing of the quantities (σq) tends in general to make the inverse set $(\sigma q)'$ infinite, we must consider the possibility of all the quantities μ_α, $\nu_{\rho\alpha}$, $\eta_{t\alpha}$, $\xi_{m\alpha}$ tending to finite limits. Using the same letters for the limits we have therefore the semi-canonical form of the limiting

* A2, pp. 38, 39.

open product, namely*

$$\Phi = \lambda_r \Big[u_r . u'_r + \sum_{\rho=1}^{s} u_{r+\rho} . u'_{r+\rho} + \sum_t u_t . u'_t \Big] + \sum_m \lambda_m u_m . u'_m$$
$$+ \sum_{\alpha=1}^{s} \Big[\mu_\alpha u_r + \sum_{\rho=1}^{s} \nu_{\rho\alpha} u_{r+\rho} + \sum_t \eta_{t\alpha} u_t + \sum_m \xi_{m\alpha} u_m \Big] . u'_{r+\alpha}. \quad (10)$$

3. To find any function $f(\Phi)$ of an open product Φ we replace each root λ by $f(\lambda)$, leaving the axes unaltered. Thus

$$f(\Phi) = f(\lambda^r) \Big[u_r . u'_r + \sum_{\rho=1}^{s} u_{r+\rho} . u'_{r+\rho} + \sum_t u_t . u'_t \Big] + \sum_m f(\lambda_m) u_m . u'_m$$
$$+ \sum_{\alpha=1}^{s} \Big[\mu'_\alpha u_r + \sum_{\rho=1}^{s} \nu'_{\rho\alpha} u_{r+\rho} + \sum_t \eta'_{t\alpha} u_t + \sum_m \xi'_{m\alpha} u_m \Big] . u'_{r+\alpha}, \quad (11)$$

where
$$\mu'_\alpha = \sum_{\sigma=1}^{s} \epsilon'_\sigma (1+\sigma 0)(\sigma\alpha)', \quad (12)$$

$$\nu'_{\rho\alpha} = \sum_{\sigma=1}^{s} \epsilon'_\sigma (\sigma\rho)(\sigma\alpha)', \quad (13)$$

$$\eta'_{t\alpha} = \sum_{\sigma=1}^{s} (\epsilon'_\sigma - \epsilon'_t)(\sigma t)(\sigma\alpha)', \quad (14)$$

$$\xi'_{m\alpha} = \sum_{\sigma=1}^{s} \{\epsilon'_\sigma - f(\lambda_m) + f(\lambda_r)\} (\sigma m)(\sigma\alpha)', \quad (15)$$

and
$$\epsilon'_k = f(\lambda_r + \epsilon_k) - f(\lambda_r). \quad (16)$$

So far approximations have only been made in the first lines of (10) and (11), where no limiting problem arises. We have now to calculate the limiting values of μ'_α, $\nu'_{\rho\alpha}$, $\eta'_{t\alpha}$, $\xi'_{m\alpha}$ in terms of those of μ_α, $\nu_{\rho\alpha}$, $\eta_{t\alpha}$, $\xi_{m\alpha}$. We first eliminate the scalars (σk), that is the modes of evanescence of the x's, leaving only the ϵ's. From (6) and (7) we have

$$\sum_{\beta=1}^{s} \mu_\beta \nu_{\beta\alpha} = \sum_{\beta=1}^{s} \sum_{\sigma=1}^{s} \sum_{\tau=1}^{s} \epsilon_\sigma (1+\sigma 0)(\sigma\beta)' \epsilon_\tau (\tau\beta)(\tau\alpha)'.$$

The summation with respect to β gives zero if $\sigma \neq \tau$ and unity if $\sigma = \tau$. Hence

$$\sum_{\beta=1}^{s} \mu_\beta \nu_{\beta\alpha} = \sum_{\sigma=1}^{s} \epsilon_\sigma^2 (1+\sigma 0)(\sigma\alpha)'.$$

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* The degenerate forms of open products of the third order have been found by different methods by J. W. Gibbs, *Scientific Papers*, Vol. 2, p. 71, and F. L. Hitchcock, *Proc. Roy. Soc. Edinburgh*, Vol. 35 (1915), p. 171.

The result can obviously be generalised. Write

$$\mu_{p\alpha} = \Sigma \mu_\beta \nu_{\beta\gamma} \nu_{\gamma\delta} \dots \nu_{\lambda\alpha} \quad (p \leqslant s),$$

where $\beta, \gamma, \dots, \lambda$ are $p-1$ of the first s positive integers, and summation is over all possible values of each, equality included. Then we can prove by induction that

$$\mu_{p\alpha} = \sum_{\sigma=1}^{s} \epsilon_\sigma^p (1+\sigma 0)(\sigma\alpha)'.$$

Equation (6) is included if we put $\mu_{1\alpha} = \mu_\alpha$. From the first s of these equations we can calculate the s coefficients

$$(1+10)(1\alpha)', \ (1+20)(2\alpha)', \ \dots, \ (1+s0)(s\alpha)'$$

and express μ'_α in terms of $\epsilon_1, \epsilon_2, \dots, \epsilon_s, \mu_\alpha, \mu^2{}_\alpha, \dots, \mu_{s\alpha}$. Write Δ for the determinant

$$\begin{vmatrix} \epsilon_1 & \epsilon_2 & \dots & \epsilon_s \\ \epsilon_1^2 & \epsilon_2^2 & \dots & \epsilon_s^2 \\ \dots & \dots & & \dots \\ \epsilon_1^s & \epsilon_2^s & \dots & \epsilon_s^s \end{vmatrix}$$

and $\Delta_{\sigma p}$ for the minor of ϵ_σ^p in Δ. Write further

$$f_p = \sum_{\sigma=1}^{s} \frac{\epsilon'_\sigma \Delta_{\sigma p}}{\Delta}.$$

Then we have without approximation

$$\mu'_\alpha = f_1 \mu_\alpha + f_2 \mu_{2\alpha} + \dots + f_s \mu_{s\alpha}.$$

If $f(\lambda)$ is holomorphic in the neighbourhood of λ_r, the quantity f_p tends to $f^{(p)}(\lambda_r)/p!$ as the ϵ's tend to zero. Hence

$$\mu'_\alpha = \mu_\alpha f'(\lambda_r) + \frac{\mu_{2\alpha}}{2!} f''(\lambda_r) + \dots + \frac{\mu_{s\alpha}}{s!} f^{(s)}(\lambda_r). \tag{17}$$

The theory for $\nu_{\rho\alpha}$ is similar. Writing

$$\nu_{p\rho\alpha} = \Sigma \nu_{\rho\beta} \nu_{\beta\gamma} \nu_{\gamma\delta} \dots \nu_{\lambda\alpha} \quad (p \leqslant s)$$

we have

$$\nu'_{\rho\alpha} = \nu_{\rho\alpha} f'(\lambda_r) + \frac{\nu_{2\rho\alpha}}{2!} f''(\lambda_r) + \dots + \frac{\nu_{s\rho\alpha}}{s!} f^{(s)}(\lambda_r). \tag{18}$$

As regards $\eta_{t\alpha}$, we have the derived formula

$$\eta_{pt\alpha} = \sum_{\sigma=1}^{s} (\epsilon_\sigma - \epsilon_t) \epsilon_\sigma^{p-1} (\sigma t)(\sigma\alpha)',$$

where $\eta_{1t\alpha} = \eta_{t\alpha}$ and

$$\eta_{pt\alpha} = \Sigma \eta_{t\beta} \nu_{\beta\gamma} \nu_{\gamma\delta} \ldots \nu_{\lambda\alpha} \quad (p \leqslant s).$$

Writing Δ for the determinant

$$\begin{vmatrix} 1 & 1 & \ldots & 1 \\ \epsilon_1 & \epsilon_2 & \ldots & \epsilon_s \\ \ldots & \ldots & \ldots & \\ \epsilon_1^{s-1} & \epsilon_2^{s-1} & \ldots & \epsilon_s^{s-1} \end{vmatrix}$$

and $\Delta_{\sigma p}$ for the minor of ϵ_σ^p, we find as before

$$\eta'_{t\alpha} = \phi_1 \eta_{t\alpha} + \phi_2 \eta_{2t\alpha} + \ldots + \phi_s \eta_{st\alpha},$$

where

$$\phi_p = \sum_{\sigma=1}^{s} \frac{\epsilon'_\sigma - \epsilon'_t}{\epsilon_\sigma - \epsilon_t} \frac{\Delta_{\sigma(p-1)}}{\Delta}.$$

It is clear from the form of equations (6), (8), (12) and (14) that if ϕ_p tends to a definite limit when ϵ_t, as well as $\epsilon_1, \epsilon_2, \ldots, \epsilon_s$, tends to zero. that limit is $f^{(p)}(\lambda_r)/p!$. This will be the case if, for every finite value of ϵ_t, ϕ_p tends to a definite value as $\epsilon_1, \epsilon_2, \ldots, \epsilon_s$ tend to zero. Let ϵ_t therefore be finite, and write $\lambda_t = \lambda_r + \epsilon_t$. Then

$$\phi_p = \sum_{\sigma=1}^{s} \frac{f(\lambda_r + \epsilon_\sigma) - f(\lambda_t)}{\lambda_r + \epsilon_\sigma - \lambda_t} \frac{\Delta_{\sigma(p-1)}}{\Delta}.$$

This expression, however, has the definite limit

$$\frac{1}{(p-1)!} \frac{\partial^{p-1}}{\partial \lambda_r^{p-1}} \left\{ \frac{f(\lambda_r) - f(\lambda_t)}{\lambda_r - \lambda_t} \right\},$$

which proves the theorem. Hence we have finally

$$\eta'_{t\alpha} = \eta_{t\alpha} f'(\lambda_r) + \frac{\eta_{2t\alpha}}{2!} f''(\lambda_r) + \ldots + \frac{\eta_{st\alpha}}{s!} f^{(s)}(\lambda_r). \tag{19}$$

The theory for $\xi_{m\alpha}$ is implicitly contained in the above, and we have

$$\xi'_{m\alpha} = \xi_{m\alpha} \frac{f(\lambda_r) - f(\lambda_m)}{\lambda_r - \lambda_m} + \frac{\xi_{2m\alpha}}{1!} \frac{\partial}{\partial \lambda_r} \left\{ \frac{f(\lambda_r) - f(\lambda_m)}{\lambda_r - \lambda_m} \right\} + \ldots$$

$$+ \frac{\xi_{sm\alpha}}{(s-1)!} \frac{\partial^{s-1}}{\partial \lambda_r^{s-1}} \left\{ \frac{f(\lambda_r) - f(\lambda_m)}{\lambda_r - \lambda_m} \right\}, \tag{20}$$

where

$$\xi_{pm\alpha} = \Sigma \xi_{m\beta} \nu_{\beta\gamma} \nu_{\gamma\delta} \ldots \nu_{\lambda\alpha} \quad (p \leqslant s).$$

The enunciation of the general theorem for any number of sets of coincident axes is cumbrous, but the theorem itself is easily understood.

Corresponding to each set of roots such as λ_r we have terms of the type

$$\lambda_r \left[u_r . u'_r + \sum_{\rho=1}^{s} u_{r+\rho} . u'_{r+\rho} + \sum_t u_t . u'_t \right]$$

$$+ \sum_{a=1}^{s} \left[\mu_a u_r + \sum_{\rho=1}^{s} \nu_{\rho a} u_{r+\rho} + \sum_t \eta_{ta} u_t + \sum_m \xi_{ma} u_m \right] . u'_{r+a},$$

where no account is taken of coincidences outside the r set, except that $u'_r \ldots$ are calculated from a complete external product in which only one axis of a coincident set is retained and the factors made up to the full number by arbitrary extensive quantities. The rules for $f(\Phi)$, or rather the part of $f(\Phi)$ belonging to r, are as above.

1. The quantity $\nu_{p\rho a}$ may be regarded as obtained by repeated application of the formula

$$\nu_{(p+q)\,\rho a} = \sum_{\beta=1}^{s} \nu_{p\rho\beta} \nu_{q\beta a} \quad (p+q \leqslant s), \tag{21}$$

and then the other quantities are given by

$$\mu_{pa} = \sum_{\beta=1}^{s} \mu_\beta \nu_{(p-1)\,\beta a}, \quad \eta_{pta} = \sum_{\beta=1}^{s} \eta_{t\beta} \nu_{(p-1)\,\beta a}, \quad \xi_{pm\beta} = \sum_{\beta=1}^{s} \xi_{m\beta} \nu_{(p-1)\,\beta a}. \tag{22}$$

Equation (21) is of the form of matricular multiplication, as it should be, for considering the special open product

$$\Psi = \sum_{a=1}^{s} \left[\sum_{\rho=1}^{s} \nu_{\rho a} u_{r+\rho} \right] . u'_{r+a}, \tag{23}$$

we have

$$\Psi^p = \sum_{a=1}^{s} \left[\sum_{\rho=1}^{s} \nu_{p\rho a} u_{r+\rho} \right] . u'_{r+a} \quad (p \leqslant s). \tag{24}$$

The possibility of identical relations between the coefficients in (10) has been left open in passing to the limit, and it remains to apply the test that $s+t+1$ of the roots are equal to λ_r. We have

$$\Phi - \lambda = (\lambda_r - \lambda) u_r . u'_r + \sum_{a=1}^{s} v_{ra} . u'_{r+a} + (\lambda_r - \lambda) \sum_t u_t . u'_t + \sum_m (\lambda_m - \lambda)\, u_m . u'_m,$$

where

$$v_{ra} = \mu_a u_r + (\lambda_r - \lambda)\, u_{r+a} + \sum_{\rho=1}^{s} \nu_{\rho a} u_{r+\rho} + \sum_t \eta_{ta} u_t + \sum_m \xi_{ma} u_m.$$

The external product of the antecedents of $\Phi - \lambda$, whose vanishing determines the roots, is obviously independent of the quantities μ, η, ξ. Considering Ψ as a special form of (10) we see that it must have all its roots

zero ; and these are all the necessary restrictions on the generality of (10). Since Ψ is an open product of order s, Ψ^s vanishes identically, and

$$\nu_{s\rho\alpha} \equiv 0. \tag{25}$$

It follows that the series (18) must stop at $\nu_{(s-1)\rho\alpha} f^{(s-1)}(\lambda_r)/(s-1)!$ at most.

Not more than s of the relations (25) are, of course, independent. Other forms may be obtained by considering that the quantities ν' satisfy equations of the same form as the quantities ν. Thus from the relation

$$\sum_{\alpha=1}^{s} \nu_{\alpha\alpha} = 0$$

we derive

$$\sum_{\alpha=1}^{s} \nu_{\alpha\alpha} = \sum_{\alpha=1}^{s} \nu_{2\alpha\alpha} = \ldots = \sum_{\alpha=1}^{s} \nu_{(s-1)\,\alpha\alpha} = 0. \tag{26}$$

To find the identical equation of lowest degree satisfied by the open product Φ we proceed as follows. Let Φ_r be a function of Φ which does not contain any of the extensive quantities u_r, $u_{r+\alpha}$, u_t (which we call the r set) as antecedents. Let Φ_v be another function similarly related to a second simple or multiple root. Then $\Phi_r\Phi_v$ does not contain the r set as antecedents, nor $\Phi_v\Phi_r$ the v set. But since Φ_r and Φ_v are functions of Φ, $\Phi_r\Phi_v = \Phi_v\Phi_r$. Hence $\Phi_r\Phi_v$ contains no member of either the r set or the v set as antecedent. Proceeding to the end we see that the produc $\Pi\Phi_r$ extended over all the roots is identically zero. Hence we have to find the function Φ_r of lowest degree having the required property with respect to u_r, $u_{r+\alpha}$, u_t, and we know from the general theory that this is a power of $\Phi-\lambda_r$.

Put $f(\Phi) = (\Phi-\lambda_r)^q$. Then

$$f^{(p)}(\lambda_r)/p! = 0 \quad \text{if} \quad p \neq q, \quad = 1 \quad \text{if} \quad p = q.$$

Thus $\mu'_\alpha = \nu'_{\rho\alpha} = \eta'_{t\alpha} = 0$ if $q > s$, and

$$\mu'_\alpha = \mu_{q\alpha}, \quad \nu'_{\rho\alpha} = \nu_{q\rho\alpha}, \quad \eta'_{t\alpha} = \eta_{qt\alpha} \quad \text{if} \quad q \leqslant s.$$

Hence $\Phi_r = (\Phi-\lambda_r)^q$, where q is the least number for which

$$\mu_{q\alpha} = \nu_{q\rho\alpha} = \eta_{qt\alpha} = 0, \tag{27}$$

provided that any such number less than s exists, failing which we have $q = s$. It follows that coalescence of the t type *always* causes reduction of the degree of the identical equation, by an amount equal to the number of terms involved, and further reduction may take place. From (21) and (22), if equations (27) are satisfied for any value of q they are satisfied for all higher values up to s, and we have also

$$\xi_{(q+1)\,m\beta} = \ldots = \xi_{sm\beta} = 0 \quad (q < s). \tag{28}$$

Hence we have what seems to be the essential point of Buchheim's second paper,* that if $(\Phi-\lambda_r)^q$ is the highest power of $\Phi-\lambda_r$ in the identical equation, differential coefficients up to the order $q-1$ occur in the corresponding part of $f(\Phi)$.

5. If $f(z)$ is q-valued and Φ a matrix of order n, $f(\Phi)$ is in general n^q-valued,† but functions of special matrices may contain arbitrary constants, or be non-existent. One cause of indeterminacy is that Φ may admit a transformation of axes (as for example the matrix unity), the multiformity of $f(z)$ causing the constants of transformation to appear in the result.‡ The other cause is more closely connected with our present subject. The q-th root of a degenerate matrix is found in practice by assuming a form of the greatest admissible generality as to roots and axes, and comparing its q-th power with the given matrix.§ That arbitrary constants may enter into the solution is clear from the preceding formulæ. Let $f(\Phi)$ in equation (11) be a q-th root of Φ, and let $\lambda_r = 0$. Then from § 4, if $q > s$, the original matrix must satisfy the conditions

$$\mu_\alpha = \nu_{\rho\alpha} = \eta_{t\alpha} = 0,$$

and then there are not equations enough to determine the assumed constants μ'_α, $\nu'_{\rho\alpha}$, $\eta'_{t\alpha}$, $\zeta'_{\iota\alpha}$. Theoretically, there is no need to assume a tentative standpoint, since the indeterminate solutions can be found by a direct limiting process. This point seems of interest, as it leads us to something approaching a general theory of functions of open products.

Starting from the finite polynomial, which is interpreted directly as the sum of a sequence of intelligible operations, we ascend to the Taylor or Laurent series with the roots $\lambda_1, \lambda_2, \ldots$ in the belt of convergence.‖ But there is no need to stop at this. The domain can, in general, be extended by quasi-analytical continuation in powers of $\Phi-\lambda$, where λ is some complex quantity, and thus a larger class of products, namely those whose roots lie within the extended domain, brought within the scope of our formulæ. If we calculate $f(\Phi)$ and then let λ_r move up to a singularity of $f(z)$, there may be either a single limiting form, or one with arbitrary constants, or no finite limit. A branch-point without infinity gives rise to the first two, any infinity of $f(z)$ to the last. We know beforehand that a pole of $f(z)$ will have this effect, since $\Phi-\lambda_r$ cannot be inverted.

* Buchheim, *loc. cit.*

† Taber, *loc. cit.*

‡ C. J. Joly, *Manual of Quaternions*, p. 99.

§ F. L. Hitchcock, *Proc. Roy. Soc. Edinburgh*, Vol. 37 (1917), p. 350.

‖ E. Weyr, *Bull. des Sciences Math.* [2], Vol. 11 (1887), p. 205.

Indeterminacy can occur even in an open product of the third order with a double root: thus $\lambda_3 u_3 . u_3' + \zeta u_3 . u_2'$ has the square root

$$\lambda_3^{\frac{1}{2}} u_3 . u_3' + \mu' u_1 . u_2' + \zeta \lambda_3^{-\frac{1}{2}} u_3 . u_2'.$$

To illustrate the general theory consider the q-th power of the open product

$$\Phi = \lambda (u_1 . u_1' + u_2 . u_2') + \lambda_3 u_3 . u_3' + (\mu u_1 + \zeta u_3) . u_2',$$

where q is real and commensurable and z^q uniformised by a radial cut, so that if $q = m/n$, $f(z)$ is the m-th power of one branch of $z^{1/n}$. From (17) and (20),

$$\Phi^q = \lambda^q (u_1 . u_1' + u_2 . u_2') + \lambda_3^q u_3 . u_3' + \left[q\mu\lambda^{q-1} u_1 + \zeta \frac{\lambda_3^q - \lambda^q}{\lambda_3 - \lambda} u_3 \right] . u_2'.$$

If $q > 1$ there is a determinate limit as $\lambda \to 0$, namely

$$\lambda_3^q u_3 . u_3' + \zeta \lambda_3^{q-1} u_3 . u_2'.$$

If $0 < q < 1$ there is a finite limit if μ tends to zero in the order λ^{1-q}, giving the indeterminate result $\lambda_3^q u_3 . u_3' + \mu' u_1 . u_2' + \zeta \lambda_3^{q-1} u_3 . u_2'$. Finally if q is negative no compensation of coefficients can give a finite limit, illustrating what has been said about the effect of a pole.

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